MODERN ALGEBRA
Structure and Method

DOLCIANI
BERMAN
FREILICH
MODERN ALGEBRA

Structure and Method: Book 1

COVER

These thousands of tiny, doughnut-shaped, ferromagnetic rings, threaded on wires, are memory units in a computer. They are a symbol of modern man's dependence on mathematics. An electrical current passing along the wire sets up a magnetic field which magnetizes the rings. Since current in the opposite direction reverses the magnetic field, the direction of the magnetic field may represent a 0 or 1, a + or -, a yes or no condition. This is the electronic mechanism for storing information in the binary number system used in most modern computers.

Knowledge of algebra has made possible these computers which initially were built to handle scientific and engineering problems. Today, electronic data processing systems are invaluable in business, industry, and research, where they help to untangle and to simplify, in a matter of seconds, calculations and paper work that formerly took days to do.

TITLE PAGE

The illustration on the title pages indicates that future vocational plans are dependent upon your high school preparation. Are you interested in business, chemistry, medicine? Do you look forward to being an engineer, an architect, a home economist, a machinist, a housewife, or a psychologist? Regardless of your vocational plans, algebra is essential to the modern educated person. More important is the fact that algebra is essential to many future vocational opportunities that do not even exist today. Algebra is equipment you will need if you are to take your place as an educated person in the modern world of today and tomorrow.
ALGEBRA

STRUCTURE AND METHOD

• BOOK ONE •

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1 Symbols and Sets

NUMBERS AND THEIR RELATIONSHIPS
- 1-1 Representing Numbers on a Line: Order Relations, 1
- 1-2 Comparing Numbers: The Sign of Equality, 5
- 1-3 Comparing Numbers: The Signs of Inequality, 7
- GROUPING NUMBERS IN SETS AND SUBSETS
- 1-4 Meaning of Membership in a Set, 10
- 1-5 Kinds of Sets, 13
- 1-6 The Graph of a Set, 16
- 1-7 How Subsets Relate to Sets, 18

USING NUMBERS IN ONE OR MORE OPERATIONS
- 1-8 Punctuation Marks in Algebra, 19
- 1-9 Order of Operations, 23

THE HUMAN EQUATION, 25
CHAPTER SUMMARY, 26
CHAPTER TEST, 27
CHAPTER REVIEW, 28
EXTRA FOR EXPERTS, 30
SURVEYORS AND MATHEMATICS, 32
JUST FOR FUN, 33

2 Variables and Open Sentences

ANALYZING ALGEBRAIC STATEMENTS
- 2-1 Evaluating Algebraic Expressions Containing Variables, 35
- 2-2 Identifying Factors, Coefficients, and Exponents, 40
- 2-3 Solving Open Sentences, 44
- PROBLEMS SOLVED WITH VARIABLES
- 2-4 Thinking with Variables: From Symbols to Words, 49
- 2-5 Thinking with Variables: From Words to Symbols, 51
- 2-6 Solving Problems with Open Sentences, 56

EXTRA FOR EXPERTS, 60
THE HUMAN EQUATION, 62
CHAPTER SUMMARY, 63
CHAPTER TEST, 64
CHAPTER REVIEW, 65
JUST FOR FUN, 67

3 Axioms, Equations, and Problem Solving

IDENTIFYING AND USING NUMBER AXIOMS
- 3-1 Axioms of Equality, 69
- 3-2 The Closure Properties, 70
- 3-3 Commutative and Associative Properties of Arithmetic Numbers, 73
- 3-4 The
Distributive Property; Special Properties of 1 and 0, 75 • TRANSFORMING EQUATIONS WITH EQUALITY PROPERTIES • 3-5 Addition and Subtraction Properties of Equality, 80 • 3-6 Division and Multiplication Properties of Equality, 83 • 3-7 Combining Terms and Using Transformation Principles, 86 • 3-8 Equations Having the Variable in Both Members, 91.

MACHINISTS AND MATHEMATICS, 96 • CHAPTER SUMMARY, 97 • CHAPTER TEST, 98 • CHAPTER REVIEW, 100 • CUMULATIVE REVIEW: CHAPTERS 1-3, 102 • EXTRA FOR EXPERTS, 105 • JUST FOR FUN, 107 • THE HUMAN EQUATION, 109.

4 The Negative Numbers 111

EXTENDING THE NUMBER LINE • 4-1 Directed Numbers, 111 • 4-2 Comparing Numbers, 114 • OPERATING WITH DIRECTED NUMBERS • 4-3 Addition on the Number Line, 116 • 4-4 The Opposite of a Directed Number, 120 • 4-5 Absolute Value, 123 • 4-6 Adding Directed Numbers, 124 • 4-7 Subtracting Directed Numbers, 128 • 4-8 Multiplying Directed Numbers, 133 • 4-9 Dividing Directed Numbers, 138 • 4-10 Averages and Directed Numbers (Optional), 142.

CHAPTER SUMMARY, 144 • CHAPTER TEST, 145 • CHAPTER REVIEW, 146 • ELECTRICAL ENGINEERS AND MATHEMATICS, 151 • EXTRA FOR EXPERTS, 152 • JUST FOR FUN, 153 • THE HUMAN EQUATION, 155.

5 Equations, Inequalities, and Problem Solving 157

OPEN SENTENCES IN THE SET OF DIRECTED NUMBERS • 5-1 Transforming Equations, 157 • 5-2 The Properties of Inequality, 159 • 5-3 Pairs of Inequalities (Optional), 164 • THE ANALYSIS OF PROBLEMS • 5-4 A Plan for Solving Problems, 166 • 5-5 Problems about Consecutive Integers, 170 • 5-6 Problems about Angles, 172 • 5-7 Uniform Motion Problems, 178 • 5-8 Mixture Problems, 182.

THE HUMAN EQUATION, 185 • CHAPTER SUMMARY, 186 • CHAPTER TEST, 187 • CHAPTER REVIEW, 188 • PSYCHOMETRISTS AND MATHEMATICS, 192 • JUST FOR FUN, 193 • EXTRA FOR EXPERTS, 194.
6 Working with Polynomials

ADDITION AND SUBTRACTION OF POLYNOMIALS 6-1 Adding Polynomials, 197 6-2 Subtracting Polynomials, 200 MULTIPLICATION OF POLYNOMIALS 6-3 The Product of Powers, 203 6-4 The Power of a Product, 204 6-5 Multiplying a Polynomial by a Monomial, 206 6-6 Multiplying Two Polynomials, 209 6-7 Problems about Areas, 211 6-8 Powers of Polynomials, 213 DIVISION OF POLYNOMIALS 6-9 The Quotient of Powers, 215 6-10 Zero as an Exponent (Optional), 218 6-11 Dividing a Polynomial by a Monomial, 219 6-12 Dividing a Polynomial by a Polynomial, 221

THE HUMAN EQUATION, 224 CHAPTER SUMMARY, 225 CHAPTER TEST, 226 CHAPTER REVIEW, 227 CUMULATIVE REVIEW: CHAPTERS 1-6, 230 EXTRA FOR EXPERTS, 232 JUST FOR FUN, 234 MERCHANDISERS AND MATHEMATICS, 235

7 Special Products and Factoring

THE DISTRIBUTIVE PROPERTY IN FACTORING 7-1 Factoring in Algebra, 237 7-2 Identifying Common Factors, 241 7-3 Multiplying the Sum and Difference of Two Numbers, 245 7-4 Factoring the Difference of Two Squares, 246 QUADRATIC TRINOMIALS 7-5 Squaring a Binomial: Plateau Section, 248 7-6 Factoring a Trinomial Square, 251 7-7 Multiplying Binomials at Sight, 253 7-8 Factoring the Product of Binomial Sums or Differences, 255 7-9 Factoring the Product of a Binomial Sum and a Binomial Difference, 257 7-10 General Method of Factoring Quadratic Trinomials, 259 EXTENSION OF FACTORING 7-11 Combining Several Types of Factoring, 261 7-12 Working with Factors Whose Product Is Zero, 263 7-13 Solving Polynomial Equations by Factoring, 264 7-14 Using Factoring in Problem Solving, 267

COUNTY AGENTS AND MATHEMATICS, 271 CHAPTER SUMMARY, 272 CHAPTER TEST, 273 CHAPTER REVIEW, 274 EXTRA FOR EXPERTS, 276 JUST FOR FUN, 278
## 8 Working with Fractions

### FRACTIONS AND RATIOS
- 8–1 Defining Algebraic Fractions, 281
- 8–2 Reducing Fractions, 283
- 8–3 Ratio, 286
- 8–4 Per Cent and Percentage Problems, 289

### MULTIPLYING AND DIVIDING FRACTIONS
- 8–5 Multiplying Fractions, 292
- 8–6 Dividing Fractions, 295
- 8–7 Fractions Involving Multiplication and Division, 296

### ADDING AND SUBTRACTING FRACTIONS
- 8–8 Combining Fractions with Equal Denominators, 297
- 8–9 Adding Fractions with Unequal Denominators, 299
- 8–10 Mixed Expressions, 302
- 8–11 Complex Fractions (Optional), 304

### FRACTIONS IN OPEN SENTENCES AND PROBLEMS
- 8–12 Open Sentences with Fractional Coefficients, 306
- 8–13 Investment Problems, 308
- 8–14 Per Cent Mixture Problems, 310
- 8–15 Fractional Equations, 312
- 8–16 Work Problems, 314
- 8–17 Motion Problems, 316

### JUST FOR FUN, HOME ECONOMISTS AND MATHEMATICS
- HOME ECONOMISTS AND MATHEMATICS, 320
- CHAPTER SUMMARY, 321
- CHAPTER TEST, 322
- CHAPTER REVIEW, 323
- CUMULATIVE REVIEW: CHAPTERS 1–8, 326
- EXTRA FOR EXPERTS, 328
- THE HUMAN EQUATION, 331

## 9 Graphs

### ORDERED PAIRS OF NUMBERS AND POINTS IN A PLANE
- 9–1 Open Sentences in Two Variables, 333
- 9–2 Coordinates in a Plane, 337

### LINEAR EQUATIONS AND STRAIGHT LINES
- 9–3 The Graph of a Linear Equation in Two Variables, 340
- 9–4 Slope of a Line, 343
- 9–5 The Slope-Intercept Form of a Linear Equation, 346
- 9–6 Determining the Equation of a Line, 349

### INEQUALITIES AND SPECIAL GRAPHS
- 9–7 Graph of an Inequality in Two Variables, 350
- 9–8 Graphs That Are Parabolas, 353

### STATISTICAL GRAPHS
- 9–9 Broken-Line and Bar Graphs, 354
- 9–10 Circle Graphs, 357

### JUST FOR FUN, CHAPTER SUMMARY, CHAPTER TEST, CHAPTER REVIEW, EXTRA FOR EXPERTS, THE HUMAN EQUATION
10 Sentences in Two Variables

SOLVING SYSTEMS OF LINEAR OPEN SENTENCES • 10–1 The Graphic Method, 367 • 10–2 The Addition and Subtraction Method, 370 • 10–3 Problems with Two Variables, 372 • 10–4 Multiplication in the Addition and Subtraction Method, 374 • 10–5 The Substitution Method, 378 • 10–6 Graphs of Pairs of Linear Inequalities (Optional), 379 • ADDITIONAL PROBLEMS • 10–7 Digit Problems, 381 • 10–8 Motion Problems, 383 • 10–9 Age Problems, 385 • 10–10 Problems about Fractions, 386 • THE HUMAN EQUATION, 388 • CHAPTER SUMMARY, 389 • CHAPTER TEST, 389 • CHAPTER REVIEW, 390 • EXTRA FOR EXPERTS, 393 • JUST FOR FUN, 395 •

11 The Real Numbers

THE SYSTEM OF RATIONAL NUMBERS • 11–1 The Nature of Rational Numbers, 397 • 11–2 Decimal Forms of Rational Numbers, 400 • IRRATIONAL NUMBERS • 11–3 Roots of Numbers, 403 • 11–4 Properties of Irrational Numbers, 407 • 11–5 Geometric Interpretation of Square Roots, 411 • RADICAL EXPRESSIONS • 11–6 Multiplication, Division, and Simplification of Radicals, 414 • 11–7 Addition and Subtraction of Radicals, 417 • 11–8 Multiplication of Binomials Containing Radicals, 419 • 11–9 Radical Equations, 421 • CIVIC PLANNERS AND MATHEMATICS, 424 • CHAPTER SUMMARY, 425 • CHAPTER TEST, 426 • CHAPTER REVIEW, 427 • CUMULATIVE REVIEW: CHAPTERS 1–11, 429 • EXTRA FOR EXPERTS, 431 • THE HUMAN EQUATION, 433 •

12 Functions and Variation

SELECTING PAIRS OF NUMBERS • 12–1 Relations, 435 • 12–2 Functions, 438 • VARIATION • 12–3 Direct Variation and Proportion, 442 • 12–4 Inverse Variation, 447 • 12–5 Joint Variation and Combined Variation (Optional), 452 • CHAPTER SUMMARY, 456 • CHAPTER TEST, 457 • CHAPTER REVIEW, 458 • PETROLEUM CHEMISTS AND MATHEMATICS, 460 • EXTRA FOR EXPERTS, 461 • THE HUMAN EQUATION, 463 •
13 Quadratic Equations and Inequalities 465

GENERAL METHODS OF SOLVING QUADRATIC EQUATIONS
• 13-1 The Square-Root Property, 465 • 13-2 Checking Solution Sets, 467 • 13-3 Completing a Trinomial Square, 469 • 13-4 The Quadratic Formula, 473 • 13-5 The Nature of the Roots of a Quadratic Equation (Optional), 476 • THE SOLUTION OF QUADRATIC INEQUALITIES • 13-6 Solving Quadratic Inequalities (Optional), 479 • 13-7 Using Graphs of Equations to Solve Inequalities (Optional), 482 •

CHAPTER SUMMARY, 484 • CHAPTER TEST, 486 • CHAPTER REVIEW, 486 • EXTRA FOR EXPERTS, 489 • ACTUARIES AND MATHEMATICS, 491 •

14 Geometry and Trigonometry 493

GEOMETRY • 14-1 Geometric Assumptions, 493 • 14-2 Rays and Angles, 495 • 14-3 Similar Triangles, 498 • NUMERICAL TRIGONOMETRY • 14-4 The Tangent Function, 501 • 14-5 The Sine and Cosine Functions, 504 • 14-6 A Table of Trigonometric Function Values, 506 • VECTORS • 14-7 Working with Vectors, 509 • 14-8 Resolving a Vector, 512 •

CHAPTER SUMMARY, 514 • CHAPTER TEST, 515 • CHAPTER REVIEW, 516 • THE HUMAN EQUATION, 519 • EXTRA FOR EXPERTS, 520 • NAVIGATORS AND MATHEMATICS, 523 •

15 Comprehensive Review and Tests 525

REVIEW YOUR ALGEBRA • 15-1 Properties of Numbers: Structure, 525 • 15-2 Algebraic Representation, 527 • 15-3 Fundamental Operations and Factoring, 528 • 15-4 Radicals, 529 • 15-5 Equations, 530 • 15-6 Functions and Variation, 531 • 15-7 Inequalities, 532 • 15-8 Problems, 532 • 15-9 Indirect Measurement: Vectors, 535 • ALGEBRAIC PRINCIPLES • 15-10 A True-False Test, 536 • 15-11 A Completion Test, 538 •

APPENDIX, 540 • INDEX, 544
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Symbols and Sets

Where is science taking us? In the direction of our dreams. Mathematics, the language of science, is the language of dreamers who plan to achieve their dreams.

The man at the drawing board is taking one of the first steps toward making a dream come true. He is translating an idea into a set of drawings (as at the left). To do this he is using symbols which are understood universally by men who turn dreams into realities. Scientists of all languages exchange ideas by using the symbols of mathematics.

You need symbols not only to help you organize your own ideas, but also to explain your ideas to others. You already have learned something about communicating with mathematical symbols. Now you will add to that knowledge and learn how these symbols help you clarify your thinking.

NUMBERS AND THEIR RELATIONSHIPS

1-1 Representing Numbers on a Line: Order Relations

In arithmetic, you learned a good deal about how to use numbers. In algebra, your initial aim will be to discover some of the properties of the numbers of arithmetic. By the word “property” is meant a distinguishing trait or an essential quality. You use this word with meaning when you say, “Sweetness is a property of sugar; hardness is a property of diamonds.”

As you will see, one of the properties of numbers — and you will learn many more — is illustrated in Figure 1-1 by the number line, sometimes also called the number scale. (This number scale has been drawn with an arrowhead to indicate the direction in which the line can be extended as far as you wish.)

\[0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}\]

Figure 1-1
You will notice that some of the points on the line have been labeled. The starting point of the scale is labeled 0 (zero). The label for any point is a numeral or name for a number that is a measure of the distance of the point from zero. So, to label any point in Figure 1–1, all you have to know is its distance from 0. You take for granted that there is always a point which is the partner of an arithmetic (ar’ith-met-ic)* number. This means that every arithmetic number has exactly one corresponding point on the line. In Figure 1–1, the labeled points are paired with whole numbers or common fractions. Later you will find that many points are partners of numbers which are neither whole numbers nor common fractions.

Although you can freely choose the unit in terms of which you scale (mark) a line, it is important that the scale be uniform. Thus, the distance between the points labeled 0 and \( \frac{1}{2} \) must be the same as the distance between the points labeled \( \frac{3}{2} \) and 2.

The number scale shows which of two numbers is the larger. For example, the number 1 is smaller than the number 5; the point labeled 1 lies to the left of the point labeled 5. Any number less than 5 corresponds to a point lying to the left of 5; whereas, any number greater than 5 corresponds to a point lying to the right of 5.

You have just discovered that the first property of arithmetic numbers is that these numbers can be “lined up” or “ordered.” That is, they can be arranged in order of magnitude (size) and associated with the points of a uniformly scaled line. The number that is paired with a point on the number line is called the coordinate (ko-or-din-it) of that point on the line. The point paired with a number is called the graph of that number.

What do you mean when you say 3 is between 1 and 5? You mean that 3 is greater than 1, but less than 5; or the graph of 3 is to the right of the graph of 1, but to the left of the graph of 5.

When you use a phrase like “the numbers between 1 and 5,” you intend to include neither 1 nor 5. The phrase “the numbers between 1 and 5, inclusive,” says that you want to include both 1 and 5. The phrase “the numbers between 1 and 5, including 5,” says you want to include 5 but not 1.

*Primary accent is shown by dark type (met), and secondary, by an accent mark (ar’).
ORAL EXERCISES

Locate the points that are the graphs of the given numbers or state the coordinates of the given points. Refer to the accompanying number line.

\[
\begin{array}{cccccccccc}
V & R & S & A & M & P & Q & T & B & K \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

**SAMPLE 1.**  \( B \)  What you say: The coordinate is 8.

**SAMPLE 2.**  \( 1 \frac{1}{4} \)  What you say: The point that is one-fourth of the way from \( R \) to \( S \).

1. 5  
2. \( P \)  
3. 3  
4. 0  
21. The point halfway between \( V \) and \( S \)
22. The point halfway between \( S \) and \( B \)
23. The point halfway between \( V \) and \( K \)
24. The whole numbers less than 6 and \( S \)
25. The whole numbers greater than \( \frac{3}{4} \) and less than 5
26. The whole numbers between 5 and 9, inclusive

WRITTEN EXERCISES

Give the coordinate for each \( R \), \( S \), and \( T \) on the following number lines.

**SAMPLE.**

\[
\begin{array}{cccc}
A & R & S & B \\
0 & \frac{1}{3} & \frac{2}{3} & 1
\end{array}
\]

Solution: \( R: \frac{1}{3} \)  \( S: \frac{2}{3} \)  \( T: 1 \frac{1}{3}, \) Answer.
Name the coordinate of the point given. Refer to the number line.

**SAMPLE.** The point that is two-thirds of the distance from $H$ to $G$

**Solution:** The length of the line from $H$ to $G$ is 2 units. The point is $\frac{2}{3} \times 2$ or $\frac{4}{3}$ units from $H$. Its coordinate is $5 + \frac{4}{3}$ or $\frac{19}{3}$.

Then $\frac{19}{3}$ or $6\frac{1}{3}$, **Answer.**

7. The point $H$
8. The point $F$
9. The point halfway from $J$ to $T$
10. The point halfway from $L$ to $W$
11. The point one-fourth of the distance from $T$ to $G$
12. The point halfway between $T$ and $C$
13. The point halfway between $L$ and $H$
14. The point one-third of the distance from $J$ to $L$
15. The point 2 units to the right of $E$
16. The point 3 units to the left of $T$
17. The point 1.5 units to the right of $J$
18. The point $4\frac{1}{3}$ units to the left of $G$

19. The point one-sixth of the distance from $E$ to $H$
20. The point one-fifth of the distance from $T$ to $L$
SYMBOLS AND SETS

21. The point two-thirds of the distance from J to G
22. The point three-quarters of the distance from W to F
23. The whole numbers greater than 2 and less than 4 1/4
24. The whole numbers between 7 and 8, inclusive
25. The whole numbers between 2 and 8
26. The whole numbers between 0 and 1, including 0
27. The whole numbers between 7/8 and 9/2, inclusive, that are exactly divisible by 3
28. The whole numbers between F and H that are exactly divisible by 5
29. The point between T and H that is three times as far from H as it is from T
30. The point between F and L that is twice as far from F as it is from L

1-2 Comparing Numbers: The Sign of Equality

You already know at least two ways of writing any whole number. For example, you can represent eight by the Roman numeral VIII or by the Arabic numeral 8. In the Braille system for the blind, eight is shown by raised points arranged like this: • • •. The flashing red light of the binary computer in the margin indicates eight. In Morse code, ——— represents eight. Spanish boys write ocho as the name for eight. Each of these representations is a symbol for the number eight. Names or symbols for numbers are called numerical expressions or numerals.

You use symbols every day of your life. Can you explain the meaning of these familiar symbols?

eight £ 5 + 3 1 + 7 = £ 5 - 3

Of course, you easily recognize the meanings of these symbols. Notice particularly that the specific signs +, −, ×, ÷ tell you what to do with numbers in arithmetic. Using these instruction symbols of arithmetic, you can express the number eight in many other ways. As a matter of fact, \( \frac{18 + 22}{5} \) and \( \frac{53 - 5}{2 \times 3} \) each designates eight. Other numerical expressions for eight are \( 1 + 7, 7 + 1, 0 + 8, 8 + 0, 2 \times 4, 4 \times 2, 4 + 0 + 4, X - II, 5 + 3. \) But, the important symbol = which stands for the word “equals” or for the words “is equal to” allows you to say that 1 + 7 and 5 + 3 designate the same number: 1 + 7 = 5 + 3; in words, “one plus seven equals five plus
three.” On the other hand, if you wrote $1 + 7 = 4 + 5$, the state-
ment would be false because $1 + 7$ and $4 + 5$ are not expressions for
the same number.

**ORAL EXERCISES**

Tell whether or not each statement is true. Give a reason for your answer.

**SAMPLE 1.**  $2 + 4 = 3 + 3$

**What you say:** True, because $2 + 4$ and $3 + 3$
each designates the number 6.

**SAMPLE 2.**  $3 \times 9 = 9 \div 3$

**What you say:** False, because $3 \times 9$ designates 27, but $9 \div 3$
designates 3.

1. $2 \times 4 = 2 \times 2 \times 2$
2. $7 \times 5 = 5 \times 7$
3. $2 + 3 = 7 - 2$
4. $6 + 0 = 5 + 1$
5. $6 \times 0 = 5 + 1$
6. $5 - 4 = 1 + 0$
7. $8 \div 4 = 4 \div 8$
8. $2 + 10 = 5 + 7$
9. $3 \times 4 = 9 + 0 + \frac{9}{3}$
10. $9 \times 0 = 1 \times 0$
11. $49 - 9 = 4 \times 5 \times 2$
12. $2 + 2 + 2 = 3 \times 2$
13. $.001 + 8 = .008 + 1$
14. $3 \times 1 = \frac{1}{8} \times 24$
15. $19 - 3 - 2 = 19 - 5$
16. $8 \times 1 = 2 \times 4 \times 1$
17. $6 \div 1 = \frac{6}{1}$
18. $10 \times \frac{1}{5} = 10 \times .2$
19. $1 + 5 + 4 = 10 - 1$
20. $12 \div 3 = 2 \times 5$
21. $19 - 2 - 2 - 2 = 19 - 6$
22. $19 - 6 = 19 - 2 - 2 - 2$
23. $\frac{3}{2} = 2 \times 1 \times 2$
24. $1 \div 5 = 5 \div 1$

**WRITTEN EXERCISES**

Tell whether or not each statement is true.

**A**

1. $8 \times 2.5 = 6 \times 4.5$
2. $.5 + .4 + .3 = .3 \times .4$
3. $\frac{1}{3} + \frac{1}{2} = \frac{3}{6} + \frac{3}{6}$
4. $\frac{1}{0.1} = \frac{2}{0.2}$
5. $\frac{5}{8} \div 15 = 24$
6. $6 \div \frac{5}{3} = 4$
7. $17 \times 1 \times 5 = 100 \div 1\frac{3}{4}$
8. $.1 \times 0 \times 8 = 0 \div \frac{4}{5}$
9. $42 - 15 = 5 \times 3$
10. \[ \frac{51 + 14}{17 - 7} = 1\frac{2}{3} + 8\frac{1}{3} - 3\frac{1}{2} \]
In each case, find a numeral to replace the question mark and make the resulting statement true.

\[ 11. \ 8 + 9 + ? = 31 - 12 \]
\[ 12. \ 32 + 19 + ? = 86 - 35 \]
\[ 13. \ 3\frac{3}{8} - ? = \frac{3}{4} + \frac{1}{3} \]
\[ 14. \ 6\frac{1}{2} - ? = 2\frac{3}{4} + 1\frac{5}{6} \]

\[ 15. \ .7 \ - \ .3 = 2 \times ? \]
\[ 16. \ .7 \ + \ .3 = .7 \div ? \]
\[ 17. \ .24 \div .6 = 1 - ? \]
\[ 18. \ 3.6 \div .9 = 1 \div ? \]

1-3 Comparing Numbers: The Signs of Inequality

To change the false statement

\[ 7 = 6 - 3 \]

into a true one, you may use the symbol \( \neq \), translated “is not equal to” or “does not equal.” Thus, a true statement is

\[ 7 \neq 6 - 3. \]

Can you tell why the following statement is also true?

\[ \frac{12 + 3}{3} \neq 12 + 1 \]

Another inequality symbol you will use is \( > \), which is read “is greater than.” Thus, \( 5 > 3 \) means “five is greater than three.”

The symbol \( < \) stands for the words “is less than.” When you write \( 3 < 5 \) you say, “three is less than five.”

The statements \( 5 > 3 \) and \( 3 < 5 \) both give the same information: 5 is a larger number than 3, and the graph of 3 is to the left of the graph of 5, or the graph of 5 is to the right of the graph of 3.

To avoid confusing the symbols \( < \) and \( > \), think of them as arrowheads always pointing to the numeral for the smaller number. For example,

\[ 39 - 33 < 14 \]
\[ 1000 > 66 \times 11 \]

are true statements,
whereas

\[ 1.732 > 17 \quad \text{and} \quad 1 < 0.9999 \]

are false statements.

You remember that the statement “3 is between 1 and 5” means that 3 is greater than 1 and also that 3 is less than 5. In symbols, you would write

\[ 3 > 1 \quad \text{and} \quad 3 < 5. \]

It is customary to put this pair of statements together and write

\[ 1 < 3 < 5 \quad \text{or graph it.} \]

This new expression is sometimes translated, “1 is less than 3, and 3 less than 5” or “3 is between 1 and 5.” This expression illustrates the neatness of mathematical symbolism and its economy of space.

**ORAL EXERCISES**

Tell whether or not the statement in each of the following exercises is true.

**SAMPLE 1.** \[ 7 + 2 = 3 \times 3 \]

*What you say:* True, because \( 7 + 2 \) and \( 3 \times 3 \) each designates the number 9.

**SAMPLE 2.** \[ \frac{10 + 2}{2} \neq 10 + 1 \]

*What you say:* True, because \( \frac{10 + 2}{2} \) designates 6, but \( 10 + 1 \)

1. \( 74 + 28 = 28 + 74 \)
2. \( 48 \div 4 \neq 6 \times 2 \)
3. \( 3 \times 3 = 3 + 3 + 3 \)
4. \( 5 + 3 = 5 - 3 \)
5. \( 53 + 47 = 57 + 43 \)
6. \( 4 + 4 = 2 \times 4 \)
7. \( 1 \times 110 \neq 110 \)
8. \( 2 \times 7 \neq 7 \times 7 \)
9. $18 + 0 = 3 \times 6$
10. $7 + 5 < 4 \times 5$
11. $\frac{5 + 1}{2} = 4 - 1$
12. $\frac{12 + 8}{5} > 2 \times 2$
13. $\frac{30 - 18}{6} = 5 - 3$
14. $\frac{48 - 12}{12} < 48 - 1$
15. $\frac{15 + 18}{3} \approx 5 + 6$
16. $\frac{12 + 5}{2} > 10$

17. $50 = \text{the number of states in the U.S.A.}$
18. $47 \times 32 \approx 37 \times 42$
19. $8 \times 6 > 40 + 8$
20. $\frac{1}{2} + \frac{3}{8} > 1$
21. $12 + 0 \neq 15 + 0$
22. $\frac{23}{1} < 23$
23. $15 \times 0 = 15$
24. $15 \times 0 \neq 15 \times 0$
25. $\frac{3}{5} + \frac{3}{5} < \frac{3}{5} + \frac{3}{5}$

**WRITTEN EXERCISES**

Make a true statement by replacing each question mark with the sign $=, <$, or $>$.  

**SAMPLE 1.** $5 \, ? \, 2$

**Solution:** $5 > 2$

**A**

1. $4 + 5 \, ? \, 10 - 1$
2. $13 \times 0 \, ? \, 16 + 0$
3. $7 \, ? \, 7$
4. $5 \, ? \, 6$
5. $0 \, ? \, 0$
6. $\frac{15}{5} \, ? \, 2 + 3$
7. $\frac{8 - 3}{5 - 2} \, ? \, 1 + .6$
8. $6 \times 0 \, ? \, 0$
9. $5 \times 1 \, ? \, 5 \times 0$
10. $2 \frac{1}{3} \, ? \, 1 + 1 \frac{2}{16}$
11. $\frac{1}{3} \, ? \, \frac{1}{6}$
12. $\frac{1}{5} + \frac{3}{4} \, ? \, \frac{6}{8} + .2$
13. $65 \times 1 \, ? \, 66$
14. $\frac{1}{3} + \frac{4}{6} \, ? \, \frac{5}{9}$
15. $3 \, ? \, 3 \times 0$
16. $\frac{1}{2} \, ? \, \frac{1}{4} + \frac{1}{2}$
17. $2 \times 10 \, ? \, 17 - 7$
18. $\frac{.75 + .25}{\frac{3}{4} - \frac{1}{3}}$
19. $\frac{\frac{5}{7} - \frac{2}{3}}{\frac{9}{16} + \frac{2}{3}}$
20. $4 \div 2 \, ? \, 8 \div 4$
21. $\frac{1}{3} \, ? \, 4 \times \frac{1}{3}$
22. $\frac{1}{3} \times 4 \, ? \, \frac{3}{4}$
23. $\frac{3}{9} - \frac{3}{8} \, ? \, \frac{3}{3} \times 3$
24. $.6 - .2 \, ? \, .2 \times 2$
Copy each of the following statements. Replace each question mark by any numeral that makes the resulting statement true.

**SAMPLE 2.** \(10 \times 2 \neq 20 + ?\)

**Solution:** \(10 \times 2 \neq 20 + \frac{1}{4}\) (or any numeral other than 0)

25. \(4 + ? = 11 - 6\)
26. \(? + 1 \neq 8\)
27. \(5 + 3 = 3 + ?\)
28. \(8 \times ? = 0\)
29. \(? \div 10 = 0\)
30. \(5 + ? = 7\)
31. \(14 = 7 + ?\)
32. \(? + 7 = 7\)
33. \(2 \times ? < 17 - 7\)
34. \(? \times 0 = 0\)
35. \(3 \times ? = 9 + 6\)
36. \(12 \times ? > 4 \times 12\)
37. \(132 \neq 11 \times ?\)
38. \(? - 8 < 8\)
39. \(? \times 15 > 0\)
40. \(? \div 8 = 1\)

**GROUPING NUMBERS IN SETS AND SUBSETS**

1–4 **Meaning of Membership in a Set**

You are accustomed to talking about all sorts of collections or sets of objects. A set of dishes, a chemistry set, the high school crowd, a pair of shoes, all the whole numbers, are familiar examples of collections of objects. The mathematician calls any such collection of objects a *set*. Each object in a set is called a *member* or *element* of the
set. For example, all the teachers in your school form a set, and your algebra teacher is a member or element belonging to that set. However, objects such as the letter \( r \), the school custodian, and the number 9, are not elements in the set of all your teachers. Thus, a set is any collection of objects so well described that you can always tell whether or not an object belongs to the set.

Suppose a set is formed of any five whole numbers. Use a capital letter, say \( R \), to name or refer to the set. You have no way of telling whether the number 3 is or is not an element of \( R \) until the five whole numbers are specified. If you specify the set by listing the objects forming the set within braces \{ \}, then you may have

\[
R = \{0, 3, 7, 8, 14\}
\]

This says, "\( R \) is the set of numbers 0, 3, 7, 8, 14." You can easily see that the number 3 is a member of this set and that the number 4 is not. We use a special symbol, \( \in \), to mean "is an element of," and \( \not\in \) to mean "is not an element of." Thus, \( 3 \in R \) and \( 4 \not\in R \).

Specifying a set by listing its elements in braces gives you a roster or list of the set. The objects named in the listing, \{our moon, the Constitution, the Alamo, California, Albert Einstein\}, form a set. Note that the elements of a set need have no relation with one another other than being listed together. Furthermore, the order of listing the elements is unimportant. What is important is that each element be named in the listing.

Often a roster is an inconvenient way of specifying a set. For example, a roster of the set of states in the U.S.A. requires the listing of all 50 of the elements within braces. This inconvenience is overcome by writing within braces a rule which describes the elements of a set. Thus,

\[
\{\text{the states of the U.S.A.}\}
\]

says, "The set of the states of the U.S.A."

**EXAMPLE.** Specify the set of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 by (a) roster, (b) rule.

**Solution:**

(a) \{1, 2, 3, 4, 5, 6, 7, 8, 9\}

(b) \{the whole numbers between 0 and 10\} or

\{one-digit numbers except 0\} or

\{whole numbers from 1 to 9, inclusive\}
Specify each of the following sets by a roster.

**SAMPLE 1.** \{the letters in the word Mississippi\}

*What you say:* \{i, m, p, s\}

1. \{the letters in the word *freshman*\}
2. \{the letters in your given name\}
3. \{the numerals on the face of a clock\}
4. \{the whole numbers less than 20\}
5. \{students in your row in the algebra class\}
6. \{states of the U.S.A. on the Gulf of Mexico\}

Specify each of the following sets by a rule.

**SAMPLE 2.** \{\(\frac{1}{17}, \frac{3}{5}, \frac{1}{7}\)\}

*What you say:* \{every fraction whose numerator is 1 and whose denominator is an odd number less than 8\}

7. \{Alaska, Hawaii\}
8. \{California, Washington, Oregon\}
9. \{2, 3, 4\}
10. \{2, 4, 6\}
11. \{Eisenhower, Truman\}
12. \{Dolciani, Freilich, Berman\}
13. \{1, 3, 5, 7\}
14. \{\(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\)\}
15. \{Illinois, Indiana, Idaho, Iowa\}
16. \{Los Angeles, San Francisco\}
17. \{a, e, i, o, u\}
18. \{Saturday, Sunday\}
19. \{20, 10, 5, 15\}
20. \{\(\frac{1}{2}, \frac{1}{6}, \frac{1}{8}, \frac{1}{4}, \frac{1}{10}\)\}
21. \{Washington (D.C.)\}
22. \{London, Paris\}
23. \{x, w, z, y\}
24. \{Jefferson Davis\}
25. \{1, 7, 4, 13, 10\}
26. \{16, 1, 11, 6, 21\}

Tell whether or not each statement is true. Give a reason for each answer.

27. \(5 \in \{\text{whole numbers less than } 5\}\)
28. \(5 \not\in \{15, 20, 25\}\)
29. \(3 \not\in \{\text{whole numbers less than } 5\}\)
30. \(8 \in \{0, 8, 9\}\)
31. \(\frac{1}{3} \in \{\text{multiples of } \frac{1}{4}\}\)
32. \(12 \not\in \{\text{even numbers}\}\)
33. \(\frac{1}{6} \in \{.25, .5, .75\}\)
34. \(7 \not\in \{1, 9, 12, 7, 21, 15\}\)
1–5 Kinds of Sets

In counting the number of eggs in the basket in Figure 1–2—one, two, three, four—you really pair each egg with a number as shown, and conclude that there are as many eggs as there are numbers in \( \{1, 2, 3, 4\} \). This pairing of eggs with numbers is a one-to-one correspondence. Two sets are in one-to-one correspondence when each member of one set has one partner in the other set, and no element in either set is without a partner. The pairing of point and number on a number line is another example of one-to-one correspondence.

Can you list all the members of the set of whole numbers? If you start to write

\[
\{0, 1, 2, 3, 4, 5, 6, \ldots\}
\]

you will never come to the end of the list. The three dots after the 6 are the mathematician’s way of indicating that the roster continues without end. A set which has so many elements that the process of counting them would never come to an end is called an infinite set. For example, you cannot list the members of

\[
\{\text{all the fractions between 0 and 1}\}
\]

although the rule enables you to identify them. Another infinite set is the set of points on a line. A set containing a large number of elements is not always an infinite set. Thus, \( \{\text{the grains of sand on the beach at Waikiki}\} \) is not an infinite set, even though it has many members.

A set is finite, or has a finite number of elements, if the process of counting the elements comes to an end. Such a set is

\[
\{\text{two-digit numbers}\} = \{10, 11, 12, \ldots, 99\}.
\]

In this example, the three dots mean and so on through.

Can a set have no elements? Consider the set of whole numbers between 8 and 9. This set contains no elements and is called the empty set or null set. Notice that this symbol \( \{0\} \) does not designate the empty set. It contains the number 0. Empty braces \( \{\} \) might be used, but a special symbol \( \emptyset \), written without braces, usually is used to designate the null set.
The notion of the empty set may seem strange at first. Still, how often have you reached into your pocket or purse to find it empty of coins? The set of coins in your pocket or purse was the empty set. By agreement there is only one null set. Thus, the set of whole numbers between 8 and 9 and the set of coins contained in your empty purse or pocket are one and the same set.

**ORAL EXERCISES**

Use a roster to specify each of the following sets.

**SAMPLE.** \{Persons now 2000 years of age\}

What you say: The empty set, \(\emptyset\).

1. \{living dogs with wings\}
2. \{even numbers\}
3. \{multiples of 7\}
4. \{letters of the alphabet\}
5. \{unit fractions\}
6. \{three-digit numbers\}
7. \{multiples of 12 less than 200\}
8. \{leap years from 1900 to 2000\}

Tell whether the members of the given sets may be paired so that the sets will be in one-to-one correspondence.

9. \{a, b, c\} and \{c, b, a\}
10. \{0, 1, 3, 5\} and \{1, 3, 5\}
11. \{vowels\} and \{a, e, u, o, i\}
12. \{\(\Delta\), \(\pi\)\} and \{\(\pi\), \(\Delta\)\}
13. \(\emptyset\) and \\{0\}\)
14. \{\(\frac{1}{4}\), \(\frac{1}{3}\), \(\frac{1}{5}\)\} and \{.5, .3, .25\}
15. \{2, 4, 6, 8\} and \{\(\frac{9}{1}\), \(\frac{12}{2}\), \(\frac{18}{3}\), 4\}
16. \{C, A, T\} and \{K, A, T\}

**WRITTEN EXERCISES**

Give a roster for each set, and state whether it is finite.

**SAMPLE.** \{multiples of 4 between 0 and 17\}

Solution: \{4, 8, 12, 16\}, finite

1. \{the whole numbers between 7 and 10\}
2. \{the whole numbers between 0 and 5, inclusive\}
3. \{the vowels in the word field\}
4. \{all five-headed people\}
5. \{U.S. cities each with populations greater than 20 million\}
6. \{ \text{the odd numbers between 0 and 10} \}
7. \{ \text{the even numbers less than 20 and greater than 9} \}
8. \{ \text{the months of the year which have fewer than 30 days} \}
9. \{ \text{the whole numbers greater than 0 but less than 1776} \}
10. \{ \text{the even numbers between 200 and 1000} \}
11. \{ \text{the multiples of 3 between 3 and 15, inclusive} \}
12. \{ \text{the multiples of 25 greater than 0} \}
13. \{ \text{the leap years after 1960} \}
14. \{ \text{the multiples of 20 between 31 and 53} \}
15. \{ \text{every fraction whose denominator is an even number chosen from \{1, 2, 3\} and whose numerator is 1} \}
16. \{ \text{all the odd numbers} \}
17. \{ \text{all the multiples of 2 greater than 0} \}
18. \{ \text{the United States Presidents who have served more than 4 terms} \}
19. \{ \text{all odd whole numbers whose squares are less than 40} \}
20. \{ \text{whole numbers less than 19 which are squares} \}
21. \{ \text{all whole numbers between 7 and 8} \}
22. \{ \text{all numbers between 3 and 8 that divide 13 exactly} \}

In Column II find a designation for each set in Column I.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>23. {x, y, z}</td>
<td>a. { whole numbers between 1 and 2 }</td>
</tr>
<tr>
<td>24. {0, 1}</td>
<td>b. { odd numbers between 1 and 7,</td>
</tr>
<tr>
<td></td>
<td>inclusive }</td>
</tr>
<tr>
<td>25. {1, 9, 25, \ldots}</td>
<td>c. { all multiples of 7 }</td>
</tr>
<tr>
<td>26. {0, 4, 16, 36, \ldots}</td>
<td>d. { numbers each of which equals its</td>
</tr>
<tr>
<td></td>
<td>square }</td>
</tr>
<tr>
<td>27. \emptyset</td>
<td>e. { the sum of 24 and 12 }</td>
</tr>
<tr>
<td>28. {0}</td>
<td>f. { z, y, x }</td>
</tr>
<tr>
<td>29. {1, 3, 5, 7}</td>
<td>g. { zero }</td>
</tr>
<tr>
<td>30. {2, 4, 6, 8}</td>
<td>h. { squares of odd numbers }</td>
</tr>
<tr>
<td>31. {0, 7, 14, 21, \ldots}</td>
<td>i. { even numbers between 2 and 8,</td>
</tr>
<tr>
<td></td>
<td>inclusive }</td>
</tr>
<tr>
<td>32. {45}</td>
<td>j. { squares of even numbers }</td>
</tr>
<tr>
<td>33. {36}</td>
<td>k. { the product of 15 and 3 }</td>
</tr>
<tr>
<td>34. { the digits in the numeral for</td>
<td>l. { the digits in the numeral for the</td>
</tr>
<tr>
<td>the product of 9 and 3 }</td>
<td>sum of 68 and 4 }</td>
</tr>
</tbody>
</table>
The Graph of a Set

Another way of specifying a set of numbers is by showing the numbers as points on the number line. The set of points corresponding to a set of numbers is called the **graph** of the set.

**EXAMPLES.**

<table>
<thead>
<tr>
<th>Set</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3}</td>
<td>![Graph 1-2-3]</td>
</tr>
<tr>
<td>{the numbers between 1 and 3, including 3}</td>
<td>![Graph 1-2-3 between 1 and 3]</td>
</tr>
<tr>
<td>{the numbers greater than 3}</td>
<td>![Graph greater than 3]</td>
</tr>
</tbody>
</table>

**Note:** A darkened circle • represents a point corresponding to a number in a set. A darkened line — is used to show that all points on it belong to the graph. Points not belonging to the graph are indicated by open circles O or appear on undarkened lines. A darkened arrow indicates that the graph continues indefinitely.

**Specifying a set: identifying its elements by**

1. the roster method — listing the elements, or
2. the rule method — describing the elements, or
3. the graphic method — locating the elements on the number line.

**ORAL EXERCISES**

Specify the graph of each set by referring to the number line below.

![Number Line](image)

**SAMPLE** \{the even whole numbers between 3 and 9, inclusive\}

*What you say:* J, B, C.

1. \{8, 2, 5\}  
2. \{1, 8, 10\}  
3. \{7, 3, 4\}  
4. \Ø\  
5. \{0\}  
6. \{4, 2, 7\}  
7. \{even whole numbers between 3 and 7\}
8. \{odd whole numbers between 3 and 7\}
9. \{even whole numbers between 5 and 6\}
10. \{whole numbers greater than 2 but less than 3\}
11. \{whole numbers less than or equal to 2\}
12. \{whole numbers between 1 and 6, inclusive\}
13. \{whole numbers between 1 and 5\}
14. \{multiples of 2 between 0 and 1\}
15. \{multiples of 5 between 4 and 6\}
16. \{whole numbers between 1 and 9\}

**WRITTEN EXERCISES**

Draw the graph of each given set.

**SAMPLE 1.** \{the whole numbers between 3 and 5, inclusive\}

*Solution:*

```
  0   1   2   3   4   5   6
```

**SAMPLE 2.** \{the numbers greater than \(2\frac{1}{2}\)}

*Solution:*

```
  0   1   2   3   4   5   6
```

1. \{the whole numbers between 2 and 8, inclusive\}
2. \{the whole numbers less than 5\}
3. \{the whole numbers less than or equal to 5\}
4. \{the numbers greater than 5\}
5. \{the numbers less than 5\}
6. \{the numbers greater than \(7\frac{1}{2}\}\}
7. \{the numbers greater than \(3\frac{1}{2}\) but less than 10\}
8. \{the numbers between 3 and 6, including 6\}
9. \{the numbers between \(4\frac{1}{2}\) and 5, including \(4\frac{1}{2}\)\}
10. \{the numbers less than 6 and greater than or equal to 2\}
11. \{the whole numbers less than 8 and greater than 3\}
12. \{the whole numbers between 3 and 4\}
13. \{3, 9, 2\}
14. \{\(\frac{1}{2}, \frac{3}{2}\)\}
15. \{0\}
16. \{\(\frac{3}{4}\)\}
17. \{2, 4, 6\}
18. \{1, 2, 3, 4, 5, \ldots, 9\}
How Subsets Relate to Sets

Suppose you form another set by removing one of the elements of \( R = \{0, 1, 2\} \). For example, remove the element 1, and form a new set,

\[ M = \{0, 2\}. \]

Notice that every element in set \( M \) is also an element in set \( R \). We say that \( M \) is a subset of \( R \). Whenever a set, such as \( M \), contains only elements which are also elements of another set, such as \( R \), the set \( M \) is said to be a subset of set \( R \). A subset, such as \( M \), which does not contain all the elements of the given set is called a proper subset of the set. Thus, \{junior high school teachers\} is a proper subset of \{all teachers\}.

See how many subsets of \( R \) you can find. By removing either one or two elements from \( R \) you can form six proper subsets of \( R \):

\[ \{0, 2\}, \{0, 1\}, \{1, 2\}, \{0\}, \{1\}, \{2\}. \]

Notice again that every element in each subset is also an element of the set \( R \), and that none of these subsets contains all the elements of \( R \).

When you remove all the elements of \( R \), you obtain the set with no elements, the empty set, \( \emptyset \). You see that \( \emptyset \) is a proper subset of every set, except of itself.

Another subset of \( R \) is formed when you remove no elements. Thus, the full set

\[ \{0, 1, 2\} \]

is also a subset of \( R \), but it is called an improper subset. You can see that every set is a subset of itself.

The notion of set appears everywhere in human thought; in mathematics the notion is consciously developed as a basic, unifying idea that you will meet over and over again as you increase your mathematical knowledge.

**ORAL EXERCISES**

Tell which statements are true and which are false. Justify your answers.

1. \{1, 3, 6\} is a subset of \{7, 6, 5, 4, 3, 2, 1\}
2. \{0, 2\} is a subset of \{1, 3, 2, 4\}
3. \{the New England states\} is a subset of \{the states of the U.S.A.\}
4. \(\{\text{red-haired people}\}\) is not a subset of \(\{\text{women}\}\)
5. \(\{\text{high school students}\}\) is a subset of \(\{\text{people studying algebra}\}\)
6. \(\{\text{people studying algebra}\}\) is a subset of \(\{\text{people studying mathematics}\}\)
7. \(\{0, 1, 2, 3, \ldots, 9\}\) is not a subset of \(\{\text{all the digits}\}\)
8. \(\{1, 3, 5, 7, 9, \ldots\}\) is a subset of \(\{1, 2, 3, 4, 5, \ldots\}\)
9. \(\{5\}\) is a subset of \(\{2, 3\}\)
10. \(\{0\}\) is not a subset of \(\{10, 6, 18\}\)
11. \(\{0\}\) is a subset of \(\{0\}\)
12. \(\emptyset\) is not a subset of \(\{0\}\)

For each set, list the largest subset of (a) odd numbers, (b) even numbers.

13. \(\{1, 2, 3, 4, 5\}\)
14. \(\{18, 19, 20, 21, 22\}\)
15. \(\{8, 16, 32, 64\}\)
16. \(\{5, 25, 125, 625\}\)
17. \(\{5, 6, 7, 8, \ldots\}\)
18. \(\{1, 4, 9, 16, \ldots\}\)
19. \(\{1, 9, 25, \ldots\}\)
20. \(\{4, 16, 36, 64, \ldots\}\)

**WRITTEN EXERCISES**

Let \(U = \{3, 15, 10\}\). List all of the subsets of \(U\) that:

1. have exactly one element
2. have exactly two elements
3. have three elements
4. have no elements
5. have multiples of 5 for elements
6. consist of elements each less than 3
7. have even numbers for elements
8. have odd numbers for elements

Specify, by rule or roster, a set of which the given set is a proper subset.

9. \(\{\text{the numbers greater than 5}\}\)
10. \(\{\text{the whole numbers}\}\)
11. \(\{\text{the fractions between 0 and 1}\}\)
12. \(\{\text{Babe Ruth}\}\)
13. \(\{\text{Juneau, Nome, Anchorage}\}\)
14. \(\{\text{Hawaii, Oahu, Molokai, Maui}\}\)

**USING NUMBERS IN ONE OR MORE OPERATIONS**

1–8 **Punctuation Marks in Algebra**

Some students were asked to rewrite the following set of words in the same order, but to insert punctuation marks and capital letters to produce a grammatically correct and meaningful sentence:

paul said the teacher is very intelligent
Mary wrote:

“Paul,” said the teacher, “is very intelligent.”

Henry wrote:

Paul said, “The teacher is very intelligent.”

Both Mary and Henry had produced correct and meaningful sentences. The differences in punctuation, however, had produced a world of difference in meaning. Without punctuation marks the original statement could be interpreted in more than one way.

In mathematics you eliminate statements which could be interpreted in more than one way by using mathematical punctuation marks, much as you use punctuation marks in English composition. For example, what number is represented by the set of symbols

\[ 3 \times 2 + 4 \]

Is it 10 or is it 18? An expression such as \( 3 \times 2 + 4 \) could be called ambiguous (am-big-u-us) because of the different interpretations.

In mathematics, one way you avoid ambiguous statements is by using parentheses. When you punctuate \( 3 \times 2 + 4 \) as follows:

\[
(3 \times 2) + 4,
\]

you mean 10. When you write

\[
3 \times (2 + 4),
\]

you mean 18.

A pair of parentheses is called a symbol of inclusion because it is used to enclose, or include, an expression for a particular number. The parentheses in the numerical expression \( 3 \times (2 + 4) \) serve to group the numerals 2 and 4 together with the symbol + and to indicate that the sum of 2 and 4 is to be multiplied by 3.

In writing \( 3 \times (2 + 4) \), it is customary to omit the symbol \( \times \) and to write simply

\[
3(2 + 4).
\]

Similarly, the product \( 3 \times 6 \) may be expressed in any of the forms

\[
3(6), \quad (3)6, \quad \text{or} \quad (3)(6).
\]

Brackets, braces, and a bar are used for the same purpose:

<table>
<thead>
<tr>
<th>Parentheses</th>
<th>Brackets</th>
<th>Braces</th>
<th>Bar</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(2 + 4)</td>
<td>[2 + 4]</td>
<td>{2 + 4}</td>
<td>32 + 4</td>
</tr>
</tbody>
</table>
SYMBOLS AND SETS

In working with fractions you have seen that the bar acts as a division sign, as well as a symbol of inclusion. For example, in the expression below, the bar groups the 16 and 4; it also groups the 4 and 1. The bar tells you that the number (16 — 4) is to be divided by the number (4 — 1).

\[
\frac{16 - 4}{4 - 1} = \frac{12}{3} = 4
\]

Each part of this statement designates the same number as the one before it, but as you carry out the operations in the indicated order the numeral becomes simpler. In simplifying an expression, you use the signs of grouping to determine the order of operation.

When you see a grouping inside of another grouping, such as \(5[3 + (7 \times 2)]\), you always simplify the numeral in the innermost symbol of inclusion and proceed to work toward the outermost grouping until all symbols of inclusion are removed, thus,

\[
5[3 + (7 \times 2)] = 5[3 + (14)] = 5[17] = 85.
\]

**ORAL EXERCISES**

Simplify each of the following expressions.

**SAMPLE.** \((8 \div 2) + 7\)  
*What you say:* \((8 \div 2) + 7 = 4 + 7 = 11*

1. \(5 + (3 \times 2)\)
2. \((8 + 3) + 5\)
3. \((5 + 3) \times 2\)
4. \(8 + (3 + 5)\)
5. \(4(3 + 7)\)
6. \(\frac{15 + 4}{6 + 4}\)
7. \(\frac{10 - 1}{1 + 2}\)
8. \(\frac{[3 \times 2] \div 4}{\text{}}\)
9. \(\frac{\left(\frac{6.0}{2}\right)}{5}\)
10. \(\frac{64}{[\frac{8}{2}]}\)
11. \(\frac{7}{\left(\frac{3}{3}\right)}\)
12. \(\frac{17 + 1}{8 - 2}\)
13. \(\frac{5 + 3 + \frac{1}{2}}{\text{}}\)
14. \(4 \times 3 + (4 \times 7)\)
15. \(2 \times \{6 + 5 + 2\}\)
16. \(5[2 + 1 - 2] + 6\)
17. \(\frac{5 - 2 \times 5 + 2}{\text{}}\)
18. \(\frac{6 \times 2}{12 - 4}\)
19. \(\frac{13 - 10}{5 \times 2}\)
20. \(\frac{15 - 5}{2 \times 10}\)
Simplify each of the following expressions.

**SAMPLE.** \[ [(14 \times 2) + 5] \div 11 \]

**Solution:** \[ [(14 \times 2) + 5] \div 11 = [28 + 5] \div 11 \]
\[ = [33] \div 11 \]
\[ = 3 \]

A
1. \[ 0 \times [5 + (2 \times 3)] \]
2. \[ 5 \times [5 - (0 \times 3)] \]
3. \[ 17 \times \frac{15 \times 2}{2} \]
4. \[ 17 \times \frac{15 \times 2}{2} \]
5. \[ \frac{49 - 25}{7 + 5} \]
6. \[ \frac{25 - 9}{5 - 3} \]

7. \[ \frac{100 - 64}{10 - 8} \]
8. \[ \frac{100 - 64}{10 + 8} \]
9. \[ (17 + 3) \times 18 \]
10. \[ (17 \times 18) + (3 \times 18) \]
11. \[ (16 + 4) \div ([3 \times 2] \div 4) \]
12. \[ [30 \div (5 \times 2)] \div 3 \]

B
13. \[ (27 \div 9) + [(27 + 9) \div 3] \]
14. \[ [(16 + 4) \div (3 \times 2)] \div 13 \]
15. \[ \left[ \frac{100 - 64}{10 + 8} \right] \div 12 \]
16. \[ \left[ \frac{50 + 25}{15} \right] \div 5 + 1 \]

17. \[ \frac{[5 \times 20 + 6 \div 3]}{51} \]
18. \[ \left[ \frac{12 + 8}{9 \times 6} - \frac{1}{54} \right] - \frac{1}{54} \]
19. \[ (3 \times 9) + [9 \times 3 + 2] \]
20. \[ \{4 \times 3\} - \{4 \div 3\} \div 8 \]

C
21. \[ \left[ \left( \frac{100 - 36}{1 + 7} - 8 \right) \times 3 \right] + 5 \times 200 \]
22. \[ ((1776 - 324 \times 5) - 5) \times 100 \]
23. \[ ([4 \times 2 + 6 \times 10 - 3 \times 20] + 56) \div 13 \]
24. \[ \left[ \left( \frac{579 + 682}{39} \right) \times 27 \right] + 9 \times 3 \div 27 \]
1–9 Order of Operations

Parentheses and the other symbols of inclusion are the customary means used to make clear the meaning of a numerical expression. However, mathematicians have agreed on a rule to fall back on if someone omits punctuation marks. This rule gives the order to be followed in performing the operations indicated in the expression. The agreement implies, for example, that

\[ 5 + 3 \times 2 \quad \text{means} \quad 5 + (3 \times 2) = 5 + 6 = 11 \]
\[ 7 \times 4 + 3 \quad \text{means} \quad (7 \times 4) + 3 = 28 + 3 = 31 \]
\[ 36 \div 4 - 1 \quad \text{means} \quad (36 \div 4) - 1 = 9 - 1 = 8 \]
\[ 6 \times 8 - 7 \times 2 \quad \text{means} \quad (6 \times 8) - (7 \times 2) = 48 - 14 = 34 \]
\[ 30 \div 10 \times 3 \quad \text{means} \quad (30 \div 10) \times 3 = 3 \times 3 = 9 \]
\[ 5 \times 3 \times 4 \quad \text{means} \quad (5 \times 3) \times 4 = 15 \times 4 = 60 \]
\[ 4 + 3 + 2 \quad \text{means} \quad (4 + 3) + 2 = 7 + 2 = 9 \]
\[ 7 - 3 - 2 \quad \text{means} \quad (7 - 3) - 2 = 4 - 2 = 2 \]
\[ 5 + 2(4 - 3) \quad \text{means} \quad 5 + 2(1) = 5 + 2 = 7 \]

In a numerical expression containing a series of numerals connected by symbols of operation, you agree to follow this order:

1. simplify the expression within each symbol of inclusion;
2. perform the multiplications and divisions in order from left to right;
3. finally, do the additions and subtractions in order from left to right.

ORAL EXERCISES

Simplify each of the following expressions.

1. \( 5 - 3 - 2 \)
2. \( 12 - 0 + 1 \)
3. \( 3(4) + 7 \)
4. \( 8 + 6 \times \frac{1}{3} \)
5. \( 20 \div 4 \div 2 \)
6. \( 10 \div 5 \times 4 \)
7. \( 10 + 9 + 3 \)
8. \( 49 + 5 \times 0 \)
9. \( 14 \div 7 \times 1 \)
10. \( 14 \div 1 \times 7 \)
11. \( 80 \div 8 - 80 \div 10 \)
12. \( 12(4) - 16 \div 2 \)
13. \( 17 \times 25 \times 4 \)
14. \( \frac{1}{2} \times 12\frac{1}{2} \times 2 \)
15. \( 6 \times 2 - 2 \div 2 \)
16. \( 12 \div 12 + 6 \times 2 \)
17. \( 18 - 6(3) + 5 \)
18. \( 5 - 2 + 4 \times 3 \)
19. \( 12 \times 6 + 12 \times 4 \)
20. \( 39 \times 9 + 39 \times 1 \)
Simplify each of the following expressions.

**SAMPLE.**

$$11 + 2(6 + 4) - 3(1 + 3)$$

**Solution:**

$$11 + 2(6 + 4) - 3(1 + 3)$$

*Step 1:* $11 + 2(10) - 3(4)$

*Step 2:* $11 + 20 - 12$

*Step 3:* $31 - 12$

19, Answer.

**A**

1. $(7 + 3 + 2) ÷ 3 + 1$

2. $(7 + 3 + 2) ÷ (3 + 1)$

3. $5(7 + 9) ÷ 4 + 3$

4. $21 + 5(7 + 3) - 20$

5. $7 + 3(5 - 1) ÷ 6$

6. $7 + (15 - 3) ÷ 6$

7. $8 ÷ 2 + 6 ÷ 3$

8. $30 - 3(7 - 2)$

**B**

17. $64 ÷ 8 ÷ 4 ÷ 2$

18. $12 × 6 ÷ 3 × 2 ÷ 48$

19. $6(7 + 2) - 15 ÷ 5$

20. $5(7 - 4) ÷ 3 + 2$

21. $9 - 5(3 - 2)$

22. $3 + 5$

23. $4(3 + 1) - 1$

**C**

30. $3 + 48 - 16 - 35 ÷ 7$

31. $\frac{3}{4}(8 ÷ 2 × 4) ÷ (8 × 4 ÷ 2)$

32. $\frac{4 × 5 × 20 - 6 ÷ 2}{13 + 3 × 1 × 5}$

33. $8 ÷ 4 × 3 - 2 + 16$

34. $\frac{1 + 44 ÷ 4 + 12 × 44}{3 × 3 - 3 ÷ 3 + 2}$
A casual visitor to the Tower of London in the year 1606 might have witnessed an unexpected sight. In the midst of this infamous prison, at a table reserved for their use, a group of men, all friends and guests of one of the prison's inmates, would congregate to discuss mathematics. The host of this unusual party was no lesser personage than the Earl of Northumberland. The leading figure in the discussions was an accomplished astronomer and mathematician, Thomas Harriot.

Harriot had come to his place at the Earl's table in the Tower by way of an eventful life. Born in 1560, he was caught up in the spirit of vigor and creativity which pervaded England during the reign of Elizabeth I. His career began with studies at Oxford, and soon after, he served as Sir Walter Raleigh's tutor in mathematics. It was Raleigh who appointed Harriot to the office of surveyor with the second expedition to Virginia. After returning to England and his mathematical studies, Harriot was awarded a life pension by the Earl of Northumberland, himself an amateur mathematician. So it was that in 1606, when the Earl came into disfavor with the Crown and was imprisoned in the Tower, Harriot was among the honored guests at his table.

Although Harriot's last years were beset by cancer, he continued to demonstrate remarkable mathematical talents. The use of the sign $=$ for equality, though introduced by another mathematician, Recorde, is partly due to Harriot, who helped persuade other mathematicians of the day to adopt this notation. To Harriot alone, moreover, we owe two of the most useful mathematical notations, the symbols $>$ and $<$.

A page from Robert Recorde's Whetstone of Witte. This was the earliest algebra written in English and contains the first use of the $=$ sign, which Thomas Harriot later helped popularize.
Chapter Summary

Inventory of Structure and Method

1. **Arithmetic numbers** can be arranged in **order** of size. Each number can be paired with a point on the **number line**. A pairing, like that of point and number, is called a 1-to-1 correspondence.

2. In **comparing numbers**, symbols of equality (=) and inequality (≠, >, <) are used. Numbers may be compared in terms of their location on the number line; larger numbers are to the right. To specify a set, identify its elements by (a) roster (list the elements) or (b) rule (describe the elements) or (c) graph (locate the elements on the number line).

3. An expression within a symbol of inclusion (grouping), such as $(9 - 5)$ in $3(9 - 5)$, is to be treated as one quantity or numeral. The **order of operations** is as follows:
   a. Simplify within each symbol of inclusion.
   b. Perform multiplications and divisions in order from left to right.
   c. Do additions and subtractions in order from left to right.

Vocabulary and Spelling

Pronounce, spell, and give the mathematical meaning or symbol of the words and expressions. The number refers to the page on which each word is first introduced.

- property (p. 1)
- number line (p. 1)
- numeral (p. 2)
- arithmetic number (p. 2)
- uniform scale (p. 2)
- order of magnitude (p. 2)
- coordinate of a point (p. 2)
- graph of a number (p. 2)
- between (p. 2)
- numerical expression (p. 5)
- sign of equality (p. 5)
- sign of inequality (p. 7)
- set (p. 10)
- member of set (p. 10)
- element of set (p. 10)
- braces (p. 11)
- $(\in), (\notin)$ (p. 11)
- specifying a set (p. 11)
- roster of a set (p. 11)
- rule for a set (p. 11)
- one-to-one correspondence (p. 13)
- infinite set (p. 13)
- finite set (p. 13)
- empty or null set ($\emptyset$) (p. 13)
- graph of a set (p. 16)
- subset (p. 18)
- proper subset (p. 18)
- improper subset (p. 18)
- symbol of inclusion (p. 20)
- parentheses (p. 20)
- brackets (p. 20)
- bar (p. 20)
- simplify (p. 21)
- order of operations (p. 23)
Chapter Test

1-1 Find the numeral corresponding to each of the following division points on the number line below:

\[ \text{A} \quad P \quad B \quad Q \quad R \]

1. \( P \)  
2. \( Q \)  
3. \( R \)

Use the number line below to find the coordinate of each of the following points:

\[ \text{C} \quad P \quad L \quad T \quad W \quad R \quad K \quad N \quad S \]

4. The point halfway between \( P \) and \( N \)  
5. The point one-fourth of the distance from \( T \) to \( R \)  
6. The point \( 3\frac{1}{3} \) units to the left of \( K \)

1-2 For each statement, tell whether or not it is true, and give a reason for your answer.

7. \( 4 \times 5 = 8 + 12 \)  
8. \( 16 - 7 = 3 \times 4 \)  
9. \( 3 + 5 + 9 = 8 + 9 \)  
10. \( 12\frac{3}{5} = 11\frac{13}{5} \)

1-3 For each statement, find a numeral that may replace the question mark and make the resulting statement true.

11. \( 8 \times 4 \neq 30 + ? \)  
12. \( ? - 10 > 10 \)  
13. \( 2 < ? < 4 \)  
14. \( \frac{27}{2.35} > ? > \frac{2.7}{.245} \)

1-4 Make a roster for each of the following sets:

15. \{the multiples of 6 between 5 and 26\}  
16. \{the three-digit numerals\}

1-5 Identify the following sets as being finite, infinite, or the empty set:

17. \{0, 1, 2, \ldots\}  
18. \{4, 6, 8\}  
19. \{whole numbers between 0 and 1\}
1–6  Draw the graph of each of the following sets:
   20. \{the whole numbers between 1 and 4, inclusive\}
   21. \{the numbers between 1 and 4\}
   22. \{the numbers greater than 1\frac{1}{2}\}

1–7  23. List that subset of \(A\), where \(A = \{2, 4, 6, 8\}\), that consists of all the elements of \(A\) that are multiples of 4.

1–8  24. Simplify: \([60 \div (3 \times 2)] \div 10\)

1–9  25. Simplify: \(10 + 15 \div 5\)  26. Simplify: \(25 \times 2 - 7 \times 7\)

Before You Go on to Chapter 2

Did you miss any of the test items? If so, note the section number that corresponds to each item you missed. Restudy that section in the chapter. Then find the section number in the Chapter Review, and do the exercises under it.

Did you get all the items correct? If so, you may turn to page 30 and enjoy the Extra for Experts.

Chapter Review

1–1  Representing Numbers on a Line: Order Relations  Pages 1–5

1. A numeral is a name for a _____.
2. The starting point of the number line is labeled _____.
3. On a number line, arithmetic numbers appear in order of _____.

Exercises 4–7 refer to the following number line:

```
A B C D F E
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
```

4. To label any point, you must know its ____ from zero.
5. The distance between the points labeled \(\frac{3}{2}\) and \(\frac{5}{4}\) is the same as the distance between the points labeled 0 and ____.
6. The coordinate of the point halfway between \(B\) and \(F\) is ____.
7. The points that are the graphs of the whole numbers between 1 and 4, including 4, are \(C\), \(D\), and ____. 
1-2 Comparing Numbers: The Sign of Equality Pages 5–7

8. Any name or symbol for a number is called a _expression_ or _symbol_.
9. When two expressions represent the same number, they are said to be _equal_.
10. The symbol for equals or _is equal to_ is _equals_.

1-3 Comparing Numbers: The Signs of Inequality Pages 7–10

11. The symbol for _does not equal_ or _is not equal to_ is _not equal to_.
12. 7 > 5 is read “7 is _greater than_ 5.”
13. “Five is less than seven” may be written in symbols as _less than_.
14. 1 < 8 < 9 reads “8 is greater than _and less than_.”
15. Replace the question mark by a whole number to make the resulting statement true: 4 < _?_ < 6.

1-4 Meaning of Membership in a Set Pages 10–12

16. Any collection of objects is called a _set_.
17. Each object in a set is called a(n) _element_ of the set.
18. The fact that 3 belongs to the set of numbers 1, 2, 3, 4, can be written _3 belongs to _{1, 2, 3, 4}_.
19. The true statement 3 ∉ {all the even numbers} means that 3 is _not an even number_.
20. When you identify a set by listing its elements within braces, you are making a _list_.
21. When you specify a set by describing the elements within braces, you are giving a _description_.
22. {the multiples of 3 between 10 and 20} = _{3, 6, 9, 12}_.

1-5 Kinds of Sets Pages 13–15

23. A nonending set of numbers is said to be _infinite_.
24. A set containing no elements is the _empty set_.
25. A set with a specific number of elements is a _finite set_.
26. Ø represents the _empty set_.

1-6 The Graph of a Set Pages 16–17

27. _represents the graph of {all numbers greater than _?_}_.
28. On a number line, show the graph of {all the numbers between 0 and 2, including 2}_.
1-7 How Subsets Relate to Sets  
29. If every element of set $A$ is also an element of set $B$, then set $A$ is a subset of set $B$.
30. The subset of $\{4, 10, 20\}$ which consists of elements that are multiples of both 4 and 5 is $\{8, 20\}$.

1-8 Punctuation Marks in Algebra  
31. Parentheses, $\{\}$, and the $\langle \rangle$ are called symbols of inclusion.
32. When an expression is enclosed in parentheses, it is to be treated as a quantity or numeral.
33. Simplify: $2 + 3 \times 4$
34. Simplify: $\frac{8 \div (4 \div 2)}{(8 \div 4) \div 2}$

1-9 Order of Operations  
35. In performing a series of operations, first simplify within each symbol of $\{\}$, then perform the indicated $\{\}$ and $\langle \rangle$; finally do the $\{\}$ and $\langle \rangle$.
36. $72 \div 6 - 2 \times 3 = \ ?$
37. $3 + 6 \times 4 = \ ?$
38. $\frac{44}{2} - 2 + 3 \times 5 = \ ?$
39. $45 - 5.5 \div 10 = \ ?$
40. $\frac{18 \div 9 + 3 \times 4}{2(3 + 4)} = \ ?$

Extra for Experts

The Arithmetic of Sets: Intersection

The intersection of two sets consists of the elements they have in common. For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$, the intersection of these sets would be $\{3, 4, 5\}$, which could be designated set $C$. The symbol for intersection is $\cap$ (read “cap”).

In words: The intersection of set $A$ and set $B$ is set $C$.

In symbols: $A \cap B = C$

or: $\{1, 2, 3, 4, 5\} \cap \{3, 4, 5, 6, 7\} = \{3, 4, 5\}$.

It should be noted that the intersection of two sets is a subset of each set.
Intersection may also be represented pictorially by closed figures called Venn diagrams. The region within a Venn diagram is assumed to represent the set being illustrated. For example,

Let \( U = \{\text{whole numbers}\} \):

\[
\begin{array}{c}
\text{U} \\
\end{array}
\]

Let \( A = \{1, 2, 3, 4, 5\} \):

\[
\begin{array}{c}
A \\
\end{array}
\]

Let \( B = \{3, 4, 5, 6, 7\} \):

\[
\begin{array}{c}
B \\
\end{array}
\]

Then \( C = A \cap B \):

(\text{the cross-hatched region, common to the circles})

Because each of the sets used in the problem is a subset of \( U \), \( U \) is called the universe or universal set. In the above, \( \{\text{people over 21 years of age}\} \) would not be a suitable universe since \( A \) and \( B \) would not be subsets of \( U \). By changing the universal set, you change the nature of the problem.

**Questions**

1. If \( A = \{1, 2, 3\} \) and \( B = \{0, 2, 3, 4\} \), give the roster of \( A \cap B \) and use Venn diagrams to picture it.
   a. What is the relationship between \( A \cap B \) and \( A \)? and \( B \)? Why?
   b. Give a rule explaining when \( A \cap B = A \) will hold.

2. If \( A = \{q, r, s, y\} \) and \( B = \{t, u, v, w\} \), specify \( A \cap B \). Use Venn diagrams to picture \( A \) and \( B \).
   a. Explain the nature of \( A \cap B \) when \( A \) and \( B \) have no common elements.
   b. What is the relationship between \( A \cap B \) and \( A \)? and \( B \)? Why?
   c. Explain why the names disjoint sets or mutually exclusive sets are appropriate for sets of this nature.
The earliest known surveys probably were made to determine property boundaries. Although these simple land surveys are still the most familiar type, surveying is now used in a great variety of situations. Whether building highways, dams, tunnels, or skyscrapers, engineers work hand in hand with surveyors. Underground surveys, for example, guide the engineer in selecting the best position for a tunnel. On the basis of these surveys, in fact, teams excavating the tunnel from opposite banks of a river meet midway under the river, within inches of each other.

Hydrographic surveys, which provide information about the form of lake and river beds as well as the deepest parts of the ocean floor, involve certain special techniques. Instead of sighting through a telescopic device, the surveyor may use sonar to send out sound waves and receive the echo which bounces off the ocean floor. He then makes a computation like that illustrated on the work pad. Since sound waves travel at a constant rate \( R \) through water, the total distance \( D \) they travel is equal to the rate \( R \) times the time \( T \) elapsed between sending and receiving the sound. Dividing the "round-trip" distance by two gives the depth to the ocean floor. A series of such soundings and computations reveals depth variations and thus reflects the configuration of the ocean floor.

Not all surveying is large scale. The surveyor in the photograph is using a transit to determine the position of the girders of a bridge. Since steel contracts when cold and expands when hot, expansion joints, that adjust to temperature changes, are often used to join steel girders. The engineers try to merge the girders within an accuracy of a millimeter and depend on surveyors' measurements during the construction of the bridge to be certain that the girders are aligned.
Cryptography

A modern use of letter symbols for numbers has become established in cryptology, the science of constructing and deciphering coded messages. Though the intelligence service uses numbers for letters, some private businesses use letters for numbers on price tags to code the initial cost of goods.

Suppose a store used the letters of the word *davenports* for the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, respectively. An *X* indicates that the digit preceding it should be repeated. Find the initial cost of the following articles: Coat: $89.95 (NTNS); Hat: $12.95 (RSX); Refrigerator $625 (VANXX).

Examine the code table and the decoded messages at the right.

<table>
<thead>
<tr>
<th>0 1 2 3 4 5 6 7 8 9</th>
<th>a b c d e f g h i j</th>
</tr>
</thead>
<tbody>
<tr>
<td>k l m n o p q r s t</td>
<td>u v w x y z</td>
</tr>
</tbody>
</table>

857836 = 7474
**SPRING IS HERE**

1829474 = 4078
**VICTORY IS OURS**

Any letter or digit in one of the columns may stand for any other letter or digit in the same column.

To decode a message, list all possible replacements under each numeral, then select those letters that form words. Can you decipher these messages?

1. 7083 = 249
2. 54024 = 507024039
3. WYNOI = PAD
\[ 5 + x = 18 \]
\[ 5 + x - 5 = 18 - 5 \]
\[ x = 13 \]
Investigate before you invest. Most people have to make decisions frequently. You usually base an important conclusion upon the investigation and consideration of several related conditions. In mathematics you are faced with decisions concerning numbers which must meet specified conditions, as the pictures indicate. In this course, you will learn a systematic procedure for using such conditions in algebraic form to find the desired numbers.

Although organized decision making is taught as part of your mathematical education, its methods are appropriate for many purposes. Analyzing conditions, evaluating them, and selecting a suitable course of action are skills which you will find useful whether your decisions are concerned with buying clothes, choosing a college, or investing a million dollars. Thus, mathematics offers opportunities for investigation and for investment in your future.

ANALYZING ALGEBRAIC STATEMENTS

2–1 Evaluating Algebraic Expressions Containing Variables

A sentence you probably have heard is, “Vegetables are good for you.” The word “vegetables” is used as a general noun which refers to elements of a set of edible plants. Replace the general noun by the name of an element of the set, and you get a specific instance of the general statement. Sentences such as

“String beans are good for you.”
“Carrots are good for you.”

are combined in the single statement, “Vegetables are good for you.”

Now, instead of vegetables, suppose you wanted to represent a multiple of three. Consider the set of multiples of three:

\[ A = \{0, 3, 6, 9, \ldots\} \]
You can express the elements of set $A$ as “three times zero,” “three times one,” “three times two,” “three times three,” and so on.

$$\{3 \times 0, 3 \times 1, 3 \times 2, 3 \times 3, \ldots\}$$

Can you write one general expression that will include all these individual elements as special cases? To do this, you need a symbol which will assume the role that was played by the word “vegetables.”

For this purpose, a letter is generally used. Thus, an element in set $A$ can be written as $3 \times n$ where $n$ stands for a member of $\{0, 1, 2, 3, \ldots\}$. It is very important that you notice that the letter $n$ is not another name for one specific number, but rather that it represents any number which is an element of $\{0, 1, 2, 3, \ldots\}$. When used in this way, $n$ is called a variable. A variable is a symbol which may represent any of the elements of a specified set. A set whose elements serve as replacements for a variable is called the replacement set for the variable. This set is also called the domain of the variable. The individual members of the replacement set, such as 0, 1, 2, 3, . . . , are called the values of the variable. A variable with just one value is called a constant.

When a product involves a variable, it is customary to omit the symbol $\times$ of multiplication. Thus, $3 \times n$ is written $3n$ and means three times $n$, and $a \times b$ is written $ab$ and means $a$ times $b$. Observe that in a product like $8 \times 7$ or $8(7)$ you do not omit the symbol of multiplication, for $87$ means eighty-seven not 8 times 7. A raised dot also is used to show multiplication. Thus, $2 \cdot n$ means $2 \times n$ and $8 \cdot 7$ means $8 \times 7$.

When you replace the variable in the expression $3n$ by each element of its replacement set in turn, the expression represents different numbers. If you replace $n$ by 0, you get an expression for 0. If you replace $n$ by 1, you get an expression for 3. If you replace $n$ by 2, you get an expression for 6. An expression which contains a variable is called a variable expression or an open expression because you have left open the decision of what number to specify for the variable. In other words, you do not know what number $3n$ names until you specify the value of $n$. Any variable expression or numerical expression is known as an algebraic expression.

If you let $r$ represent, in turn, each of the elements of the replacement set $\{2, 3, 5, 8\}$, you evaluate the expressions $\frac{1}{2}r$ and $5r$ as follows:

\[
\begin{align*}
\frac{1}{2}r &= \frac{1}{2}(2) = 1 & 5r &= 5(2) = 10 \\
\frac{1}{2}r &= \frac{1}{2}(3) = \frac{3}{2} & 5r &= 5(3) = 15 \\
\frac{1}{2}r &= \frac{1}{2}(5) = \frac{5}{2} & 5r &= 5(5) = 25 \\
\frac{1}{2}r &= \frac{1}{2}(8) = 4 & 5r &= 5(8) = 40
\end{align*}
\]
This process of determining the number an algebraic expression represents is called **evaluating the expression** or **finding its value**. An algebraic expression written as a product or quotient of numerals or variables or both is called a **term**. The expression $3n$ is a term, and so are $7$, $2d$, $3xy$, and $\frac{m}{n - 1}$. In $7xy + 3(x - y) + \frac{y - x}{5}$ there are three terms, $7xy$, $3(x - y)$, and $\frac{y - x}{5}$.

**ORAL EXERCISES**

If $a$, $b$, and $x$ have the values 15, 3, and 2, respectively, tell the value of each of the following; also, tell how many terms each contains.

1. $3a$
2. $7x$
3. $a + x$
4. $a - b$
5. $\frac{1}{b}a$
6. $30 \div x$
7. $ab$
8. $xx$
9. $2x - b$
10. $\frac{1}{b}(a + b)$
11. $42 \div (a - 1) + x$
12. $bbb$
13. $b + b + b$
14. $3b$
15. $a \div b + x$
16. $a \div (b + x)$
17. $ab - x$
18. $ax - b$
19. $ax + bx$
20. $ab - bx$
21. $\frac{bx}{a}$
22. $\frac{ax}{b}$
23. $\frac{a + b}{x}$
24. $\frac{a - b}{x}$

**WRITTEN EXERCISES**

In each algebraic expression, let $r = 1$, $s = 3$, $t = 12$, $u = 0$, $v = 5$, and $w = \frac{1}{2}$. Tell how many terms each expression contains; then evaluate it.

**SAMPLE.**

$$\frac{5s + 3r}{v + 2w}$$

**Solution:**

$$\frac{5 \cdot 3 + 3 \cdot 1}{5 + 2 \cdot \frac{1}{2}} = \frac{15 + 3 \cdot 1}{5 + 1} = \frac{18}{6} = 3$$

Number of terms, 1; value, 3, Answer.
Evaluate the following algebraic expressions found in geometry, statistics, physics, and machine shop work by replacing the variables as indicated.

**A**

1. Tailstock offset for cutting a full-length taper: \( \frac{D - d}{2} \); let \( D = 4.375'' \) and \( d = 3.625'' \).

2. Specific gravity of an object which floats in water: \( \frac{U}{W - V} \); let \( U = 5 \text{ lb.}, W = 12 \text{ lb.}, V = 6 \text{ lb.} \)

3. Degrees centigrade to degrees Fahrenheit: \( \frac{9}{5}C + 32 \); let \( C = 20° \).

4. Length of an open belt over pulleys of equal diameters: \( 2L + \pi D \); let \( L = 12.250', D = 3.000', \pi = 3.142 \).

**B**

5. Area of a trapezoid: \( \frac{h(b + B)}{2} \); let \( h = 4'', b = 6'', B = 7'' \).

6. Sum, \( 1 + 2 + 3 + \cdots + n: \frac{n(n + 1)}{2} \); let \( n = 7 \).

7. Degrees Fahrenheit to degrees centigrade: \( \frac{9}{5}(F - 32) \); let \( F = 77° \).

8. Distance traveled in a given second under acceleration: \( \frac{a}{2}(2t - 1) \); let \( a = 20 \text{ ft./sec./sec.}, t = 6 \text{ sec.} \).
9. Mechanical advantage of a differential pulley: \( \frac{2C}{C - c} \); let \( C = 15'' \), \( c = 12'' \).

10. Slope of a line: \( \frac{Y - y}{X - x} \); let \( Y = 7, y = 3, X = 6, x = 2 \).

11. Interior angle of a regular polygon: \( 180^\circ \left( \frac{n - 2}{n} \right) \); let \( n = 12 \).

12. Linear expansion of a heated rod: \( a(l - t) \); let \( a = .000023, l = 10', T = 80^\circ, t = 20^\circ \).

13. Area of a circular cross section: \( \pi(R - r)(R + r) \); let \( R = 24'', r = 20'', \pi = \frac{22}{7} \).

14. Square of the area of a triangle: \( s(s - a)(s - b)(s - c) \); let \( a = 12'', b = 10'', c = 8'', s = 15'' \).

15. Surface area of a box: \( 2(lw + wh + lh) \); let \( l = 7', w = 6', h = 5' \).

16. Electric current through three resistances in parallel: \( E \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) \); let \( E = 110 \) volts, \( A = .10 \) ohm, \( B = .25 \) ohm, \( C = .20 \) ohm.

17. Tailstock offset for cutting a partial length taper: \( \frac{L}{T} \left( \frac{D - d}{2} \right) \); let \( L = 12.250'', T = 6.125'', D = 1.875'', d = 1.125'' \).

18. Electric current delivered by battery cells in series: \( \frac{nE}{R + nr} \); let \( n = 3, E = 1.5 \) volt, \( R = 12 \) ohms, \( r = 0.1 \) ohm.

19. Sum, \( 1 + 4 + 9 + \cdots + n^2 \); let \( n = 7 \).

20. Approximate length of an open belt over pulleys of unequal diameters: \( 2L + 3.25 \left( \frac{D + d}{2} \right) \); let \( L = 14', D = 2.5', d = 1.2' \).
2—2 Identifying Factors, Coefficients, and Exponents

When two or more numbers are multiplied, each of the numbers is called a factor of the product. Thus, 3 and 7 are factors of 21; two other factors are 1 and 21. Note that in factoring whole numbers you usually consider only whole number factors. Thus, the product $6x$ has 1, 2, 3, 6, x, 2x, 3x, and, of course, $6x$ itself as factors.

Each factor of a product is the coefficient (ko'-e-fish-ent) of the product of the other factors. In the product $\frac{1}{2}xy$, $\frac{1}{2}$ is the coefficient of $xy$, $\frac{1}{2}x$ is the coefficient of $y$, and $\frac{1}{2}y$ is the coefficient of $x$. Frequently, the numerical part of a term is called the coefficient of the term. For example, the coefficient of $343x^2$ is 343. Also, the coefficient of $a$ is 1, since $a = 1a$.

Sometimes a number appears more than once as a factor in a product. The product $s \cdot s$ is commonly written $s^2$. The term $s^2$ may be read: $s$ squared or $s$-square. The small raised number is an exponent (ek-spo-nent). It shows that $s$, which is called the base, is to be used twice as a factor. The base is the expression used as a factor one or more times (as indicated by the exponent).

To compare an exponent with a coefficient, compare $s^2$ and $2s$ when you replace $s$ by 15.

\[
\begin{align*}
{s^2} &= s \cdot s \\
{s^2} &= 15 \cdot 15 \\
{s^2} &= 225 \\
{2s} &= 2 \cdot s \\
{2s} &= 2 \cdot 15 \\
{2s} &= 30
\end{align*}
\]

An exponent tells how many times another number, called the base, is to be used as a factor. A coefficient is a factor.

A number which can be expressed by means of a base and exponent is called a power. The exponent 1, which is seldom written, means that the base is used only once; therefore, $x^1$ — the first power of $x$ — is the same as $x$. Here are some other powers of $x$:

Third power: \[x^3 = x \cdot x \cdot x\] (read $x$ cubed or $x$-cube)
Fourth power: \[x^4 = x \cdot x \cdot x \cdot x\] (read $x$ fourth or $x$ exponent 4)
Fifth power: \[x^5 = x \cdot x \cdot x \cdot x \cdot x\] (read $x$ fifth or $x$ exponent 5)
In an expression such as $3a^2$, the 2 is the exponent of the base $a$. In an expression such as $(3a)^2$, the 2 is the exponent of the base $3a$, because you enclosed the expression in a symbol of inclusion. Compare the examples that follow:

$$rs^3 = r \cdot s \cdot s \cdot s$$

$$(rs)^3 = rs \cdot rs \cdot rs$$

$4 \cdot 5^3 = 4 \cdot 5 \cdot 5 \cdot 5 = 500$$

$$(4 \cdot 5)^3 = 20 \cdot 20 \cdot 20 = 8000$$

$$5 - n^2 = 5 - (n \cdot n)$$

$$(5 - n)^2 = (5 - n)(5 - n)$$

$$15 - 3^2 = 15 - 9 = 6$$

$$(15 - 3)^2 = (12)^2 = 144$$

**ORAL EXERCISES**

Read each of the following expressions as a product.

**SAMPLE 1.** $7(y + 3)$  
*What you say:* 7 times the sum $y$ plus 3.

**SAMPLE 2.** $\frac{a}{4}$  
*What you say:* One-fourth times $a$.

1. $2y$
2. $3x$
3. $11n$
4. $\frac{3}{2}z$
5. $\frac{1}{3}r$
6. $0.3b$
7. $cd$
8. $2(x + 5)$

In Exercises 15–22, name at least two factors of the given terms.

15. $ab^2$
16. $rs^4$
17. $12$
18. $7$
19. $2c^2d^2$
20. $\frac{ab^2c^3}{2}$
21. $\frac{rs^4}{3}$

In Exercises 23–34, name the coefficient of $z$.

23. $4z$
24. $19z$
25. $\frac{1}{8}z$
26. $\frac{1}{2}z$
27. $z$
28. $xyz$
29. $z(3 + 1)$
30. $(a + b)z$
31. $0.7z$
32. $1.4z$
33. $2z + 5$
34. $a + bz$
In Exercises 35–44, name the numerical coefficient, the base, and the exponent.

**SAMPLE.** \(9(u + 6)^2\) *What you say:* The coefficient is 9; the base is \((u + 6)\); the exponent is 2.

35. \(2z^2\)  
37. \(x^7\)  
39. \(3(V + 2)^3\)  
41. \((a + b)^3\)  
43. \(5t\)

36. \(4y^3\)  
38. \(w^6\)  
40. \(4(u - 3)^2\)  
42. \((a - b)^4\)  
44. \(17s\)

Tell the meaning of each of the following terms; then give its value.

**SAMPLE.** \(y^2; y = 9\).  
*What you say:* \(y^2\) means \(y\) times \(y\); when \(y = 9\), \(y^2 = 9 \times 9 = 81\).

45. \(k^2; k = 5\)  
46. \(n^3; n = 10\)  
47. \(a^4; a = 176\)  
48. \(u^4; u = 2\)

49. \((3x)^3; x = 2\)  
50. \((2y)^2; y = 3\)  
51. \(5h^2; h = 10\)  
52. \(2J^2; J = 5\)

53. \((a + 2)^2; a = 3\)  
54. \((b - 7)^3; b = 9\)  
55. \((m - 9)^2; m = 13\)  
56. \((S + Q)^2; Q = 4\)

**WRITTEN EXERCISES**

Rewrite each of the following expressions in a shorter form.

**A**

1. \(b \cdot b\)
2. \(c \cdot c \cdot c\)
3. \(a\) cubed
4. \(d\) squared
5. \(7 \cdot n \cdot n \cdot n\)
6. \(14 \cdot m \cdot m \cdot m\)
7. \(n\)\(\text{n}\)
8. \(8\)\(\text{n}\)
9. \(E\) fourth
10. \(F\) cubed
11. \(R\) used as a factor 5 times
12. \(s\) used as a factor 6 times
13. Five times the cube of \(y\)
14. Eight times the square of \(z\)
15. One-half the second power of \(g\)
16. One-fourth the fifth power of \(h\)
17. The square of \(2P\)
18. The square of \(8i\)
19. The cube of \(xy\)
20. The cube of \(ab\)
21. The cube of \((a - 1)\)
22. The cube of \((1 - a)\)
23. The cube of the sum \(r\) plus 2
24. The square of the sum \(t\) plus 7

Find the value of each of the following expressions.

**B**

25. \(m^2; m = \frac{1}{2}\)
26. \(n^2; n = \frac{1}{4}\)
27. \(4p^2; p = 3\)
28. \(8r^2; r = 5\)
29. \((9x)^2; x = \frac{1}{3}\)
30. \((8y)^2; y = \frac{1}{4}\)
31. \(2x^2 + 4x + 5; x = 3\)
32. \(5y^2 - 3y + 4; y = 1\)
33. \(7z^3 + z^2 - z; z = 2\)
34. \(a^3 - 2a^2 + a + 4; a = 5\)
35. \(v^5 + 3v^4 - v^3 + v; v = 0\)
36. \(w^{10} + w^5 + w + 9; w = 0\)

Let \(x = 5, y = 2,\) and \(z = 3,\) and evaluate the following expressions.

37. \(x^2 + y^2 + z^2\)
38. \(x - y^2 + z^2\)
39. \(x^2 + y + z^2\)
40. \(x - y + z^2\)
41. \(x^2 + y^2 - 2z^2\)

42. \(x^2 + y - z^2\)
43. \(x^2 - y^2 + z^2\)
44. \(\frac{2x^2 + xy}{20z}\)
45. \(\frac{z^3 - 27}{xy}\)
46. \(\frac{x^2 + z^2}{y^2}\)

47. \(\frac{7y^2z - 2x^2}{xy}\)
48. \((xz)^3 + y^6\)
49. \(\left(\frac{6y^3}{z}\right) + 3x^2\)
50. \(\frac{(x - z)^4 - y^4}{x - y^2}\)
51. \(\frac{(2y - z)^3 + y^3}{(2x - z + y)^3}\)

Evaluate each expression.

**A**

1. Area of a square: \(s^2;\) let \(s = 15\) cm.
2. Volume of a cube: \(s^3;\) let \(s = 15\) cm.
3. Conversion of mass to energy: \(mc^2;\) let \(c = 300,000,000\) meters per second and \(m = 254\) g.
4. Electrical power: \(I^2R;\) let \(I = 15\) amp., \(R = .01\) ohm.
5. Distance traveled during acceleration: \(\frac{at^2}{2};\) let \(a = 13.7\) ft./sec./sec., \(t = 25\) sec.

**B**

6. Volume of a sphere: \(\frac{4\pi R^3}{3};\)
   let \(R = 24''\), \(\pi = 3.14.\)

7. Heat radiation: \(kT^4;\) let \(k = .000000822, T = 6000^\circ\)K.
8. Kinetic energy: \( \frac{mv^2}{2} \); let \( m = 25 \text{ g.}, \ v = 100 \text{ cm./sec.} \).

9. Volume of a circular cylinder: \( \pi r^2 h \); let \( \pi = 3.14, \ r = 1.25\text{''}, \ h = 12.0\text{''}. \)

10. Illumination: \( \frac{C}{D^2} \); let \( C = 300 \text{ candle power}, \ D = 500 \text{ feet}. \)

11. Centripetal force: \( \frac{mv^2}{r} \); let \( m = 15 \text{ lb.}, \ v = 20 \text{ ft./sec.}, \ r = 10 \text{ ft.} \)

12. Resistance of an electrical conductor: \( \frac{kl}{d^2} \); let \( k = 10.37, \ l = 200 \text{ ft.}, \ d = 25 \text{ mil}. \)

13. Law of gravitation: \( \frac{GmM}{r^2} \); let \( G = 0.0000000667, \ m = 100,000 \text{ g.}, \ M = 900,000 \text{ g.}, \ r = 1000 \text{ cm}. \)

14. Heat energy from electricity: \( 0.238I^2Rt \); let \( I = 20 \text{ amp.}, \ R = 10 \text{ ohms}, \ t = 300 \text{ sec}. \)

15. Length of a pendulum: \( g \left( \frac{t}{2\pi} \right)^2 \); let \( g = 32.2 \text{ ft./sec./sec.}, \ t = .50 \text{ sec}. \)

2–3 Solving Open Sentences

Consider this sentence:

\( w \) is a city in Texas.

This sentence, as written, is neither true nor false. Suppose the replacement set of \( w \) is the cities of the U.S.A. Replacing \( w \) by Dallas produces a true statement. Putting New York in place of \( w \) leads to a false statement. This sentence becomes true or false as the variable is replaced by one of the values from its replacement set. In general, a sentence containing a variable may be neither true nor false, as the value of the variable is left open. Consequently, a sentence containing a variable is called an open sentence. The open sentence serves as a pattern for the various sentences, some true, some false, which you obtain by substituting in it the different values of the variable.
An **algebraic sentence** is a statement composed of algebraic expressions related by one of the symbols $=, \neq, >, <, \leq, \text{ or } \geq$. ($\leq$ is read is less than or equal to; $\geq$ is read is greater than or equal to.) These symbols of relationship are equivalent to the verbs in a sentence.

Any sentence using the symbol $=$ is called an **equation**. Consider the equation

$$3x + 1 = 16$$

**Left member** \hspace{1cm} **Right member**

The equation states that the left member (L.M.) expression, $3x + 1$, and the right member (R.M.) expression, $16$, designate the same number. If $\{5, 6\}$ is the replacement set for $x$, you can determine the values of $x$ which make the given equation a true statement, as follows:

$$3x + 1 = 16$$
$$3 \cdot 5 + 1 = 16$$
$$15 + 1 = 16$$
$$16 = 16, \text{ True}$$

The subset of the domain of the variable consisting of the elements of the domain which make the open sentence true is called the **solution set** of the open sentence. The solution set of $3x + 1 = 16$ is $\{5\}$. The process of determining the solution set is called **solving** the open sentence. Each member of the solution set is called a **root** of the open sentence; thus $5$ is a root of $3x + 1 = 16$.

Consider the open sentence $3x + 1 > 16$, and let $x \in \{5, 6\}$. It is called an **inequality** because it uses one of the symbols $\neq, >, <, \geq, \leq$ to relate the left member expression and the right member expression.

To solve the inequality, you may proceed as follows:

$$3x + 1 > 16$$
$$3 \cdot 5 + 1 > 16$$
$$15 + 1 > 16$$
$$16 > 16, \text{ False}$$

$$3x + 1 > 16$$
$$3 \cdot 6 + 1 > 16$$
$$18 + 1 > 16$$
$$19 > 16, \text{ True}$$

\[ \therefore (\text{Read "therefore"}) \text{ The solution set is } \{6\}. \]

The solution set of an equation or of an inequality may be shown in rule form, roster form, or by graph. The graph of the solution set of an open sentence is called the graph of the sentence.
CHAPTER TWO

EXAMPLE 1. \(3t + 2 = 14; \ t \in \{3, 4\}\)

Solution:

\[
\begin{align*}
3t + 2 & = 14 \\
3 \cdot 3 + 2 & = 14 \\
11 & = 14, \text{ False} \\
\end{align*}
\]

\[
\begin{align*}
3t + 2 & = 14 \\
3 \cdot 4 + 2 & = 14 \\
14 & = 14, \text{ True} \\
\end{align*}
\]

\[\therefore \text{ The solution set is } \{4\}, \text{ Answer.}\]

EXAMPLE 2. \(2 < x \leq 5; \ x \in \{4, 5, 8\}\)

Solution:

\[
\begin{align*}
2 & < x \leq 5 \\
2 & < 4 \leq 5, \text{ True} \\
2 & < 5 \leq 5, \text{ True} \\
2 & < 8 \leq 5, \text{ False} \\
\end{align*}
\]

\[\therefore \text{ The solution set is } \{4, 5\}, \text{ Answer.}\]

ORAL EXERCISES

In Exercises 1–16, replace the variable by names of elements of the given replacement set, and tell whether the resulting sentences are true.

SAMPLE 1. \(2w - 3 < w + 1; \ w \in \{5, 4\}\)

What you say:

\[
\begin{align*}
2 \cdot 5 - 3 & < 5 + 1, \text{ False} \\
2 \cdot 4 - 3 & < 4 + 1, \text{ False} \\
\end{align*}
\]

1. \(t - 7 = 6; \ t \in \{9, 13\}\)
   6. \(6 < f - 1; \ f \in \{10, 9\}\)
2. \(10 = n + 5; \ n \in \{5, 2\}\)
   7. \(2k = 3 + k; \ k \in \{2, 4\}\)
3. \(2y + 7 = 17; \ y \in \{0, 4\}\)
   8. \(s - 2 = s \div 2; \ s \in \{8, 2\}\)
4. \(9 \neq 3q - 6; \ q \in \{5, 8\}\)
   9. \(m \div 2 > m + 4; \ m \in \{2, 0\}\)
5. \(d + 3 > 5; \ d \in \{3, 4\}\)
   10. \(18 \div b < 18 + b; \ b \in \{9, 3\}\)
11. \(n + 5 < 100; \ n \in \{2, 4, 6, \ldots, 100\}\)
12. \(4m > 6; \ m \in \{1, 2, 3, 4, \ldots, 10\}\)
13. \(\frac{1}{2}u + 1 > \frac{1}{2}(u + 2); \ u \in \{2, 4, 6, 8\}\)
14. \(\frac{s}{2} - 1 < 10; \ s \in \{2, 4, 6, \ldots, 20\}\)
15. \(2(a + 3) = 2 \times a + 3; \ a \in \{1, 5\}\)
16. \(2(b + 3) = 2 \times b + 2 \times 3; \ b \in \{1, 5\}\)
In Exercises 17–30, determine the solution set from the replacement set of \{the numbers of arithmetic\}.

**SAMPLE 2.** \(1 < x \leq 2\)  
*What you say:* The solution set is \{the numbers between 1 and 2, including 2\}.

**SAMPLE 3.** \(y + 3 = y + 2\)  
*What you say:* The solution set is the empty set.

17. \(x + 4 = 9\)  
18. \(z - 3 \neq 7\)  
19. \(v + 5 \neq 11\)  
20. \(r - 2 = 10\)  
21. \(m > 6\)

22. \(0 < d < 4\)  
23. \(v + 21 = 32\)  
24. \(29 - p = 16\)  
25. \(4x \neq 20\)  
26. \(7y = 46\)

27. \(0 \leq x \leq 2\)  
28. \(3 \leq r \leq 5\)  
29. \(\frac{1}{2}r > 8\)  
30. \(3t > 9\)

**WRITTEN EXERCISES**

In Exercises 1–10, substitute members of the given replacement set in the open sentences, tell whether the resulting sentences are true, and give each solution set.

**SAMPLE 1.**  
\(1 < x < 7; x \in \{0, 2, 4, 6, 8\}\)

*Solution:*  
\(1 < 0 < 7\), False  
\(1 < 2 < 7\), True  
\(1 < 4 < 7\), True  
\(\therefore\) The solution set is \{2, 4, 6\}.

1. \(x + 1 = 5; x \in \{1, 2, 3, 4, 5\}\)  
2. \(x > 3; x \in \{0, 2, 4, 6, 8\}\)  
3. \(y < 1; y \in \{0, 1, 2, 3, 4\}\)  
4. \(2z \neq 8; z \in \{2, 4, 6, 8\}\)  
5. \(2x \neq 0; x \in \{0, 2, 4, 6\}\)  
6. \(2y + 1 = 5; y \in \{0, 2, 4, 6\}\)  
7. \(2z < 20; z \in \{8, 9, 10, 11\}\)

8. \(3y > 25; y \in \{7, 8, 9, 10\}\)  
9. \(x + (y + 2) = (x + y) + 2\)  
   a. \(x \in \{10\}, y \in \{7\}\)  
   b. \(x \in \{2\}, y \in \{0\}\)  
10. \(x - (y + 1) > (x - y) + 1\)
    a. \(x \in \{10\}, y \in \{6\}\)
    b. \(x \in \{5\}, y \in \{3\}\)

Determine the solution set from \{numbers of arithmetic\}. Give the solution set of each equation in roster form. Graph the solution set of each inequality.

**SAMPLE 2.**  
\(3t + 1 = 10\)

*Solution:*  
\(3 \cdot 3 + 1 = 10, \therefore \{3\}, Answer.\)
SAMPLE 3. \[ 2 \leq x < 5 \] Solution: 

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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</table>

SAMPLE 4. \[ 2n > 1 \] Solution: 

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
</table>

11. \[ t + 2 = 5 \]  
12. \[ 5 - z = 4 \]  
13. \[ m + 40 = 51 \]  
14. \[ 2 = 5 - t \]  
15. \[ x = 0 \]  
16. \[ 14 - s = 14 \]  
17. \[ k \leq 5 \]  
18. \[ x \leq 6 \]  
19. \[ 1 < x < 3 \]  
20. \[ 7 \leq y \leq 9 \]  
21. \[ 2 \leq r < 7 \]  
22. \[ 5 < n \leq 8 \]  
23. \[ z + 1 = z + 4 \]  
24. \[ y - 4 = y - 5 \]  
25. \[ 2 > m \]  
26. \[ x \geq 0 \]  
27. \[ 1 + x = x + 1 \]  
28. \[ y + y = 2y \]  
29. \[ 2t + 7 = 23 \]  
30. \[ 3k + 11 = 26 \]  
31. \[ x + 1 > 5 \]  
32. \[ y - 1 < 2 \]  
33. \[ 4 < a < 4 \]  
34. \[ 0 \leq k \leq 0 \]  
35. \[ 3w - 1 = 20 \]  
36. \[ 1 = 4s - 3 \]  
37. \[ 3s + 7 \neq 10 \]  
38. \[ v \neq 3v \]  
39. \[ 2m + 1 > 5 \]  
40. \[ 3k - 1 \leq 8 \]  

In Exercises 41–56, substitute the members of the given replacement set in the open sentences, and tell whether the resulting sentences are true.

SAMPLE 5. \[ 5t + 9v = 14tv; t \in \{1, 2\}, v \in \{4, 5\} \] Solution:  

a. \[ 5 \cdot 1 + 9 \cdot 4 = 14 \cdot 1 \cdot 4, \text{ False} \]  
b. \[ 5 \cdot 1 + 9 \cdot 5 = 14 \cdot 1 \cdot 5, \text{ False} \]  
c. \[ 5 \cdot 2 + 9 \cdot 4 = 14 \cdot 2 \cdot 4, \text{ False} \]  
d. \[ 5 \cdot 2 + 9 \cdot 5 = 14 \cdot 2 \cdot 5, \text{ False} \]  

41. \[ 9(m + t) = 9m + 9t; m \in \{3, 5\}, t \in \{2, 4\} \]  
42. \[ 12(cd) = (12c)d; c \in \{1, 3\}, d \in \{9, 11\} \]  
43. \[ (r + 1)(k + 3) > 3rk; k \in \{10, 21\}, r \in \{2, 4\} \]  
44. \[ 12d + 8 < 20d + 8e; d \in \{4, 7, 0\}, e \in \{0, 1\} \]  
45. \[ (t)(u + v) = tu + tv; t \in \{1, 2\}, u \in \{1, 2\}, v \in \{1, 2\} \]  
46. \[ x + yz = (x + y)z; x \in \{0, 2\}, y \in \{3\}, z \in \{5, 3\} \]  
47. \[ 5t + 9v = 14tv; t \in \{2, 1\}, v \in \{3, 1\} \]  
48. \[ 12d + 8d < 20d; d \in \{4, 7\} \]  
49. \[ (9 + m) : n = 9 : (m + n); m \in \{3, 9\}, n \in \{3, 1\} \]  
50. \[ 10 + 8y - 2 < 2y + 12 + 4y; y \in \{2, 1\} \]
51. \[10 - t + s + 2 > 8 \div t + 8(2 \div s); t \in \{4, 8\}, s \in \{2, 4\}\]

52. \[\frac{dd - hh}{d + h} = d - h; h \in \{4, 8\}, d \in \{6, 8\}\]

53. \[7r + 5s + 2t > 12r + t + t\]
   a. 3 for \(r\); 3 for \(s\); 0 for \(t\)
   b. 4 for \(r\); 1 for \(s\); 2 for \(t\)

54. \[(5 + x) + 7(2 + y) + (r + y) = 19 + x + 8y + r\]
   a. 2 for \(x\); 3 for \(y\); 5 for \(r\)
   b. 5 for \(x\); 2 for \(y\); 3 for \(r\)

55. \(k(m - n) \neq km - n\)
   a. 5 for \(k\); 12 for \(m\); 4 for \(n\)
   b. 1 for \(k\); 6 for \(m\); 0 for \(n\)

56. \[\frac{hk + m}{h} = k + m\]
   a. 15 for \(k\); 5 for \(m\); 4 for \(h\)
   b. 12 for \(k\); 2 for \(m\); 1 for \(h\)

**PROBLEMS SOLVED WITH VARIABLES**

**2–4 Thinking with Variables: From Symbols to Words**

Can you find an English phrase that is represented by the algebraic expression \(x + 1\)?

You may say: "Ron is a year older than his sister Sue. If \(x\) represents Sue's age in years, then \(x + 1\) represents Ron's age."

Another interpretation: "Let \(x\) represent a number; then \(x + 1\) represents that number increased by 1 or it represents the sum of \(x\) and 1 or it is the number one more than \(x\)."

Why may the variable \(x\), in the second interpretation, represent any number in the set of numbers of arithmetic, while in the first case \(x\) may not be 0?

The expression \(2w - 3\),

that is, the difference between twice a number \(w\) and 3, could arise in the following way. The length of a room is 3 feet less than twice its width. Hence, if \(w\) represents the number of feet in the width of the room, \(2w - 3\) represents the number of feet in its length. Since \(2w - 3\) must be greater than 0, \(2w\) must be greater than 3. Thus, \(w\) must represent a number greater than \(1\frac{1}{2}\). Why?
ORAL EXERCISES

Find two interpretations for each of the following algebraic expressions.

1. $2x$  
   2. $10y$  
   3. $z + 3$  
   4. $n - 1$  
   5. $\frac{g}{3}$  
   6. $\frac{1}{2}h$  
   7. $d + 5$  
   8. $u - 1.7$  
   9. $2p + 1$  
   10. $2q - 1$  
   11. $\frac{1}{2}s + \frac{1}{2}$  
   12. $\frac{1}{3}r + \frac{1}{3}$  
   13. $3m - 2$  
   14. $5n + 6$  
   15. $\frac{u + 1}{3}$  
   16. $\frac{v - 1}{4}$  
   17. $2t + 7$  
   18. $5w - 3$  
   19. $9s + 42s$  
   20. $30r - 19r$

WRITTEN EXERCISES

Give two interpretations for each of the following algebraic expressions. In each case identify the replacement set of the variable.

SAMPLE. $\frac{v}{2} - 5$

Solution: 1. Half a given number $v$ decreased by 5, with $v \geq 10$.

2. Mike and Bill belong to different boys’ clubs. Bill’s club has 5 fewer than half as many members as Mike’s club has. Let $v$ represent the number of boys in Mike’s club. Then $\frac{v}{2} - 5$ represents the number in Bill’s club. Here, the value of $v$ can be any even whole number exceeding 10.

1. $7x$  
2. $9y$  
3. $x + 15$  
4. $y - 10$  
5. $2a + 3$  
6. $3b - 7$  
7. $4(x + 1)$  
8. $5(y - 2)$  
9. $\frac{x}{4} - 2$  
10. $\frac{1}{3}n - 3$  
11. $2(2v + 1)$  
12. $u(3u - 1)$  
13. $z + (2z + 5)$  
14. $m + 2(m + 5)$
In solving a problem your initial job will be to translate English phrases into the language of algebra. Consider the following examples:

**EXAMPLE 1.** If whole milk costs 5 cents more per quart than skimmed milk, express the cost of a quart of whole milk in terms of the cost of a quart of skimmed milk.

**Solution:** Since the cost in cents of skimmed milk could be any number greater than 0 (the milkman is not giving it away), use a variable to represent the cost:

\[ m = \text{the cost in cents of 1 qt. of skimmed milk} \]

\[ m + 5 = \text{the cost in cents of 1 qt. of whole milk} \]

Here you are employing the symbol \( = \) in place of the word represent or represents.

**EXAMPLE 2.** Ken has 4 fewer than one-third as many stamps as Len has. Write an algebraic expression for the number of Ken’s stamps.

**Solution:** Let \( s = \) the number of Len’s stamps.

Then, \( \frac{s}{3} = \) one-third the number of Len’s stamps.

\[ \frac{s}{3} - 4 = \text{the number of Ken’s stamps} \]

Here \( s \) is a variable whose value may be any of the multiples of 3 greater than or equal to 12. Why?
ORAL EXERCISES

Answer each question by giving an algebraic expression. In each case identify a suitable replacement set of the variable.

SAMPLE. After reaching the green, a golfer used his putter to measure the distance of his ball from the cup. He found the distance to be 5 club lengths. Using a variable, represent the number of feet from the ball to the cup.

What you say: Let \( c \) = the number of feet in 1 club length.
The replacement set of \( c \) is the set of the numbers of arithmetic appropriate for the length of a putter.

1. A chemistry teacher wishes to have a shelf built which will hold exactly 8 reagent bottles. If each bottle is \( d \) inches in diameter, how long should the shelf be?

2. Mrs. Lund wishes to buy some fancy paper for her cupboard shelves. How much will she need for the shelves sketched?

3. Each of the 13 stripes of a flag is \( w \) inches wide. What algebraic expression represents the hoist (width) of the entire flag?
4. To panel one wall of a recreation room required 6 sections of paneling, each \( p \) inches wide. How long was the wall?

5. A hexagonal entry is floored with tile as shown. If each tile is \( t \) inches high, what is the width of the entry?

6. On a piano keyboard, a white key is \( i \) inches wide. How many inches are there in the width of an octave, such as the stretch of 8 white keys from \( C \) to \( C \), inclusive?

7. A box holds 12 Christmas ornaments, each \( d \) inches in diameter. State the length, width, and height of the box.

8. One tree is half as tall as another. Let \( h \) be the height of the taller tree. Represent the height of the smaller tree.

9. The distance across a baseball diamond from first base to third base is 1.4 times the distance from home plate to third base, which is \( x \) feet. Represent the distance from first base to third base.

10. The linoleum floor in a kitchen is made up of squares each \( k \) inches on a side. If the floor has eight tiles on one side and twelve on the other, what are its dimensions?

11. A farmer made his chicken lot one-fourth as wide as it was long. Let \( q \) be the number of feet in its length. Express the number of feet in its width in terms of \( q \).

12. The height in inches of a certain triangle is 2 less than the number of inches in its base. Represent the number of inches in the base of this triangle; then, the number of inches in its height, using that variable.

13. John owns \( k \) books, and Sue owns 1 more than twice as many. What algebraic expression stands for the number of Sue's books?

14. Joy is \( x \) years old. Tom is 1 year younger. What algebraic expression represents Tom's age?
15. A man earns \( d \) dollars a month. He gives one-tenth of his earnings to charity. Represent the amount he gives per month.

16. In making pastry, the amount of fat used is one-third the amount of flour. Represent the amount of fat to be used with \( c \) cups of flour.

**WRITTEN EXERCISES**

Translate into symbols.

**A**

1. 5 added to \( x \)
2. \( c \) added to \( d \)
3. \( 2x \) increased by 4
4. \( 5y \) increased by \( 2x \)
5. 15 decreased by \( n \)
6. \( 3x \) decreased by 2
7. 1 less than \( z \)
8. \( d \) less than \( x \)

**B**

1. The difference between \( 2x \) and 5
2. The difference between \( 5k \) and 5
3. \( 3x \) times \( y \)
4. The product of 6 and \( a \)
5. The sum \( r + 6 \), divided by 3
6. One-third of the difference, \( r - 6 \)
7. One-half of the sum, \( x + y \)
8. The difference \( a - b \), divided by \( c \)

17. 5 times the sum of 2 and \( y \)
18. \( a \) times the difference between \( a \) and \( b \)
19. 5 more than the product of \( x \) and the sum \( x + 2 \)
20. The difference between twice \( u \) and half of \( v \)

**PROBLEMS**

Wherever a drawing will help you answer a question, make one.

**A**

1. A runway at a large airfield is twice as long as a runway at a smaller field. Draw lines to represent the two runways, and express the length of each runway in terms of the same variable.

2. Amy used some cotton material to make a blouse. She required a piece three times as long to make a dress. Draw lines to represent the two lengths of material. Label the lines in terms of the same variable.
3. The floor of a schoolroom is made of 90 boards, laid side by side. If \( w \) equals the number of inches in the width of one board, express the width of the room.

4. There are five steps at the entrance to the school building, each \( i \) inches in height. Use an algebraic expression to represent the height of the door sill above the street level.

5. Mr. Landon divided his rectangular piece of land into three lots, each \( w \) feet in width. Express the width of the land before it was divided.

6. Mr. Paxton bought a box of apples which contained three layers. Each layer contained \( n \) apples. How many apples did he buy?

7. In a ten-year period, the total number of miles of airmail routes in the United States was tripled. Represent the number of miles in the routes at the beginning and at the end of the period.

8. There are twelve rows of seats in a high-school stadium. Each row is higher than the row below it by the distance the bottom row is above the ground. If a first row seat is \( s \) inches above the ground, how far above the ground is a seat in the tenth row?

9. During its second year of operation a company produced 400 units of a certain commodity. If its production had increased by \( q \) units during the second year, express its production during the first year.

10. A firm produced \( h \) units of a certain commodity during its first year. Production increased by 5000 units during each succeeding year; express the number of units produced during the third business year.

11. A tennis court is 6 feet longer than twice its width. Represent the number of feet in (a) the width, (b) the length.

12. Express the number that is 17 less than 3 times a given number.

13. The base of a triangle with two equal sides is 4 feet less than the sum of the two equal sides. Represent the number of inches in (a) each of the equal sides, (b) the base, (c) the perimeter.

14. The sum of two numbers is 57. Let \( w \) represent the smaller number. Using these facts, find an algebraic expression for the larger number.

15. One number is \( \frac{1}{3} \) of another number. Using \( s \) to represent the larger number, write an equation stating that the sum of the two is 108.

16. Represent the number that is 5 more than the sum of 3 times a certain number and 7 times that number.

17. A notebook costs 28 cents more than a pencil. Represent the cost of 5 pencils and 2 notebooks.

18. Bob is twice as old as Emma. Kent is 3 years older than Bob. If Emma is \( x \) years old, how old is (a) Bob? (b) Kent?
In a “word problem” you are told how numbers, some of which are described by word phrases, are related to one another. To solve the problem, you determine the described numbers so that the indicated relationships will be true.

**EXAMPLE 1.** The main body of the Air Force Titan missile in 1960 was eight times as long as the nose cone. The entire missile was 90 feet in length. How long was the nose cone?

*Solution:*

1. The first step in solving this problem is to choose a symbol to represent the number you want to find: the number of feet in the nose cone. Let \( n \) represent this number. You know that the main body of the missile is 8 times as long as the nose cone; so \( 8n \) represents the number of feet in the main body of the Titan. The only values admissible for \( n \) are the numbers of arithmetic, since \( n \) represents a measurement (number of feet).

2. The second step is to write an open sentence. You show the two expressions for the total length of the missile as:

\[
\frac{\text{length of cone}}{n} + \frac{\text{length of body}}{8n} = \frac{\text{entire length}}{90}
\]

or

\[
9n = 90
\]

3. The third step is to determine the solution set of the open sentence. The only value in the replacement set of \( n \) that produces a true statement for the sentence \( 9n = 90 \) is 10. The possible length of the nose cone is, therefore, 10 feet.

4. The fourth step is to use the words of the problem to check your answer.

<table>
<thead>
<tr>
<th>Length of nose cone:</th>
<th>10 feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of main body (8 times as much):</td>
<td>80 feet</td>
</tr>
</tbody>
</table>

Entire length: 90 feet

\( \therefore \) 10 feet, Answer.

**EXAMPLE 2.** The number of boys on a special committee is supposed to be more than three times the number of girls. If 12 boys are put on the committee, show on the number line the possible values for the number of girls.
Solution:

1. Let \( x \) = the number of girls; \( x \in \{\text{whole numbers}\} \)
   \[
   \therefore 3x = \text{three times the number of girls}
   \]

2. Form an open sentence.
   
   The number of boys is more than three times the number of girls
   
   \[
   12 > 3x
   \]

3. Find the solution set. Since \( x \) must represent a whole number, the solution set is \( \{0, 1, 2, 3\} \).

4. Since the largest value for the number of girls is 3, and since 12 is greater than 3 times 3, each element of the solution set does satisfy the requirements of the problem. Why is 4 not an element of the solution set? On the number line:

   Answer.

Four possible steps in solving a problem:

1. Choose a variable with an appropriate replacement set, and use the variable in representing each described number.
2. Form an open sentence by using facts given in the problem.
3. Find the solution set of the open sentence.
4. Check your answer with the words of the problem.

ORAL EXERCISES

Give an open sentence to fit each of the following exercises.

SAMPLE.

Multiply a number by 3, then add 8 to the product and you get 23.

What you say: \( 3n + 8 = 23 \)

1. Multiply a number by 2, and you get 6.
2. Multiply a number by 9, and you get 45.
3. Double a number, and you get 52.
4. Add 5 to a number, and you get 11.
5. Subtract 4 from a number, and you get 15.
6. Add 20 to a number, and you get 39.
7. Subtract 2 from a number, and you get 5.
8. Multiply a number by one, and you get 8.
9. Multiply a number by 3, then multiply this product by 2, and you get 30.
10. Multiply a number by 5, then multiply the product by \( \frac{1}{5} \), and you get 13.
11. Add 4 to a number, subtract 4 from the sum, and you get 39.
12. Add 7 to a number, then subtract 4 from the sum, and you get 5.
13. Multiply half a number by 2, and you get 9.
14. Double a number, add 2 to the product, and you get 15.

**PROBLEMS**

Solve the following problems by the four-step method on page 57.

**SAMPLE.** Jim is 3 years more than twice as old as his sister June. If June is 6 years old, how old is Jim?

*Solution:*

1. Let \( x \) represent Jim's age; \( x \in \{ \text{whole numbers} \} \)
2. \[ x = 2(6) + 3 \]
3. \[ x = 15 \]
4. 15 equals 3 more than \( 2 \) times 6. \( \checkmark \)

\[ x \]

\[ \therefore \text{Jim's age is 15 years, Answer.} \]

1. A baseball team won 3 times as many games as it lost. It won 84 games. How many games did it lose? (Let \( x \) represent the number lost.)
2. A class assessed each member 5 cents to buy flowers for entertainment. The total was 170 cents. How many members were there?
3. Mr. Jonas got a roll of 50 pennies to use only for parking meters. If he used 5 pennies daily, how many days did the roll last?
4. The number of boys in a certain class is seven times the number of girls. The number of boys is 28. How many girls are in the class?

5. A house cost $18,200. It cost seven times as much as the lot on which it was built. What was the cost of the lot?

6. A man takes a position at a monthly salary. At the end of nine months he has earned $4050. What is his monthly salary?

7. Si's age is \( \frac{1}{3} \) that of his aunt. If Si is 8 years old, how old is his aunt?

8. In a certain city \( \frac{1}{3} \) of the girls are blonde. Find the number of girls in the city if 10,195 of them are blonde.

9. The perimeter of a square is 50 inches. How long is one side?

10. The area of a rectangle that is 4 feet wide is 68 square feet. Find the length of the rectangle.

11. After Jack deposited $55.25, his total bank balance was $1342.70. How much did Jack have in his account before that deposit?

12. After a $15.75 bank withdrawal Phil's balance was $672.39. How much did he have on deposit before making the withdrawal?

13. Jane's weight is 11 pounds more than normal for her height and age. If Jane weighs 109 pounds, what is the normal weight?

14. A thermometer reads 56 degrees. What is the temperature, if this reading is 3.7 degrees less than the true reading?

15. Fred earns $7.50 per week more than Bill. If Fred's weekly salary is $115, what does Bill earn per week?

16. Dave's golf score was 3 less than Mark's. If Dave scored 89, what was Mark's score?

17. A man traveled a certain number of miles by automobile, and then nine times as far by airplane. His total trip was 500 miles in length. How far did he travel by automobile?

18. The number of Central High School freshmen studying French is one-fourth the number studying Spanish. The total number of students enrolled in these languages is 150. How many freshmen elect Spanish?

19. A certain number was doubled. Then the product was multiplied by 3. If the result was 84, find the number.

20. After delivering his first dozen bottles of milk, a milkman had fewer than 75 bottles left. At most, how many bottles had he originally?

21. Sue owns 1 more than twice as many books as John. If Sue owns 59 books, how many books does John own?

22. The number of bolts produced daily by machine A is 600 less than four times the number produced by machine B. If machine A's output is 4800 bolts per day, what is B's daily output?
23. Mary's bowling score was 10 more than half Jay's score. If Mary bowled 100, find Jay's score.

24. If one-third of a certain number is diminished by 16, the result is 21. Find the number.

25. The length of a picture is 4 feet less than twice its width. To frame it, 76 feet of framing are needed. Find the dimensions of the picture.

26. Mr. Tripp completed a journey of 640 miles. The average speed of the jet plane taken by Mr. Tripp was 15 times that of the automobile he used to get to the airport. If he traveled an hour by auto and an hour by jet, how far did he travel by automobile?

27. Linda said: "I sold 3 more than twice the number of tickets Jo sold." Maria replied: "I sold 32 tickets, and that is more than you sold." What is the largest possible number of tickets Jo sold?

28. Helen weighs twice as much as her sister. If the sum of their weights is less than or equal to 165 pounds, what is the most that Helen may weigh?

29. There are three numbers such that the second is twice the first and the third is 1 less than three times the first. If the sum of the numbers is 35, find the largest number.

30. One side of a triangular lot is 13 feet less than 3 times the second. The third side is 18 feet more than the second. To fence the lot 130 feet of fencing are required. Find the length of each side of the lot.

Extra for Experts

The Arithmetic of Sets: Union

The union of two sets consists of all the elements of both sets, but no element is listed more than once. For example, if the union of the sets $A = \{2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$ is called set $D$, then:

$$A \cup B = D$$

$\{2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7\} = \{2, 3, 4, 5, 6, 7\}$

This example may be illustrated pictorially:
Note that the union of two sets has each of the original sets as a subset; also, that the symbol for union is a stylized U.

Questions

1. If \( A = \{1, 2, 3\} \) and \( B = \{0, 2, 4, 3\} \), give the roster of \( A \cup B \), and show the union pictorially.
   a. What is the relationship between \( A \cup B \) and \( A \) and \( B \)? Why?
   b. Give a rule to determine when \( A \cup B = B \) will hold. Illustrate.

2. Under what conditions would \( A \cup B = A \cap B \)?
   a. Define a set \( A \) and a set \( B \) which would satisfy this condition.
   b. For these sets, draw Venn diagrams of \( A \cup B \) and \( A \cap B \).

3. In the accompanying diagram, the universal set \( G = \{\text{all graduates of Newville High School}\} \), \( H = \{\text{honor graduates}\} \), \( A = \{\text{award winners}\} \), and \( S = \{\text{scholarship recipients}\} \). For each item, copy the diagram, and shade it to show the indicated subset.
   a. Honor graduates
   b. Award-winning honor graduates
   c. Award-winning honor graduates who received scholarships
   d. Award-winning graduates and honor graduates
   e. Graduates receiving no honors, awards, or scholarships
The time was the sixteenth century. France and Spain were at war. As in every war, both sides sent their messages in code to hide their plans from the enemy. Obviously, secrecy was important.

But the Spanish secret could not be kept. Not that the Spaniards didn’t try. When the French captured a Spanish messenger, they read the message as accurately as any Spaniard could have read it. How could this thing be? The Spaniards knew their codes were baffling. How could a mere Frenchman decipher them? In fact, how could any man, unless he had the key? The conclusion was obvious. Something more than man must be at work. The French must be in league with the Devil. They must be using black magic!

The Spaniards complained to the Pope. But the Pope was too wise to interfere, for it was not the Devil who was breaking the codes; it was a French lawyer named Vieta. Nor was it by magic that he did his work, but by mathematics. For Vieta was a lawyer with a hobby, and his hobby was algebra. Code-breaking was nothing to him but solving equations.

The French king owed Vieta a debt of gratitude. So do generations of algebra students. For Vieta not only broke the Spanish code; he simplified the whole subject of algebra. Before his time, there was practically no use of signs and symbols; everything was done the hard way — in words. Vieta introduced the use of letters as variables (he used vowels for unknown numbers and consonants for known numbers). He used signs of operation — to show whether to add, subtract, multiply, or divide. So great were these and other contributions that Vieta, though only an amateur, is known today as "the father of algebra."

A portrait of Vieta, the French lawyer who used algebra for code-breaking.
Chapter Summary

Inventory of Structure and Method

1. In algebraic expressions, multiplication may be indicated by no sign, \(ab\), parentheses, \(8(7)\), or a raised dot, \(9 \cdot 5\). In a term having no other numerical factor, 1 is listed as the numerical coefficient, as in \(ab\) which is equal to \(1ab\). An exponent tells how many times another number (the base) is a factor, but a coefficient is, itself, a factor.

2. An algebraic sentence represents a condition which relates two expressions. The two expressions are called the left member and the right member of the equation or inequality. An open sentence, becomes true or false as the variables are replaced by numerals. Solving an open sentence in one variable consists in determining the elements of the replacement set of the variable for which the sentence is true.

3. The steps in solving problems algebraically are as follows:
   1 — Choose a variable with an appropriate replacement set and use the variable in representing each described number.
   2 — Form an open sentence by using facts given in the problem.
   3 — Find the solution set of the open sentence.
   4 — Check your answer with the words of the problem.

Vocabulary and Spelling

variable (p. 36) base (p. 40)
replacement set (domain) (p. 36) power (p. 40)
value of a variable (p. 36) open sentence (p. 44)
constant (p. 36) algebraic sentence (p. 45)
variable (open) expression (p. 36) equation (p. 45)
algebraic expression (p. 36) left & right members (p. 45)
value of an expression (p. 37) inequality (p. 45)
evaluate an expression (p. 37) solution set (p. 45)
term (p. 37) solve (p. 45)
factor (p. 40) root (p. 45)
coefficient (p. 40) graph of an open sentence (p. 45)
exponent (p. 40)
Chapter Test

2-1 Evaluate the following expressions if \( r = 3 \) and \( s = \frac{1}{3} \).

1. \( 8r \)  
2. \( 6s + r \)  
3. \( 6(r + s) - 3rs \)  
4. \( (r + s)(r - s) \)

Evaluate the following expressions if \( a = 2 \) and \( b = 3 \).

5. \( ab^3 \)  
6. \( (ab)^2 \)

2-2 Give the set of factors of each of these expressions.

7. \( a \)  
8. \( 12mn \)  
9. \( \frac{pq}{3} \)

Give the missing coefficients, as indicated.

10. \( 8qrs = (\_\_\_)rs \)  
11. \( \frac{st}{5} = (\_\_\_)st \)

12. Identify the numerical coefficient, the base, and the exponent in \( 3(x - 1)^2 \).

13. Write in mathematical symbols, \( h \) used as a factor three times.

2-3 The replacement set for \( x \) is \( \{3, 6, 9, 12\} \). Which of the elements make each of the following open sentences true?

14. \( x - 5 = 1 \)  
15. \( x - 5 > 1 \)

Let \( m = 9 \) and \( n = 1 \) in each of the following open sentences. For each resulting sentence, state whether it is true or it is false.

16. \( m - 5n \geq 6 \)  
17. \( m - (n + 8) = (m - n) + 8 \)

From \( \{1, 2, 3, \ldots, 10\} \) determine the solution set of each sentence.

18. \( 3t + 6 = 9 \)  
19. \( 3t - 6 > 9 \)

On the number line, graph each of the following inequalities.

20. \( 1 \leq y < 4 \)  
21. \( z > 3 \)

2-4 For each expression, give two interpretations.

22. \( x + 5 \)  
23. \( 2x - 5 \)

2-5 24. Write an algebraic expression for the amount of money saved in one year by Mr. Jones if his weekly income is \( s \) dollars, and his average monthly expenses, including taxes, are \( e \) dollars.

2-6 25. The length of a rectangular swimming pool is 5 feet less than twice its width. If the pool is 35 feet in length, find its width.
2-1 Evaluating Algebraic Expressions Containing Variables

1. A variable is a _?_ which represents each of the elements of a specified set.
2. The set whose elements may be used to replace a variable is called the _?_ set or _?_ of the variable.
3. Evaluate \((5r + s)(5r - s)\), letting \(r = 2\) and \(s = 1\).
4. An expression written as a _?_ or _?_ is called a term.

2-2 Identifying Factors, Coefficients, and Exponents

5. Each factor of a term is called the _?_ of the other factors.
6. The usual way of writing \(ln\) is _?_.
7. The meaning of \(s^2\) is _?_ times _?_.
8. An _?_ tells how many times another number is to be used as a factor.
9. In mathematical symbols, \(c\) used as a factor 5 times is _?_.
10. In the expression \(3a^2\), the numerical coefficient is _?_, the base is _?_, and the exponent is _?_.
11. The exponent of \(r\) in \(7r\) is _?_.
12. Find the value of \(e^3\) and the value of \(3e\) when \(e\) is 15.

2-3 Solving Open Sentences

Exercises 13-15 refer to the general statement: \(1 \cdot n = n\).

13. The variable in the given statement is _?_.
14. The _?_ of the variable is the elements of \(\{\text{all the numbers of arithmetic}\}\).
15. The expression \(1 \cdot n\) is the _?_ member of the equation.
16. An open sentence may become a true statement or a false statement depending on the replacement for the _?_.
17. Each of the open sentences \(3x \neq 6, 2p > 8,\) and \(r - 4 < 10\) is called an _?_.

In Exercises 18–21, write numerical sentences using \(y = \{0, 1, 2, 3, 4\}\) as the replacement set for the variable. State which are true and which false.

18. \(y - 1 = 3\)  
19. \(y < 5\)  
20. \(3y \neq 9\)  
21. \(y > 4\)
22. The set of numbers which belong to the replacement set of the variable and which make the sentence true is called the _set of the sentence._

23. A number which satisfies an open sentence is called a _solution_ of the open sentence.

24. Since 4 satisfies the condition expressed in \(5x - 1 = 19\), it is a _solution_ of the equation.

25. Of the following graphs, which represents the solution set of the inequality \(1 < x \leq 3\)?

![Graphs (a), (b), and (c)]

26. Of the inequalities (a) \(x \neq 2\), (b) \(x > 2\), (c) \(x \geq 2\), the one whose solution set is represented by the adjoining graph is _graph (c)._ 

2-4 Thinking with Variables: From Symbols to Words  
Find an interpretation for each of the following algebraic expressions. In each case, identify the replacement set of the variable.

27. \(3x\)  
28. \(\frac{x}{3} + 3\)  
29. \(3(x - 1)\)

2-5 Thinking with Variables: From Words to Symbols  
In Exercises 30–35, translate from words to symbols.

30. 5 less than \(a\)  
31. \(2b_{}\) _increased by 3_  

32. The difference between \(3c_{}\) and 3  
33. One-third of the sum \(d_{}\) and \(e_{}\)

34. A line is divided into three equal parts. Using \(i_{}\) as the number of inches in one part, represent the length of the entire line.

35. Find an expression for the number of cents Julie received in change from a one-dollar bill after buying \(n_{}\) five-cent articles.

2-6 Solving Problems with Open Sentences  
In Exercises 36–35, use the number in each described number.

36. When you have a problem to solve, first select a _variable_ and use it in representing each described number.

37. After forming an open sentence and finding its solution set, _interpret_ each root with the words of the problem.
Write an open sentence expressing the conditions described in Exercises 38–40; then find the solution set.

38. Eighteen times a certain number is 198.
39. In a school cafeteria one week, 1440 bottles of milk were sold. Three times as many bottles of milk as ice cream bars were sold.
40. A class of 25 boys wishes to donate from $3 to $5 to a charity. If each boy is to contribute the same amount, $a$, express as an inequality the amount each boy may give.

**Just for Fun**

**Be a Magician with Numbers**

If you practice a bit, you will be able to mystify your family and friends with your seemingly magical knowledge of numbers.

Tell a friend that you can give his age (or any other whole number he might choose) if he will follow a few instructions. Ask him to do the following silently, as you give him directions. If he will then give you the result, you can tell him his age (or the number he chose).

<table>
<thead>
<tr>
<th>You say</th>
<th>You think (or write)</th>
<th>He thinks (if his age is 13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply your age by 3.</td>
<td>$3a$</td>
<td>$13 \times 3 = 39$</td>
</tr>
<tr>
<td>Add 10 to the result.</td>
<td>$3a + 10$</td>
<td>$39 + 10 = 49$</td>
</tr>
<tr>
<td>Subtract twice your age.</td>
<td>$3a + 10 - 2a = a + 10$</td>
<td>$13 \times 2 = 26$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$49 - 26 = 23$</td>
</tr>
<tr>
<td>Subtract 6.</td>
<td>$a + 10 - 6 = a + 4$</td>
<td>$23 - 6 = 17$</td>
</tr>
<tr>
<td>Tell me the answer.</td>
<td>$a + 4 = 17$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$13 + 4 = 17$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a = 13$</td>
<td></td>
</tr>
</tbody>
</table>

Try this with your own age before you try it on anyone else. After a while, you can vary your directions provided you keep track of the steps, using a variable to stand for the number that you want to find. Remember, if your directions say to multiply the number by 5, for example, you must multiply all your terms by 5. Or, if the directions say to divide by 3, you must divide all your terms by 3.
In the diamond you recognize a symbol of value. You also know that diamonds of the same size may have different values. Why is this so? The beautiful pattern you see in the upper photograph is the outward expression of a regular internal structure. This regular pattern gives the diamond (being examined in lower photo) its beauty and hardness, its decorative and practical values.

Mathematics, too, has a regular structure which makes it a source of pleasure to those who understand its internal beauty and discover its many applications. Just as a diamond cutter cannot bring out the hidden beauty of a gem until he understands its internal structure, you cannot make full use of mathematics until you understand its structure.

Because of the importance of structure to both diamonds and mathematics, a diamond is used throughout this book to mark those ideas which form the basic structure of algebra.

IDENTIFYING AND USING NUMBER AXIOMS

3–1 Axioms of Equality

You learned to perform many operations with numbers because you abided by certain rules. These rules, which are statements accepted as true, are called assumptions, axioms, or postulates. Though some of these assumptions may seem simple, you must be able to understand and use these rules in solving complicated problems.

The first fundamental assumptions that you will meet are the axioms of equality which govern your work with equations. Perhaps the simplest of all axioms is the reflexive property of equality, which says that any number is equal to itself.

For any number \( a, a = a \).

The symmetric property of equality states that an equality is reversible.

For any numbers, \( a \) and \( b \), if \( a = b \), then \( b = a \).
The transitive property of equality makes it possible for you to identify two numbers as equal if each of them equals a third number.

For any numbers, \(a, b,\) and \(c,\) if \(a = b,\) and \(b = c,\) then \(a = c.\)

**ORAL EXERCISES**

In each exercise, name the property of equality which is illustrated.

**SAMPLE 1.** If \(5 + 6 = 14 - 3,\) then \(14 - 3 = 5 + 6\)

*What you say:* The symmetric property.

**SAMPLE 2.** Given \(5 + 6 = 11,\) and \(11 = 14 - 3,\) then \(5 + 6 = 14 - 3\)

*What you say:* The transitive property.

1. Given that \(17 - 2 = 15;\) therefore, \(15 = 17 - 2.\)
2. Given that \(4 + 6 = 10\) and \(10 = 5 + 5;\) therefore, \(4 + 6 = 5 + 5.\)
3. \(r + s = r + s.\)
4. Given that \(8\frac{2}{3} + 4\frac{1}{3} = 12\frac{1}{3}\) and that \(12\frac{1}{3} = 13;\) therefore, \(8\frac{2}{3} + 4\frac{1}{3} = 13.\)

Name the property of equality which is illustrated in each of the successive conclusions in the following examples.

5. Given that \(5(3 + 4) = 5(7),\) that \(5(7) = 35,\) and that \(35 = 15 + 20;\) therefore, \(5(3 + 4) = 35\) and \(5(3 + 4) = 15 + 20.\)
6. Given that \(5(1 + 0) = 5,\) and that \(5 + 0 = 5;\) therefore, \(5 = 5 + 0\) and \(5(1 + 0) = 5 + 0.\)
7. Given that \(17 - 6\frac{2}{3} = 16\frac{1}{4} - 6\frac{2}{3},\) and that \(10\frac{1}{4} = 16\frac{1}{4} - 6\frac{2}{3};\) therefore, \(16\frac{1}{4} - 6\frac{2}{3} = 10\frac{1}{4}\) and \(17 - 6\frac{2}{3} = 10\frac{1}{4}.\)
8. Given that \(\frac{7}{3} + \frac{5}{8} = \frac{7}{3} \times \frac{5}{8},\) that \(\frac{7}{3} \times \frac{5}{8} = \frac{35}{24},\) and that \(\frac{35}{24} = 3\frac{11}{24};\) therefore, \(\frac{7}{3} \times \frac{5}{8} = 3\frac{11}{24}\) and \(\frac{7}{3} \div \frac{5}{8} = 3\frac{11}{15}.\)

3–2 The Closure Properties

When you add two whole numbers, is the result *always* a whole number? To try every example would be an endless task. After checking a large number of varied examples:

\[
138 + 51 = 189; \quad 174 + 236 = 410;
\]

and so on, you would probably *assume* that the answer is yes.
Is this true also for multiplication? Again you try many examples:

\[ 5 \times 37 = 185; \quad 23 \times 48 = 1104; \]

and so on. Again you will, no doubt, assume that the product of two whole numbers is always a whole number.

Any set \( S \) is said to be **closed under an operation** performed on its elements, provided that each result of the operation is an element of \( S \). This is known as the **closure property** of a set under an operation. Calculations in arithmetic are based on the often unstated assumption that the set of numbers is closed under addition and multiplication.

The **closure property for addition** is stated:

> For every number \( a \) and every number \( b \), the sum \( a + b \) is a unique number (one and only one number).

The **closure property for multiplication** is stated:

> For every number \( a \) and every number \( b \), the product \( ab \) is a unique number.

Closure under any operation depends on both the particular **operation** and the **domain** of numbers used. For example, the set of odd numbers is closed under multiplication but not under addition (\( 3 \cdot 5 = 15, \ 3 + 5 = 8 \)). On the other hand, under division the set of whole numbers is not closed, but the set of arithmetic numbers other than 0 is.

An operation on elements of a specified set may not be possible unless that set is closed under the operation. For example, if you try to subtract any number of arithmetic from a smaller number, you know of no arithmetic number which could be the result. The set of numbers of arithmetic is not closed under subtraction.

Important also is the assumption that an indicated sum or product of numbers does not depend on the particular names designating the numbers.

\[ 3(2 + 5) = 3 \cdot 7 \quad \text{and} \quad 4 \cdot 99 = 4(100 - 1) \]

because

\[ 2 + 5 = 7 \quad \text{and} \quad 99 = 100 - 1. \]

These examples illustrate the **substitution principle**:

> For any numbers \( a \) and \( b \), if \( a = b \), then \( a \) and \( b \) may be substituted for each other.
You used the substitution principle often in arithmetic.

Add: \[ 13 + 7 + 4 + 12 \]

Note: Each red numeral was substituted for an expression which it equaled:
\[ 13 + 7 = 20; \quad 20 + 4 = 24; \quad \text{and} \quad 24 + 12 = 36. \]

**ORAL EXERCISES**

Which of the following sets are closed under the specified operations? Why?

**SAMPLE 1.** The even numbers, \( \{0, 2, 4, 6, \ldots\} \), multiplication

*What you say:* Closed, because the product of two even numbers is even.

**SAMPLE 2.** \( \{0, 1\} \), addition

*What you say:* Not closed, because \( 1 + 1 = 2 \) and \( 2 \not\in \{0, 1\} \).

1. \( \{0, 1, 2, 3\} \), addition
2. \( \{0, 1\} \), multiplication
3. \( \{1\} \), multiplication
4. \( \{2\} \), subtraction
5. \( \{0, 2\} \), subtraction
6. \( \{\frac{1}{2}, 1, 2\} \), division
7. \( \{0, 1\} \), division
8. \( \{0, 1, 2, 3, 4, \ldots\} \), subtraction
9. \( \{0, 2, 4, 6, 8, \ldots\} \), addition
10. \( \{1, 3, 5, 7, \ldots\} \), multiplication
11. \( \{1, 3, 5, 7, 9, \ldots\} \), addition
12. \( \{3, 6, 9, 12, \ldots\} \), addition
13. \( \{1, \frac{1}{2}, 2, \frac{1}{4}, 4, \frac{1}{8}, 8, \ldots\} \), division
14. \( \{1, 3, 5, \ldots\} \), multiplying by 5

**WRITTEN EXERCISES**

Which of the following sets are closed under each of the operations of addition, multiplication, subtraction, and division? When the set is not closed, give an example which shows this.

**SAMPLE.** \{fractions from 0 to 1\}

*Solution:* Addition — not closed, as \( \frac{2}{3} + \frac{1}{3} \) is not in the set
Subtraction — not closed, as \( \frac{1}{2} - \frac{1}{2} \) is not in the set
Division — not closed, as \( \frac{1}{2} \div \frac{1}{4} \) is not in the set
Multiplication — closed
Axioms, Equations, and Problem Solving

1. \{0\}  
2. \{1\}  
3. \{2\}  
4. \{3\}  
5. \{0, 1, 2, 3\}  
6. \{0, 1\}  
7. \{1, 2\}  
8. \{multiples of 5\}  
9. \{numbers between 0 and 2\}  
10. \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}  
11. \{nonzero numbers of arithmetic\}  
12. \{all fractions of arithmetic which are not whole numbers\}

3–3 Commutative and Associative Properties of Arithmetic Numbers

You know that

$$6 + 3 = 3 + 6; \quad 7 + 1 = 1 + 7; \quad 9 + 2 = 2 + 9.$$ 

In arithmetic you assume that when you add two numbers, you get the same sum no matter what order you use in adding them. This commutative (ka-mu-ta-tiv) property of addition may be stated:

For every number \(a\), and every number \(b\), \(a + b = b + a\).

Likewise, \(6 \times 3 = 3 \times 6\) and \(6 \cdot n = n \cdot 6\). When you multiply numbers, you obtain the same product, regardless of the order of the factors. The commutative property of multiplication is written:

For every number \(a\), and every number \(b\), \(ab = ba\).

Notice that subtraction and division do not have the commutative property. For example, \(6 - 3 \neq 3 - 6\) and \(6 \div 3 \neq 3 \div 6\).

To find the sum of \(252 + 60 + 40\), you probably first add 60 and 40, obtaining 100, and then add 252 to the result, getting 352. But if you add 252 and 60, and to that sum add 40, you obtain the same total. That is,

$$252 + (60 + 40) = (252 + 60) + 40.$$ 

Thus, you are free to choose any adjacent pair in addition, for the answer is the same. This associative (a-so-she-ay’tiv) property of addition states that:

For every number \(a\), every number \(b\), and every number \(c\),

$$a + (b + c) = (a + b) + c.$$
The associative property of multiplication is:

For every number $a$, every number $b$, and every number $c$, $a(bc) = (ab)c$.

Are subtraction and division associative? No, because

$24 - (6 - 2) \neq (24 - 6) - 2$ and $24 \div (6 \div 2) \neq (24 \div 6) \div 2$.

The commutative (order) and associative (grouping) properties permit you to omit parentheses in a sum because the numbers may be added in any groups of two and in any order.

$101 + 33 + 46 + 67 + 14 + 99 = 101 + 99 + 33 + 67 + 46 + 14$

$= 200 + 100 + 60$

$= 360$

**ORAL EXERCISES**

Name the property illustrated in each of the following true sentences. Every variable has the set of the numbers of arithmetic as its replacement set.

**SAMPLE.**

$3 \times (7 \times 9) = (7 \times 9) \times 3$

**What you say:** Commutative property of multiplication.

1. $6 + 2 = 2 + 6$
2. $(12 + 4) + 5 = 12 + (4 + 5)$
3. $\frac{1}{2} \times 6 = 6 \times \frac{1}{2}$
4. $8 \times (0 \times 4) = (8 \times 0) \times 4$

5. For each $a$, $5(6a) = 30a$
6. For each $m$, $m \times 3 = 3 \times m$
7. $\frac{1}{2} \times (2 + 9) = (2 + 9) \times \frac{1}{2}$
8. $9 + 0 = 0 + 9$

9. For each $x$, $7 + 9 + x = 16 + x$
10. $17.99 + 15 + 1.01 = 17.99 + 1.01 + 15$
11. $(58 + 11) + 139 = 58 + (11 + 139)$
12. For each $z$, $(32 + 17z) + 33z = 32 + (17z + 33z)$
13. For each $z$, $11 + (4 + z) = 15 + z$
14. $\frac{3}{5} \times (9 \times 7) = (\frac{3}{5} \times 9) \times 7$
15. For each $u$ and $w$, $5u + (3u + w) = (5u + 3u) + w$
16. For each $r$, $(r + 3)19 = 19(r + 3)$
17. $(11 \times 17\frac{1}{2}) \times 2 = 11 \times (17\frac{1}{2} \times 2)$
18. $25 \times (4 \times 93) = (25 \times 4) \times 93$
19. $\frac{2}{7} \times (\frac{5}{2} \times 16) = (\frac{2}{7} \times \frac{5}{2}) \times 16$
20. For each $a$ and each $b$, $7 \times (4a) \times b = 28ab$
Name the property that justifies each lettered step of these chains of equality. A check (√) shows that the step is justified by the substitution principle.

21. \(17 + (38 + 3) = 17 + (3 + 38)\)  
   \(= (17 + 3) + 38\)  
   \(= 20 + 38\) (√)  
   \(= 58\) (√)

22. \(4 \times (59 \times 25) = 4 \times (25 \times 59)\)  
   \(= (4 \times 25) \times 59\)  
   \(= 100 \times 59\) (√)  
   \(= 5900\) (√)

23. For each \(m\),  
   \(5 + (m' + 3) = 5 + (3 + m)\)  
   \(= (5 + 3) + m\)  
   \(= 8 + m\) (√)

24. For each \(n\),  
   \(5 \times (n \times 3) = 5 \times (3 \times n)\)  
   \(= (5 \times 3) \times n\)  
   \(= 15n\) (√)

25. For each \(r, s,\) and \(t\),  
   \(r(st) = r(ts)\)  
   \(= (rt)s\)  
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Either way you get the same result; that is,

\[ 95(3 + 7) = 95 \times 3 + 95 \times 7. \]

Note that 95, the coefficient (multiplier) of the sum \((3 + 7)\), is distributed as a multiplier of each term of \((3 + 7)\). The property shown in this example is called the distributive (dis-trib-u-tiv) property of multiplication with respect to addition and is stated:

For every number \(a\), every number \(b\), and every number \(c\),

\[ a(b + c) = ab + ac \quad \text{or} \quad ab + ac = a(b + c). \]

The following show how the distributive property is used:

\begin{enumerate}
\item \(28 \left( \frac{1}{2} + \frac{1}{4} \right) = 28 \times \frac{1}{2} + 28 \times \frac{1}{4} = 14 + 7 = 21\)
\item \(9 \times 4 \frac{2}{3} = 9 \left( 4 + \frac{2}{3} \right) = 9 \times 4 + 9 \times \frac{2}{3} = 36 + 6 = 42\)
\item \(\frac{15}{4} + \frac{9}{4} = \frac{15 + 9}{4} = 6, \text{ or} \frac{1}{4} \left( 15 \right) + \frac{1}{4} \left( 9 \right) = \frac{1}{4} \left( 15 + 9 \right) = \frac{1}{4} \left( 24 \right) = 6\)
\end{enumerate}

You can readily show that the following sentences involving subtraction and multiplication are true.

\[14 \left( \frac{3}{2} - 1 \right) = 14 \times \frac{3}{2} - 14 \times 1 = 7\]
\[20 \cdot 8 - 20 \cdot 5 = 20(8 - 5) = 60\]

These two sentences illustrate the distributive property of multiplication with respect to subtraction:

For each \(a\) and each \(b\) and each \(c\) for which \(b - c\) is a number,

\[a(b - c) = ab - ac, \text{ or } ab - ac = a(b - c).\]

Often you will use these properties to simplify variable expressions. For example, to show that \(5x + 3x = 8x\) for each number \(x\):

\[5x + 3x = x5 + x3 \quad \text{Commutative property of multiplication}\]
\[= x(5 + 3) \quad \text{Distributive property}\]
\[= x8 \quad \text{Substitution principle}\]
\[= 8x \quad \text{Commutative property of multiplication}\]
Similarly,

\[ 7ab - 4ab = (7 - 4)ab = 3ab. \]

The distributive property enables you to write the sum \((5x + 3x = 8x)\) or the difference \((7ab - 3ab = 4ab)\) of similar terms as a single term. Terms such as 7 and 9, 5x and 3x, or 7ab and 4ab are called similar terms or like terms. Similar terms are numerical terms or variable terms whose variable factors are the same.

Terms such as 8x and 3ab are unlike terms, because their variable factors are different. Also, the terms 9 and 7b are unlike terms. Hence, expressions such as 8x + 3ab and 7b - 9 cannot be written in simpler form.

\[ 1 \times 3 = 3 \times 1 = 3 \]
illustrates the multiplicative property of 1 (mul-ti-pli-kay'tiv): one times any given number equals the given number itself.

For each number \(a\),

\[ 1 \cdot a = a \cdot 1 = a. \]

Since the given number and the product are always identical when you use the multiplicative property of 1, do you see why 1 is the multiplicative identity element?

Likewise, \(0 + 3 = 3 + 0 = 3\) illustrates that the additive identity element is 0. The additive property of 0 states that when 0 is added to any given number, the sum is the given number itself, or simply:

For each number \(a\),

\[ 0 + a = a + 0 = a. \]

The multiplicative property of 0, shown in \(0 \times 3 = 3 \times 0 = 0\), states that when one of the factors of a product is 0, the product itself is 0.

For each number \(a\),

\[ 0 \cdot a = a \cdot 0 = 0. \]

This multiplicative property of 0 affects the use of 0 as a divisor. The statement \(\frac{6}{3} = 2\) means that \(6 = 3 \times 2\). Likewise, \(\frac{a}{0} = b\) should mean that \(a = 0 \times b\). If \(a \neq 0\), no value of \(b\) can make the latter statement true, since \(0 \times b = 0\) for each \(b\). If \(a = 0\), every value of \(b\) makes the statement true for the same reason. Thus, the fraction \(\frac{a}{0}\) either has no value or is indefinite in value. A consequence of the multiplicative property of 0 is that you may not divide by 0.
ORAL EXERCISES

Name the property of numbers which justifies each step in the following exercises.

1. \(7 \times (4 \times \frac{1}{2}) = 7 \times (\frac{1}{2} \times 4)\)
   \[= (7 \times \frac{1}{2}) \times 4\]
   \[= 1 \times 4\]
   \[= 4\]

2. \((\frac{3}{4} - \frac{1}{6})48 = 48(\frac{3}{4} - \frac{1}{6})\)
   \[= 48(\frac{3}{4}) - 48(\frac{1}{6})\]
   \[= 36 - 8\]
   \[= 28\]

3. \(973(101) = 973(100 + 1)\)
   \[= 973(100) + 973(1)\]
   \[= 97,300 + 973\]
   \[= 98,273\]

4. \(24 \times 5\frac{1}{2} = 24(5 + \frac{1}{2})\)
   \[= 24(5) + 24(\frac{1}{2})\]
   \[= 120 + 12\]
   \[= 132\]

5. For each \(r, s,\) and \(t,\)
   \[(r + s)t = t(r + s)\]
   \[= tr + ts\]
   \[= rt + st\]

6. For each \(r,\)
   \[\frac{3r + 15}{3} = \frac{1}{3}(3r + 15)\]
   \[= \frac{1}{3}(3r) + \frac{1}{3}(15)\]
   \[= \left(\frac{1}{3} \cdot 3\right) r + 5\]
   \[= 1 \cdot r + 5\]
   \[= r + 5\]

7. For each \(x,\)
   \[x + (7 + 6x) = x + (6x + 7)\]
   \[= (x + 6x) + 7\]
   \[= (1 \cdot x + 6x) + 7\]
   \[= (1 + 6)x + 7\]
   \[= 7x + 7\]
   \[= 7(x + 1)\]

Read each expression with similar terms combined.

**SAMPLE.** \(8z + 3 - 3z\)

<table>
<thead>
<tr>
<th>Expression</th>
<th>What you say</th>
</tr>
</thead>
<tbody>
<tr>
<td>8. (3x + 5x)</td>
<td>5(x + 3)</td>
</tr>
<tr>
<td>9. (5a + 6a)</td>
<td>1(1\cdot a + 3\cdot a)</td>
</tr>
<tr>
<td>10. (45s - 15s)</td>
<td>1(5\cdot s - 3\cdot s)</td>
</tr>
<tr>
<td>11. (56y - 16y)</td>
<td>1(7\cdot y - 2\cdot y)</td>
</tr>
<tr>
<td>12. (5x + 4x + x)</td>
<td>2(2\cdot x + 3\cdot x)</td>
</tr>
<tr>
<td>13. (8n + 2n + n)</td>
<td>3((4n + n + n))</td>
</tr>
<tr>
<td>14. (6m + 5m + 7)</td>
<td>4(4\cdot m + 3\cdot m + 1)</td>
</tr>
<tr>
<td>15. (9y + y - 3n)</td>
<td>5((3 \cdot y - 3\cdot y))</td>
</tr>
<tr>
<td>16. (10y + n - 5y)</td>
<td>6(5\cdot y + 1\cdot n - 4\cdot y)</td>
</tr>
<tr>
<td>17. (x + 4x - 5x)</td>
<td>7((6 \cdot x - 4 \cdot x))</td>
</tr>
<tr>
<td>18. (8p + 2p - 10p)</td>
<td>8(8\cdot p + 2\cdot p - 10\cdot p)</td>
</tr>
<tr>
<td>19. (2(a + 1) - 2)</td>
<td>9((2\cdot a + 2\cdot 1) - 2\cdot 1)</td>
</tr>
<tr>
<td>20. (3 + 3(b - 1))</td>
<td>10(3 + 3\cdot b - 3)</td>
</tr>
<tr>
<td>21. (5(2t + 3) + t)</td>
<td>11(10\cdot t + 1\cdot 3 + 1\cdot t)</td>
</tr>
<tr>
<td>22. (4(4s + 5) + 3s)</td>
<td>12(16\cdot s + 3\cdot 5 + 3\cdot s)</td>
</tr>
<tr>
<td>23. (7k + 3(2 - k))</td>
<td>13(7\cdot k + 6\cdot 2 - 3\cdot k)</td>
</tr>
<tr>
<td>24. (8v + 5(7 - v))</td>
<td>14(56\cdot v + 35\cdot 7 - 5\cdot v)</td>
</tr>
<tr>
<td>25. (6v + 7(3 + m))</td>
<td>15(18\cdot v + 21\cdot 3 + m)</td>
</tr>
<tr>
<td>26. (x - x + x - x)</td>
<td>16(x \cdot 1 - x \cdot 1)</td>
</tr>
<tr>
<td>27. (9y + 10y + 7 - 4)</td>
<td>17(19\cdot y + 3\cdot 7 - 1\cdot 4)</td>
</tr>
<tr>
<td>28. (3x + 5y - 6u - 4)</td>
<td>18(3\cdot x + 5\cdot y - 6\cdot u - 4)</td>
</tr>
<tr>
<td>29. (3y + 12 + y + 8)</td>
<td>19(4\cdot y + 12 + 8)</td>
</tr>
</tbody>
</table>
Simplify each of the following expressions by combining similar terms.

**SAMPLE.** \(8x + 3y + 2x - 3y\)

**Solution:** \((8x + 2x) + (3y - 3y) = 10x + 0\)

\[= 10x\]

**A**

1. \(17x + 39x\)
2. \(48y + 37y\)
3. \(100a - 35a\)
4. \(75b - 52b\)
5. \(13a + a + 16a\)
6. \(15x - x - 7x\)
7. \(5x + 7x + 6\)
8. \(9n + 6n + 3\)
9. \(2s + 3s + s + r\)
10. \(3x + y + x + y\)

**B**

11. \(2a + 3b + 4c - d\)
12. \(6u - 5v - 3w - 4\)
13. \(15n + 13 - 5n + 8\)
14. \(18z + 27 + 8z - 17\)
15. \(5a + 6b - a + b - 4a\)
16. \(3x + x + 5 + 24 + 1\)
17. \(16 + 16a + 5b - 15 - b\)
18. \(19 + s + 6t - t - 17\)
19. \(5ab + 7a + 6ab - a\)
20. \(7xy + 3x - 2xy + 9x\)

**C**

21. \(3(a + b) + 7(a + b)\)
22. \(9(m + n) + 7(2m + n)\)
23. \(8(3 + a) + 4(3 + a)\)
24. \(2(a + 5) + 3(a - 2)\)
25. \(4(d + 5) + 3(2d - 1)\)

**D**

31. \(3[8a + 5(3 - a)] - 17\)
32. \(19 + 2[4b + 3(5b - 2)]\)
33. \(6(2r + s) + 2[5r + 3(4s - r + 1)]\)
34. \(5[4(2m + n + 3) - 6m - 1] + 2(5m - n)\)

**Determine the value of each numerical expression. Whenever you can, use the properties of numbers to simplify the calculation.**

35. \(28 \times 37 + 22 \times 37\)
36. \(77 \times 19 + 23 \times 19\)
37. \(50 \left(\frac{3}{2} - \frac{1}{2}\right)\)
38. \(779(11) - 779\)
39. \(4 \times (0 \times 25)\)
40. \((68 \times 32) \times 0\)

39. \(7 \times 46 - 5 \times 46 - 2 \times 46\)
CHAPTER THREE

TRANSFORMING EQUATIONS WITH EQUALITY PROPERTIES

3–5 Addition and Subtraction Properties of Equality

Certain properties of equality can be proved from the properties of equalities given in Section 3–1. A knowledge of these properties will help you to solve complicated equations more readily. Consider the following illustration.

Two men receive equal salaries. Each gets the same $500 raise. Their salaries change, but the new salary of man $A$ is equal to the new salary of man $B$.

This example of the addition property of equality shows that if the same number is added to equal numbers, the sums are equal:

$$
\text{Man } A \quad \text{Man } B
$$

Two men receive equal salaries.Each gets the same $500 raise.Their salaries change, but the new salary of man $A$ is equal to the new salary of man $B$.

For each $a$, each $b$, and each $c$, if $a = b$, then $a + c = b + c$.

This new property of equality follows from facts already learned. The reasoning leading from the assumption $a = b$ to the conclusion $a + c = b + c$ is shown in the following sequence of statements, each justified by the indicated reason:

$$
\begin{align*}
    a + c & \text{ is a number} \\
    a + c & = a + c \\
    a & = b \\
    a + c & = b + c
\end{align*}
$$

Closure property of addition

Reflexive property of equality

Given

Substitution principle

This form of logical reasoning, from known facts and assumptions to conclusions, is called a proof.

Can you prove the subtraction property of equality: if the same number is subtracted from equal numbers, the differences are equal, provided the indicated subtraction is possible?

For each $a$, each $b$, and each $c$ for which $a - c$ is a number, if $a = b$, then $a - c = b - c$. 

"
The addition and subtraction properties of equality are used to solve equations. To see how to use them, first notice:

\[ 100 - 70 + 70 = 100 \quad \text{and} \quad n - 6 + 6 = n \]

In general, \( a - c + c = a \) and \( c \) is the number you add to \( a - c \) to produce \( a \). To undo a subtraction, you add.

Similarly, \( 8 + 5 - 5 = 8 \) and \( y + 3 - 3 = y \). In general, \( a + c - c = a \), and \( c \) is the number you subtract from \( a + c \) to obtain \( a \). To undo an addition you subtract.

Because the operations of adding and subtracting the same number are opposite in effect, they are called **inverse operations**. Can you think of another pair of inverse operations?

Study the following sequence of equations:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) ( n - 6 = 19 )</td>
<td>Original equation</td>
</tr>
<tr>
<td>(2) ( n - 6 + 6 = 19 + 6 )</td>
<td>( n = 25 )</td>
</tr>
</tbody>
</table>

Because you obtain the third equation by adding 6 to each member of the first equation, while you can obtain the first by subtracting 6 from each member of the third equation, the addition and subtraction properties of equality imply that these equations have the same solution set, \( \{25\} \), and are **equivalent equations**. This process which uses addition to transform an equation such as (1) into a simple equivalent equation such as (3) is called **transformation by addition**.

To check the value found, substitute it for the variable in equation (1). If the resulting sentence is true, this value satisfies the equation and is a root. Because numerical errors may occur in transforming an equation, you should check all values found.

*Check:* \( n - 6 = 19 \) \( \leftarrow \) original equation

\[ 25 - 6 \overset{?}{=} 19 \] \( \leftarrow \) **The question mark above the equals sign means, “Is this statement true?”**

\[ 19 = 19 \checkmark \] \( \leftarrow \) **The check (\( \checkmark \)) means “Yes, it is.”**

\( \therefore \) The solution set is \( \{25\} \), Answer.
EXAMPLE 1.  \( x - 2 = 5 \)

Solution:  
\[
x - 2 = 5
\]
\[
x - 2 + 2 = 5 + 2
\]
\[
x = 7
\]
Check:  
\[
7 - 2 = 5
\]
\[
5 = 5 \checkmark
\]
\[\therefore \text{The solution set is } \{7\}, \text{ Answer.}\]

EXAMPLE 2.  \( m - 8 = 0 \)

Solution:  
\[
m - 8 = 0
\]
\[
m - 8 + 8 = 0 + 8
\]
\[
m = 8
\]
Check:  
\[
8 - 8 = 0
\]
\[
0 = 0 \checkmark
\]
\[\therefore \text{The solution set is } \{8\}, \text{ Answer.}\]

EXAMPLE 3.  \( y + 3 = 14 \)

Solution:  
\[
y + 3 = 14
\]
\[
y + 3 - 3 = 14 - 3
\]
\[
y = 11
\]
Check:  
\[
11 + 3 = 14
\]
\[
14 = 14 \checkmark
\]
\[\therefore \text{The solution set is } \{11\}, \text{ Answer.}\]

EXAMPLE 4.  \( 9 = a + 2 \)

Solution:  
\[
a + 2 = 9
\]
\[
a + 2 - 2 = 9 - 2
\]
\[
a = 7
\]
Check:  
\[
9 = 7 + 2
\]
\[
9 = 9 \checkmark
\]
\[\therefore \text{The solution set is } \{7\}, \text{ Answer.}\]

Examples 3 and 4 illustrate transformation by subtraction, a process leading to equivalent equations because of the subtraction and addition properties of equality.

**ORAL EXERCISES**

Solve each equation, first stating what number must be subtracted from each member of the equation or what number must be added to each member of the equation.

1.  \( x + 2 = 6 \)
2.  \( n + 5 = 7 \)
3.  \( x - 1 = 8 \)
4.  \( y - 6 = 7 \)
5.  \( k + 7 = 11 \)
6.  \( r + \frac{2}{3} = \frac{7}{3} \)
7.  \( n - 9 = 11 \)
8.  \( h - 18 = 6 \)
9.  \( k + \frac{2}{3} = \frac{5}{3} \)
10.  \( m + 3 = 3 \)
11.  \( a - 40 = 8 \)
12.  \( p - 8 = 5 \)
13.  \( n - .8 = 1.1 \)
14.  \( b - .5 = 3.2 \)
15.  \( x + 4 = 4 \)
16.  \( t + .2 = .7 \)
17.  \( 18 = x - 6 \)
18.  \( 65 = n - 5 \)
19.  \( 75 = 60 + x \)
20.  \( 14 = 7 + n \)
21.  \( \frac{7}{2} = w - \frac{5}{2} \)
22.  \( \frac{2}{3} = z - \frac{5}{3} \)
23.  \( \frac{1}{3} = \frac{2}{3} + n \)
24.  \( \frac{17}{11} = s + \frac{5}{11} \)
Solve each equation by using addition or subtraction.

1. \( x + 32 = 81 \)
2. \( y + 17 = 45 \)
3. \( 27 + R = 105 \)
4. \( 91 + S = 91 \)
5. \( 39 = t + 39 \)
6. \( 73 = u + 73 \)
7. \( v - 47 = 47 \)
8. \( w - 37 = 73 \)
9. \( 175 = z + 82 \)
10. \( 314 = 165 + a \)
11. \( b - 17 = 0 \)
12. \( c - 51 = 0 \)
13. \( d + 3.2 = 7.8 \)
14. \( f + 2.7 = 3.1 \)
15. \( h - 0.78 = 9.2 \)
16. \( k - 0.37 = 4.1 \)
17. \( \frac{3}{5} + m = \frac{18}{5} \)
18. \( \frac{7}{7} + n = \frac{16}{7} \)
19. \( \frac{3}{3} = Q - \frac{5}{3} \)
20. \( \frac{3}{10} = T - \frac{13}{10} \)
21. \( m - 750 = 5 \)
22. \( 0.04 + p = 1 \)
23. \( a - 0.85 = 0.15 \)
24. \( b - 5 = \frac{3}{5} \)
25. \( h - \frac{3}{5} = 3 \)
26. \( d - 2.5 = 3.95 \)
27. \( y - \frac{3}{8} = 2 \)
28. \( p - \frac{5}{6} = 3\frac{1}{6} \)
29. \( \frac{8}{8} = n - 75.7 \)
30. \( \frac{3}{4} = .75 + r \)

### 3–6 Division and Multiplication Properties of Equality

The **division property of equality** states that when equal numbers are divided by the same number, the quotients are equal (recall that 0 may not be used as a divisor):

For each \( a \), each \( b \), and each nonzero \( c \), if \( a = b \), then \( \frac{a}{c} = \frac{b}{c} \).

The **multiplicative property of equality** says that when equal numbers are multiplied by the same number, the products are equal:

For each \( a \), each \( b \), and each \( c \), if \( a = b \), then \( a \cdot c = b \cdot c \).

These properties permit you to apply *transformation by division* and *by multiplication* to an equation without changing its solution set.

**EXAMPLE 1.** \( 6k = 84 \)

**Solution:**

\[
\begin{align*}
6k &= 84 \\
\frac{6k}{6} &= \frac{84}{6} \\
k &= 14
\end{align*}
\]

**Since** \( 6k \) shows \( k \) multiplied by 6, you use the inverse operation, division; that is, divide each member by 6, the coefficient of the variable.

**Check:**

\[
\begin{align*}
6(14) &= 84 \\
84 &= 84 \checkmark \\
\therefore \text{The solution set is } \{14\}, \text{ Answer.}
\end{align*}
\]
CHAPTER THREE

EXAMPLE 2. \[ \frac{n}{4} = 6 \]

Solution:

\[ \frac{n}{4} = 6 \]

\[ 4 \cdot \frac{n}{4} = 4 \cdot 6 \]

\[ n = 24 \]

Check:

\[ \frac{24}{4} \cdot 6 \]

\[ 6 = 6 \; \checkmark \]

\[ \therefore \text{The solution set is} \; \{24\}, \text{Answer.} \]

EXAMPLE 3. \[ \frac{2x}{5} = 86 \]

Solution:

\[ \frac{2x}{5} = 86 \]

\[ 2 \cdot \frac{2x}{5} = 5 \cdot 86 \]

\[ 2x = 430 \]

Check:

\[ \frac{2(215)}{5} \cdot 2 \]

\[ x = 215 \]

\[ \therefore \text{The solution set is} \; \{215\}, \text{Answer.} \]

You know that you may not divide by zero, but do you see why you may not use zero as a multiplier in transforming an equation? Consider the equation \( \frac{n}{4} = 6 \) whose solution set is \( \{24\} \). If you multiplied by 0, the equation \( 0 \cdot \frac{n}{4} = 0 \cdot 6 \) would be satisfied by any value of \( n \); its solution set would be the set of all arithmetic numbers. Thus, the original equation \( \frac{n}{4} = 6 \), and the derived equation \( 0 \cdot \frac{n}{6} = 0 \cdot 6 \) would have different solution sets; they would not be equivalent equations. Therefore, multiplication by zero cannot be used because it does not produce an equivalent equation.

**WRITTEN EXERCISES**

Solve each of the following.

A

1. \[ \frac{n}{19} = 6 \]

2. \[ \frac{r}{27} = 8 \]

3. \[ 17s = 187 \]

4. \[ 13t = 169 \]

5. \[ 1.5n = 3 \]

6. \[ .7 = .7m \]

7. \[ \frac{f}{1.2} = 15 \]

8. \[ \frac{x}{1.8} = 1.9 \]

9. \[ \frac{a}{3} = \frac{1}{2}x \]

10. \[ \frac{3}{8} = \frac{1}{2}y \]

11. \[ 16 = 12t \]

12. \[ 20 = 16z \]

13. \[ .8 = \frac{y}{5} \]

14. \[ 1.6 = \frac{h}{1.5} \]

15. \[ \frac{4}{3} = 6c \]

16. \[ 12d = \frac{3}{4} \]

17. \[ .36(y) = 4.8 \]

18. \[ .14(y) = 14 \]
19. \( \frac{k}{3.2} = 0 \)
22. \( \frac{p}{.03} = .2 \)
26. \( .75 = x \div 15 \)
27. \( \frac{3}{2} y = 6 \)
28. \( \frac{5}{4} w = 20 \)
29. \( \frac{1}{2} r = 3 \frac{1}{2} \)
30. \( \frac{1}{3} v = 4 \frac{3}{4} \)

31. \( 4 + 8y = 52 \)
33. \( 9r - 9 = 27 \)
35. \( 2n - 5 = 8.5 \)
36. \( 29 = 7h + 7.79 \)

Supply the reasons for each step in these proofs. Assume that \( m = n \) and \( r = s \) and that \( m + r, mr, \frac{m}{r} \), and \( m - r \) represent numbers.

**SAMPLE.**

\begin{align*}
\text{a.} & \quad m + r = m + r \quad \text{a. Reflexive property} \\
\text{b.} & \quad m = n \quad \text{b. Given} \\
\text{c.} & \quad m + r = n + r \quad \text{c. Substitution principle} \\
\text{d.} & \quad r = s \quad \text{d. Given} \\
\text{e.} & \quad \therefore m + r = n + s \quad \text{e. Substitution principle}
\end{align*}

37. a. \( mr = mr \)
   b. \( m = n \)
   c. \( \therefore mr = nr \)
38. Assume \( r \neq 0 \).
   a. \( \frac{m}{r} = \frac{m}{r} \)
   b. \( m = n \)
   c. \( \therefore \frac{m}{r} = \frac{n}{s} \)
39. a. \( m - r = m - r \)
   b. \( m = n \)
   c. \( \therefore m - r = n - r \)
40. a. \( mr = mr \)
   b. \( m = n \)
   c. \( mr = nr \)
   d. \( r = s \)
   e. \( \therefore mr = ns \)
41. Assume \( r \neq 0 \).
   a. \( \frac{m}{r} = \frac{m}{r} \)
   b. \( m = n \)
   c. \( \therefore \frac{m}{r} = \frac{n}{s} \)
42. a. \( m - r = m - r \)
   b. \( m = n \)
   c. \( \therefore m - r = n - r \)
   d. \( r = s \)
   e. \( \therefore m - r = n - s \)
3-7 Combining Terms and Using Transformation Principles

To analyze the solution of an equation, study the steps used in building it to its present form. The following example shows that in the solution, you undo the operations used in forming the equation, but in reverse order.

<table>
<thead>
<tr>
<th>Building an Equation</th>
<th>Solving the Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v = 5 )</td>
<td>Given</td>
</tr>
<tr>
<td>( 2 \cdot v = 2 \cdot 5 )</td>
<td>Multiply by 2</td>
</tr>
<tr>
<td>( 2v = 10 )</td>
<td></td>
</tr>
<tr>
<td>( 2v + 7 = 10 + 7 )</td>
<td>Add 7</td>
</tr>
<tr>
<td>( 2v + 7 = 17 )</td>
<td></td>
</tr>
</tbody>
</table>

You may use the following as a guide in solving equations:

1. Combine any similar terms in either member of the equation.
2. If there are still indicated additions or subtractions, use inverse operations to undo them.
3. If there are any indicated multiplications or divisions in the variable term, use the inverse operations to find the value of the variable.
4. Check by substituting the value in the given equation to see whether it satisfies that equation.

**EXAMPLE 1** Solve \( 7x + 3x - 4 = 12 + 39 \)

**Solution:**

\[
\begin{align*}
7x + 3x - 4 &= 12 + 39 \\
10x - 4 &= 51 \\
10x &= 55 \\
x &= 5.5
\end{align*}
\]

**Check:**

\[
\begin{align*}
7(5.5) + 3(5.5) - 4 &= 12 + 39 \\
38.5 + 16.5 - 4 &= 51 \\
51 &= 51 \checkmark
\end{align*}
\]

\( \therefore \) The solution set is \( \{5.5\} \), Answer.
EXAMPLE 2

Eighteen-carat gold contains three times as much pure gold as copper. How much of each metal is there in 19.6 grains (gr.) of 18-carat gold?

Solution:

1. Choose a symbol and use it in representing each described number.

Let \( p \) = the number of grains of copper.

Then \( 3p \) = the number of grains of pure gold.

2. Form an open sentence by using the facts given in the problem.

<table>
<thead>
<tr>
<th>copper</th>
<th>added to</th>
<th>pure gold</th>
<th>gives</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>+</td>
<td>( 3p )</td>
<td>=</td>
<td>19.6</td>
</tr>
</tbody>
</table>

\[ p + 3p = 19.6 \]
\[ 4p = 19.6 \]
\[ p = 4.9 \text{ gr. copper} \]
\[ 3p = 3(4.9) = 14.7 \text{ gr. pure gold} \]

3. Find the solution set

\[ p = 4.9 \text{ gr. copper} \]
\[ 3p = 14.7 \text{ gr. pure gold} \]

Is there three times as much gold as copper?
Yes, because \( 14.7 = 3(4.9) \)

Is the total weight 19.6 gr.?
Yes, because \( 14.7 + 4.9 = 19.6 \)

: 4.9 gr. copper, 14.7 gr. pure gold, Answer.

EXAMPLE 3

Solve \( n - \frac{3}{5}n + 6 = 27 - 9 \)

Solution:

\[ n - \frac{3}{5}n + 6 = 27 - 9 \]
\[ \frac{2}{5}n + 6 = 18 \]
\[ \frac{2}{5}n + 6 - 6 = 18 - 6 \]
\[ \frac{2}{5}n = 12 \]
\[ 5 \cdot \frac{2}{5}n = 5 \cdot 12 \]
\[ 2n = 60 \]
\[ \frac{2n}{2} = \frac{60}{2} \]
\[ n = 30 \]

(continued on page 88)
Check:  
\[ 30 - \frac{3}{5} \cdot 30 + 6 = 18 \]
\[ 30 - 18 + 6 = 18 \]
\[ 18 = 18 \checkmark \]

\[ \therefore \text{The solution set is } \{30\}, \text{ Answer.} \]

**WRITTEN EXERCISES**

Solve the following equations.

**A**

1. \[ 3x + 5x = 34 - 10 \]
2. \[ 7n - 3n = 40 + 16 \]
3. \[ 28 = 8x + 10x + 10 \]
4. \[ 45 = 7x + 8x + 30 \]
5. \[ x - \frac{1}{2}x - 10 = 0 \]
6. \[ \frac{5}{6}y - y - 5 = 0 \]
7. \[ 13a + 2a + 5a + 85 = 95 \]
8. \[ 8b + 2b + 7b + 173 = 221 \]
9. \[ 5.6x + 2.4x + 176 = 176 \]
10. \[ 8.3y + 2.7y + 154 = 154 \]
11. \[ 57 = 8y + 25 \]
12. \[ 127 = 45t + 37 \]
13. \[ 0 = 17n - 102 \]
14. \[ 0 = 19n - 57 \]
15. \[ 26 = 14 + 4n \]
16. \[ 36 = 17 + 6r \]
17. \[ \frac{3}{4}p + \frac{1}{6} = 2 \]
18. \[ \frac{5}{6}m + \frac{3}{4} = 7 \]
19. \[ \frac{3}{5}z - \frac{3}{5} = 12 \]
20. \[ \frac{3}{4}k - \frac{3}{4} = 24 \]
21. \[ 11.3n + 4.2n - 129.5 = 10 \]
22. \[ 15.8x + .5x - 130.4 = 16.3 \]
23. \[ 23.6x - x = 45.2 \]
24. \[ 4.5a - a = 70 \]
25. \[ \frac{1}{3}z + \frac{1}{3}z = 3.4 \]
26. \[ \frac{k}{2} - \frac{k}{6} = 2.9 \]
27. \[ \frac{17}{9}m - \frac{5}{9}m = 1 \]
28. \[ \frac{3}{2}b + \frac{12}{5}b = 8 \]

**B**

29. \[ 3(x + 5) - 2x = 51 - 25 \]
30. \[ 67 - 48 = 7(y - 3) - 6y \]
31. \[ \frac{3n}{2} + \frac{5n}{3} - \frac{13n}{6} - 2 = \frac{5}{6} \]
32. \[ \frac{3k}{4} - \frac{11k}{20} + \frac{4k}{5} - \frac{3}{4} = \frac{17}{20} \]
33. \[ .3t + .2 = 1.7 \]
34. \[ .7t - .5 = 2.3 \]
35. \[ 11.4 + 2n + 4n = 11.4 \]
36. \[ 4.4 + 7x - x = 10.4 \]
37. \[ 1.5r - r - 2.4 = 17.4 \]
38. \[ 3h - 7.2 - .5h = 32.8 \]
39. \[ 4a + 3 - \frac{1}{2}a = 10 \]
40. \[ 4z + \frac{1}{3} - \frac{3}{2}z = 7 \]

**C**

41. \[ z + 4 + 2z + 5 + 4z = 51 \]
42. \[ 5a + 6 - 2a + 2 - a = 56 \]
43. \[ 11 + 4b - 2b = 73 - 6 \]
44. \[ 25 + 6a + a + 7 = 32 \]
45. \[7n + 5 - 3 - n = 8\]  
46. \[25s - 15 - 5s = 0\]  
47. \[z + 7 + 3 - z = 10\]  
48. \[2t + 5 - t + 1 - t = 6\]  
49. \[y + 5 - y + 1 = 4\]  
50. \[5x + 7 - 3x - 2x = 8\]

**Problems**

Solve these problems with variables, using the four steps identified on page 57. Make a sketch whenever you can do so.

1. The sum of twice a number and 16 is 86. Find the number.
2. The sum of three times a number and 17 is 98. Find the number.
3. If six times a number is diminished by 5, the result is 67. Find the number.
4. Find a number such that 17 less than twice the number is 109.
5. A tennis court 78 feet long is 6 feet longer than twice its width. What is the width?
6. A badminton court 44 feet long is 4 feet longer than twice its width. What is the width?
7. When sirloin steak was priced at 21 cents more per pound than round steak, Mrs. Carney bought 4 pounds of round steak and 2 pounds of sirloin steak, for $4.50. Find the cost per pound of the sirloin.
8. Mary sold 19 more women's sport coats at a special sale than half the number Pat sold. The two sold 157 coats. How many did each sell?

Find the numbers described in Problems 9-14.

9. Three less than 4 times the number is 325.
10. Twenty-one more than six times the number is 177.
11. Twice the number is increased by 17. The result is 27.6.
12. If four times the number is diminished by 13, the result is 31.8.
13. Five more than the sum of 3 times a number and 7 times the number is 385.
14. If the sum of 3 times a number and twice the number is decreased by 15, the result is 165.
15. The sum of two numbers is 78. If three times the smaller is increased by the larger, the result is 124. Find the smaller number.
16. The sum of two numbers is 121. When the larger number is added to 4 times the smaller, the sum is 235. Find the larger number.
17. Bob is twice as old as Emma; Kent is 16 years older than Emma. The sum of their ages is 60 years. Find the age of each.
18. A rectangular house lot is twice as long as it is wide. The sum of its four sides is 222 feet. Find the dimensions of the lot.

19. The length of a rectangular house is three times its width. The distance around the house is 192 feet. Make a sketch, and find the length and width of the house.

20. A farmer’s rectangular hen yard is three times as long as it is wide. It is enclosed by 72 feet of chicken wire. What are its dimensions?

21. When hydrogen and oxygen unite to form water, the weight of the oxygen is eight times that of the hydrogen. How many grams of oxygen are in 126 grams of water?

22. Two boys publish a neighborhood news sheet. The boy owning the press is to receive twice as much of the profits as the other. How much will each receive when a week’s profits are $1.05?

23. A girl bought a jacket and a skirt for $15. The jacket cost 1.5 times as much as the skirt. How much did each cost?

24. A quart of kerosene weighs 0.8 as much as a quart of water. If the combined weight of 1 quart of water and 1 quart of kerosene is 3.6 pounds, find the weight of (a) the water (b) the kerosene.

25. Helen had a hamburger sandwich and a glass of milk which totaled 495 calories. The milk contained half as many calories as the sandwich. Determine the number of calories in the (a) milk (b) sandwich.

26. The total weight of a space capsule put into orbit was 840 pounds. If the regulating devices in the capsule weighed twice as much as the container, while the recording equipment weighed half as much as the container, find the weight of each of the three parts.

27. Information in numerical form is stored in the memory units of an electronic computer. The storage capacity of Computer B is twice that of Computer A, while Computer C has a capacity four times as great as that of Computer B. If the total storage capacity of the three machines is 22,000 words, find the capacity of each computer.

28. A certain type of concrete contains twice as much sand as cement, and two and a half times as much gravel as sand. How many kilograms of cement are needed to make 300 kilograms of the dry concrete mixture?

29. The longest side of a triangle is twice the shortest side. The third side of the triangle is 1 inch shorter than the longest side. If the perimeter of the triangle is 149 inches, how long is each side?

30. In a day, Machine A produces one and a half times as many cartons as Machine B. Machine C produces 100 more cartons than Machine A. If the total production is 5020, how many cartons does each produce?

31. Ted has 4 times as many dimes as nickels and half as many pennies as dimes. If he has $4.70, how many coins of each kind has he?
32. In a den, there are three lamps of equal size, a radio which uses one-third the number of watts used by a lamp, and a heater which uses fifteen times as many watts as a lamp. When all are in use, the total electrical power is 1.1 kilowatts. How many watts does the radio use? (1 kilowatt = 1000 watts)

3–8 Equations Having the Variable in Both Members

An equation such as $3t = 2t + 16$ differs from the open sentences you have met, since the variable appears in both members of the equation. Fortunately, transformation by addition or subtraction allows you to add to or to subtract from each member any product of a number and the variable, without changing the solution set.

**EXAMPLE 1.** Solve $3t = 2t + 16$

*Solution:* 

$$3t = 2t + 16$$

$$3t - 2t = 2t + 16 - 2t$$

$$t = 16$$

*Check:* 

$$3(16) \text{?} 2(16) + 16$$

$$48 \text{?} 32 + 16$$

$$48 = 48 \checkmark : \text{ The solution set is } \{16\}, \text{ Answer.}$$

**EXAMPLE 2.** Solve $x + 4x - 8 = 6 + 2x + 1$

*Solution:* 

$$x + 4x - 8 = 6 + 2x + 1$$

$$5x - 8 = 7 + 2x$$

$$5x - 8 - 2x = 7 + 2x - 2x$$

$$3x - 8 = 7$$

$$3x - 8 + 8 = 7 + 8$$

$$3x = 15$$

$$\frac{3x}{3} = \frac{15}{3}$$

$$x = 5$$

*Check:* 

$$5 + 4 \cdot 5 - 8 \text{?} 6 + 2 \cdot 5 + 1$$

$$5 + 20 - 8 \text{?} 6 + 10 + 1$$

$$17 = 17 \checkmark$$

:: The solution set is $\{5\}$, Answer.
EXAMPLE 3. Solve $1 + 3(2s + 4) = 15 + 6s$

Solution:

\[
\begin{align*}
1 + 3(2s + 4) &= 15 + 6s \\
1 + 3 \cdot 2s + 3 \cdot 4 &= 15 + 6s \\
1 + 6s + 12 &= 15 + 6s \\
6s + 13 &= 15 + 6s - 6s \\
13 &= 15
\end{align*}
\]

Check: For every replacement of the variable $s$, the equation will be converted into a false statement; thus, the equation has no root.

\[\therefore \text{ The solution set is } \emptyset, \text{ Answer.}\]

EXAMPLE 4. Solve $9d + 3(5 + 2d) = 15(d + 1)$

Solution:

\[
\begin{align*}
9d + 3(5 + 2d) &= 15(d + 1) \\
9d + 3 \cdot 5 + 3 \cdot 2d &= 15 \cdot d + 15 \cdot 1 \\
9d + 15 + 6d &= 15d + 15 \\
15d + 15 &= 15d + 15
\end{align*}
\]

\[\therefore \text{ The solution set is } \{ \text{all numbers} \}, \text{ Answer.}\]

Any replacement for $d$ converts this equation into a true statement. Such an equation is called an identity.

EXAMPLE 5. Len and Vito together collected 98 pounds of waste paper in the clean-up drive. Vito announced, “Four pounds more than the amount I collected is exactly twice the amount Len collected.” How much did each collect?

Solution:

1. Let $p = \text{number of pounds Vito collected}$

Then $98 - p = \text{number of pounds Len collected}$

2. \[
\begin{align*}
4 \text{ more than Vito’s share} &= 2 \text{ twice Len’s share} \\
4 + p &= 2(98 - p)
\end{align*}
\]
The "check" is left for you. Show that Vito collected 64 lb., and Len, 34 lb.

**ORAL EXERCISES**

Solve each equation, first telling what term must be added to or subtracted from each member of the equation.

1. $2x = 3 + x$
2. $3a = 8 + 2a$
3. $c = 9 - 2c$
4. $4m = 10 - m$
5. $3y = 8 - y$
6. $5r = 2r + 30$
7. $2x + 5 = 7x$
8. $3z = 2z + 1$
9. $1.5x = .5x + 4$
10. $2.9t = 12 - .1t$
11. $6x - 9 = 5x$
12. $4v - 10 = 3v$
13. $7s = 7 - 7s$
14. $18 = \frac{3}{4}r - \frac{1}{2}r$
15. $x = -5 + 3x$
16. $11b = 6 + 5b$
17. $4 + 2a = 14$
18. $5z = 2z + 9$

**WRITTEN EXERCISES**

Solve each equation.

A

1. $7v = 45 + 2v$
2. $9u = 6u + 39$
3. $6l = 18 - 3l$
4. $5b = 28 - 2b$
5. $11x = 8 + 5x$
6. $4y = 8 + 2y$
7. $12 - 3r = 3r$
8. $7h - 35 = 0$
9. $9x - 24 = 3x$
10. $5 + 2b = 7b$
11. $8a = 5a + 18$
12. $4x + 18 = 10x$
13. $7y - 9 = 2y$
14. $b + 28 = 6b$
15. $5x - 6 = 3x$
16. $12 - 3h = 9$
17. $5x - 5 = 13$
18. $12 - y = 5y$

B

19. $4z + 2 = 2z + 8$
20. $5w + 2 = w + 7$
21. $7c - 7 = 15 - c$
22. $12a - 3 = 4 - 2a$
23. $4r + 2 = 2r + 2$
24. $7f - 1 = 29 + f$
25. $8u + 2 = 13 - 3u$
26. $16 + 4y = 10y - 20$
27. $19r + 4 = 19 + 14r$
28. $5x + 1 = 4x + 2$
29. $24x - 24 = 12 + 2x$
30. $12n + 8 = 18 - 6n$
31. $5y + 10 = 6 + 6y$
32. $10w + 6 = 504 + 8w$

33. $6 + 4(2 - t) = 3t$
34. $7m + 5(3 - m) = 19$
35. $12 \left( \frac{x}{3} - \frac{1}{2} \right) = x + 21$
36. $\frac{3}{5}(a + 10) = a$
37. $3t + 4 = 3(t + 2)$
38. $5 - b = b + 5$

C
39. $10 + 18x - 2 = 2x + 12 + 4x$
40. $9y + 3 - 2y = 12 - 6y + 4$
41. $4(b + 1) + 9 = 2(3b - 4) + b$
42. $5(2y + 3) - 4y = 3(2 + y) + 39$
43. $\frac{2}{3}(x + 1) + \frac{3}{7}(x + 1) = 14$
44. $2(t + 4) - 3 = \frac{1}{2}(10 + 4t)$

**PROBLEMS**

Use variables to solve these problems. Make a sketch when possible.

**A**

1. Paul said: “I am thinking of a number which equals its double decreased by 1. What is the number?”
2. Roger asked: “Can you tell me the number I am thinking of? When I multiply it by 2 and then add 3, I get the same answer as when I multiply it by 3 and then subtract 2.”
3. Dividing a certain number by 2 yields the same result as subtracting 15 from 3 times the number. Find the number.
4. Tom has a set of blocks, each having the same weight. He found that if he put three of the blocks in one pan of a beam balance, and put one block together with a 7-pound weight in the other pan, then the pans would balance each other. How much did each block weigh?
5. Harry earns three times as much per week as does Tom, while Dick earns $80 a week more than Tom. If Dick and Harry have the same salary, how much does each of the three men earn?
6. The sum of two numbers is 46. Five times the smaller number is 6 more than twice the larger. Find the numbers.
7. Bill is twice as old as Mary. If he is also exactly 10 years older than Mary was last year, how old are Bill and Mary?
8. Divide 73 into two parts, such that twice the larger number is 4 less than three times the smaller.
9. Sam challenged: “Tell me my number. When I subtract 3 from it and then multiply the result by 2, I get the same result as when I divide my number by 2 and then add 18 to the quotient.”

10. A deluxe ball-point pen costs $1 more than the standard model. Peter bought 3 standard pens and 5 deluxe pens. His bill totaled $6.20. What is the cost of each model of the pen?

11. The length of one rectangle is 2 feet more than the length of a second rectangle. The width of the first rectangle is 3 feet; the width of the second is 7 feet. If the total area of the two rectangles is 121 square feet, determine the length of each rectangle.

12. The sum of the length and width of a rectangle is 42 inches. Twice the length is 1 inch less than 3 times the width. Find the dimensions of the rectangle.

13. A purchase of 50 stamps, some costing 7 cents each and the rest 4 cents each, cost $2.15. How many of each kind were bought?

14. The base of a triangle has the same length as a side of a square. The second side of the triangle is 2 inches longer than the base, and the third side is 6 inches longer than the base. If the perimeter of the triangle equals that of the square, find the longest side of the triangle.

15. A company has 4 large buses and 5 small ones. Each large bus has 12 seats more than a small bus. The total seating capacity of all the buses is 336. How many seats are in a large bus?

16. Ten less than the sum of 4 times a number and 2 times the number is 260 more than 4 times the number. Find the number.

17. Twice the sum of half a number and 3 times the number is 27 more than half of 5 times the number. Find the number.

18. The temperature in Omaha was twice the temperature in Juneau. If the temperature in Juneau were increased by 15° and the temperature in Omaha decreased by 15°, the temperatures would have been the same. What were the temperatures in those cities?

19. A baseball bat costs half a dollar more than a ball. When the team bought 2 bats and 5 balls, it paid $11.50. Find the cost of a bat.

20. Nine pounds of potatoes cost the same as 6 pounds of apples. At the same time 1 pound of potatoes costs twice as much as a pound of onions, while 1 pound of apples costs 8 cents more than a pound of onions. Determine the cost of one pound of each commodity.

21. “I am 3 times as old as Dave is,” said Frank to Kay. “On the other hand, I am 15 years older than Joe, while Dave is 1 year younger than Joe. How old are Joe, Dave, and I?”
Machinists and Mathematics

Machinists are skilled mechanics who cut, file, and grind metals to shape them into specified sizes and forms. Working closely with engineers and scientists, machinists must be able to understand blueprints and mathematical equations in order to make tools and parts to exact specifications.

The example illustrated is typical of the machinists' problems involving the use of mathematics. The photograph shows a machinist using a boring mill to rough cut a collar for a steel shaft. In such an operation the rotation speed of the lathe, which turns the cylinder to be cut against the stationary drill, must be carefully set. Cutting too rapidly would burn the steel, while cutting too slowly would be inefficient and might produce a jagged edge.

To determine the proper lathe speed, the machinist performs the calculation illustrated on the work pad. The number of revolutions per minute \( R \) is given by the equation

\[
C = \frac{\pi RD}{12}
\]

In this example, the machinist is using a drill made of tool steel to cut a machine-steel cylinder. The optimum cutting speed \( C \) for this combination of metals is 50–70 ft. per minute. The diameter \( D \) of the drill is one inch. By first substituting 50 for \( C \), the machinist finds that the lathe speed should be at least 191 revolutions per minute (r. p. m.). Then, substituting 70 for \( C \), he determines that the speed of the lathe should not exceed 267 r. p. m. As shown, the speed should be set somewhere between 191 and 267.
Chapter Summary

Inventory of Structure and Method

1. The use of the equality sign is governed by these assumptions:
   The reflexive property: Any number is equal to itself.
   The symmetric property: The members of an equation may be inter¬
   changed.
   The transitive property: If two numerical expressions are both equal to
   a third expression, the two are equal to each other.
   The axiom of closure states that the sum and product of every pair of
   arithmetic numbers are unique arithmetic numbers.

2. Arithmetic numbers have properties involving operations. The commu¬
   tative property for addition: For each \( a \) and each \( b \), \( a + b = b + a \).
   The commutative property for multiplication: For each \( a \) and each \( b \),
   \( ab = ba \). The commutative properties for addition and multiplication
   show that you get the same sum or same product of two numbers, no
   matter what order you use.
   The associative property for addition: \( a + (b + c) = (a + b) + c \) for each
   \( a \), \( b \), and \( c \). The associative property for multiplication: For every \( a \), \( b \),
   and \( c \), \( a(bc) = (ab)c \). The associative properties for addition and multi¬
   plication show that you get the same sum or same product of three
   numbers, no matter which adjacent numbers are grouped.
   The distributive property for multiplication with respect to addition or
   subtraction: For each \( a \), each \( b \), and each \( c \), \( a(b + c) = ab + ac \) and
   \( a(b - c) = ab - ac \). These properties are used in simplifying expres¬
   sions, especially in combining similar terms.
   The numbers 1 and 0 have special properties. The multiplicative property
   of 1 (multiplicative identity): For every \( a \), \( 1 \cdot a = a \cdot 1 = a \). The additive
   property of 0 (additive identity): For every \( a \), \( 0 + a = a + 0 = a \). The
   multiplicative property of 0: For every \( a \), \( 0 \cdot a = a \cdot 0 = 0 \). As a result
   of the multiplicative property of zero, you may not divide by zero.
   The substitution principle states: If two numerical expressions are equal,
   one may be substituted for the other.

3. Properties of equality involving operations can be derived from previous
   axioms and properties. In the following four properties, \( a \), \( b \), and \( c \)
   represent any three numbers.
   Addition property of equality: if \( a = b \), then \( a + c = b + c \). Subtraction
   property of equality: if \( a = b \), then \( a - c = b - c \), provided
   \( a - c \) represents an element in the set. Multiplicative property of equality:
   if \( a = b \), then \( ac = bc \). Division property of equality: if \( a = b \), then
\[
\frac{a}{c} = \frac{b}{c}, \text{ if } c \text{ is not zero. Thus, if equal numbers are added to equal numbers, or subtracted from equal numbers, or multiplied by equal numbers, or divided by equal nonzero numbers, the results are equal.}
\]

4. You solve an equation by transforming it into an **equivalent equation**, repeating this process until you derive an equation which shows the solution set. You may transform an equation by **addition**, by **subtraction**, by **multiplication** by a nonzero number, and by **division** by a nonzero number. Each time you transform an equation, you undo an indicated operation by using the **inverse operation**. To **check** the tentative value, you **substitute** it for the variable in the **original equation** and see whether it satisfies the equation by evaluating each member separately. If the result is the same for each member, the equation is true for the value substituted, and this value is a root of the equation, or an element in its solution set. Suggested steps in solving an equation are as follows: (a) **Simplify** by carrying out indicated multiplications where possible; combine similar terms in each member. (b) **Undo** any indicated addition or subtraction by using the inverse operation. (c) **Undo** any indicated multiplication or division by using the inverse operation. (d) **Check** the tentative answer in the original equation.

**Vocabulary and Spelling**

- assumption (p. 69)
- axiom (p. 69)
- postulate (p. 69)
- reflexive property (p. 69)
- symmetric property (p. 69)
- transitive property (p. 70)
- closure (p. 71)
- substitution principle (p. 71)
- commutative properties (p. 73)
- associative properties (p. 73)
- distributive property (p. 76)
- similar terms (p. 77)
- unlike terms (p. 77)
- multiplicative properties (p. 77)
- identity element (p. 77)
- additive property (p. 77)
- proof (p. 80)
- equivalent equations (p. 81)
- transformation (p. 81)
- check (p. 81)
- inverse operation (p. 81)
- identity (p. 92)

**Chapter Test**

3-1 Which property of equality does each of these examples illustrate?

1. \[\frac{217}{5} = \frac{432}{5}\]  
2. If \(x = y\), then \(y = x\).
3–2 State whether the given set is closed under the indicated operation.
3. \( \{1, 2\} \), multiplication
4. \( \{1, 3, 5, 7, 9, \ldots\} \), addition of any three elements

3–3 In the chains of equalities of Exercises 5–7, give a property of numbers that justifies each lettered step. For each \( n \),
5. \( 4 + (n + 7) = 4 + (7 + n) \) (a)
   \[ = (4 + 7) + n \] (b)
   \[ = 11 + n \]
6. \( 4 \times (n \times 7) = 4 \times (7 \times n) \) (a)
   \[ = (4 \times 7) \times n \] (b)
   \[ = 28n \]

3–4 For each \( r \),
7. \( 5(r + 1) + 3(r + 1) = 5 \times r + 5 \times 1 \\
   + 3 \times r + 3 \times 1 \) (a)
   \[ = 5 \times r + 5 + 3 \times r + 3 \] (b)
   \[ = 5 \times r + 3 \times r + 5 + 3 \] (c)
   \[ = (5 + 3)r + 5 + 3 \] (d)
   \[ = 8r + 8 \] (e)
   \[ = 8(r + 1) \]

8. Simplify and combine like terms: \( 3x + 4(x + 2) - 5x - 8 \).
9. Find the value of \( \frac{3}{4}(39) + 9\frac{1}{4}(39) \).

3–5 Solve: 10. \( s - 73 = 91 \) 11. \( 42 = t + 18 \)

3–6 Solve: 12. \( 7y = 28 \) 13. \( \frac{3}{2}y = 48 \)

3–7 14. State whether the indicated value of the variable is a root of the equation. Show work. \( \frac{2}{3}m - 4 = 3 - .3m; m = 10 \)
15. In a Junior Achievement enterprise, Ben invests three times as much as Amy. At the end of the year, the total net profit is $46.28. Find Ben’s proportionate share of the profit.
16. Solve: \( 3d + 2(6d - 5) = 5 \)
17. A Bonrite pen and pencil set costs $2.78, the pen costing $0.80 more than the pencil. Find the cost of each.

3–8 18. Solve: \( 3(2h + 7) - 5 = 2(5h - 4) + 4h \)
19. The length of a rectangle is 3 inches less than the length of a smaller rectangle. The larger rectangle is 9 inches wide; the smaller is 4 inches wide. If the area of the larger rectangle is 48 square inches more than the area of the smaller, find the length of the larger rectangle.

Chapter Review

3-1 Axioms of Equality

1. \( a + b = a + b \) illustrates the ___ property of equality.
2. If \( b = d \), then \( d = b \) illustrates the ___ property.
3. \( 23 - 15 = 8 \), \( 2 \cdot 4 = 8 \), \( 23 - 15 = 2 \cdot 4 \) illustrates the ___ property.

3-2 The Closure Properties

4. The sum of two numbers of arithmetic will be a ___.
5. A set \( R \) is closed under multiplication if the product of any two of its elements is an ___ of \( R \).
6. The set of numbers of arithmetic is closed under ___ and ___.

State whether each given set is closed under the indicated operation.

7. \{3, 6, 9, 12, \ldots\}, division
8. \{6, 4, 2, 0\}, subtraction

3-3 Commutative and Associative Properties of Arithmetic Numbers

9. You know that \( 4n + 1 = 1 + 4n \) because of the ___ property for ___.
10. For any numbers \( m \) and \( n \), \( mn = \ldots \) because multiplication is ___.
11. You know that \( 8 \div 4 \neq 4 \div 8 \) because division does not have the ___ property.
12. \( (8 + 5) + 4 = 8 + 9 \) because addition is ___.
13. \( (157 \times 4) \times 25 = 157 \times 100 \) because multiplication is ___.
14. \( 16 - (8 - 2) \neq (16 - 8) - 2 \) because subtraction is not ___.

3-4 The Distributive Property; Special Properties of 1 and 0

15. \( 5(x - y) = 5x - 5y \) because of the ___ property for multiplication with respect to ___.
16. When you combine $4w + w$ to get $5w$, you are using the ___ property.

17. The expression $4 \cdot 1 \cdot 1 = 4$ because of the ___ property of 1.

18. $0 \cdot 5 = 0$ because of the ___ property of ___.

19. When the product of several factors is 0, at least one of the factors must be ___.

20. Zero, or any expression whose value is ___, may never be used as a ___.

In the following chain of equality, give a property of numbers that justifies each lettered step. For each $n$,

21. $6n - 6n + 3 \times (4 \times \frac{1}{3}) = n(6 - 6) + 3 \times (4 \times \frac{1}{3})$
   
   (a) $= n \cdot 0 + 3 \times (\frac{1}{3} \times 4)$
   
   (b) $= 0 + 3 \times (\frac{1}{3} \times 4)$
   
   (c) $= 0 + (3 \times \frac{1}{3}) \times 4$
   
   (d) $= 0 + 1 \times 4$
   
   (e) $= 0 + 4$
   
   (f) $= 0 + 4$
   
   (g) $= 4$

Simplify by combining similar terms.

22. $7r + 3(2r + 1) - 13r - 2$

23. $2k + 3[5 + 2(k - 1)]$

3-5 Addition and Subtraction Properties of Equality Pages 80-83

24. The properties of equality involving operations are not axioms because they can be ___.

25. Equations having the same solution set are ___ equations.

26. The expression on one side of an equation is called a ___ of the equation.

27. Solve $0 = h - 6$.

28. Solve $n + 8 = 8$.

29. Check a root by ___ it for the variable in the given equation.

30. In checking, evaluate each ___ of the equation separately.

31. Check an answer to a problem by showing that it satisfies the ___ of the problem.

3-6 Division and Multiplication Properties of Equality Pages 83-85

32. To find $x$, ___ each member of the equation $5x = 3$ by ___.

33. The inverse of subtraction is ___; the inverse of ___ is division.

34. You may not multiply each member of an equation by 0 because the resulting equation will not be ___ to the original equation.
35. Solve $12k = 3$.  
36. Solve $18 = \frac{2}{3}m$.

3-7 Combining Terms and Using Transformation Principles
Pages 86–91

37. Solve $9t - 3t = 9 + 3$.  
38. Solve $1 + y = \frac{4}{3}$.

39. The sum of a number $n$ and 6 times that number may be represented by $n + \underline{\ ?}$, or $\underline{\ ?}$.

Solve.

40. Three times a number decreased by half the number gives 10. Find the original number.

41. Undo indicated $\underline{\ ?}$ and $\underline{\ ?}$ before considering multiplications and divisions.

42. $27 = 7.2a - 5 + .8a$  
43. $38 + 7b - 26 + 5b = 16\frac{1}{3}$

44. A man is 2 years younger than three times his daughter's age. Their ages total 50 years. Find the age of each.

3-8 Equations Having the Variable in Both Members
Pages 91–95

45. When an equation has the variable in each member, transform it into an equation containing the variable in only $\underline{\ ?}$ member.

Solve.

46. $2n + 7 = 3(2n - 1) + 6$  
47. $\frac{3}{5}(9r - 4) = r + 3(r + \frac{3}{5})$

48. A rectangle is 6 inches wide. If a rectangular strip 4 inches long were cut from the end, the area of the remaining rectangle would be $\frac{3}{4}$ of the original area. Find the dimensions of the original rectangle.

49. At a County Fair, 200 ice cream cones were sold in one day, some at 15 cents each, the rest at 10 cents. If the proceeds from the sale of cones were $23.75, how many of each were sold?

Cumulative Review: Chapters 1–3

State whether each of the following sets is (a) finite, or (b) infinite.

1. \{points on a circumference\}  
2. \{all the trees in the world\}  
3. \{odd numbers between 3 and 5\}  
If the set below is given by a roster, specify it by a rule; if given by a rule, specify the set by a roster.

5. \{the states in the U.S.A. whose names begin with the letter \(K\)\}

6. \{5, 11, 17, 23, \ldots 47\}

7. \{1, 3, 9, 27, 81\}

Draw the graph of each set described below.

8. \{the numbers between 2 and 4.5, inclusive\}

9. \{the numbers between 1 and \(2\frac{1}{2}\)\}

10. \{the numbers greater than or equal to 3\}

11. Simplify \(8 \div 8 \times 2\).

12. Simplify \(3[9 - 2(1 + 1.2)]\).

Given the sets: \(A = \{0, 2, 6, 10, 20\}\) and \(B = \{1, 2, 3, 9, 12\}\), find the set specified as follows:

13. \{the elements in \(A\) or \(B\), or in both \(A\) and \(B\)\}

14. The subset of \(B\) containing all multiples of 3

15. The set of those elements which are in both \(A\) and \(B\)

In each case, give the property which makes the conclusion true.

16. Given: \(b = m; m = 6\)  
   Conclusion: \(b = 6\)

18. Given: \(3 + x = 5x - 1\)  
   Conclusion: \(5x - 1 = 3 + x\)

17. \(5(s - 2) = 5s - 10\)

19. \(5(t + 2) = (t + 2)5\)

In the following chain of equalities, give the property that justifies each lettered step. For each \(n\),

20. \(\{2(5n + 1)\} - 10n = \{10n + 2\} - 10n\)
    \(= \{2 + 10n\} - 10n\)
    \(= 2 + \{10n - 10n\}\)
    \(= 2 + n(10 - 10)\)
    \(= 2 + n \cdot 0\)
    \(= 2 + 0\)
    \(= 2\)

Simplify and combine like terms.

21. \(7n + 3[4n + 5(2 - n) - 3]\)

22. \(\frac{1}{6}(d - 4) - \frac{3}{6}(d - 4)\)

Graph the solution set of each sentence.

23. \(2x - 1 \neq 5\)

24. \(3r \geq 12\)

State whether each of the given sets is closed under the indicated operation.

25. \(\{0, 2, 4, 6, \ldots\}\), multiplication by 3

26. \(\{4, 3, 2, 1\}\), subtraction
If \( a = 1, b = 3\frac{1}{2}, c = 4, \) and \( d = 0, \) find the value of each expression.

27. \( 4(b + c) - 3d(a + b) \)
28. \( \frac{2b + c}{2b - c} \)
29. \( \frac{3a^4 + 3c^2 - 2}{4b^2} \)
30. \( 16(c - b)^3 \)

Express in algebraic symbols.

31. Twice a number \( n, \) decreased by 3
32. One-fifth of the sum of \( a \) and \( b \)
33. The larger of two numbers when it is 5 more than the smaller, \( s \)
34. The larger of two numbers when their sum is 50 and the smaller is \( x \)
35. The square of the sum of two numbers (\( R \) and \( r \))
36. Twice the sum of the squares of two numbers (\( R \) and \( r \))

If the replacement set for \( n \) is \( U = \{0, 1, 2, 3\}, \) find the subset of \( U \) whose elements make each of the following sentences true.

37. \( 2n - 1 = 5 \)
38. \( 2(n - 1) = 5 \)
39. \( 3n > 0 \)
40. \( 4n + 1 \geq 6 \)
41. \( n(5n) = n \cdot 5 \cdot n \cdot n \)
42. \( n(n - 1) = n \times n - n \)

Write an algebraic expression for each of the following.

43. \( a. \) A whole number that is 5 more than a given number, \( t \)
   \( b. \) Give the replacement set for \( t. \)
44. The third side of a triangle whose perimeter is 27, and whose second side is twice the first, \( a \)

Solve each equation.

45. \( 8m - 13 = 67 + 3m \)
46. \( 7s - 4 - s = 31 + 2s - 5 \)
47. \( \frac{3}{4}(5y - 4) = 13 + \frac{1}{4}(6y + 8) \)
48. \( \frac{2}{3}w + 1 + \frac{5}{6}w = \frac{13}{3}w - 1 \)
49. \( 3(2t + 5) - 7 = 9t - 4 \)
50. \( 12 + 3(2k - 1) = 2(3k + 1) \)

Solve each problem.

51. Find a number such that the difference of twice the number and two-thirds of the number is 68.
52. The sum of two numbers is 20. Four times the larger is 1 less than five times the smaller. Find the numbers.
53. For Mother's Day, Tony bought three kerchiefs. For Father's Day, he bought two ties. If a tie cost 50 cents more than a kerchief and Tony spent a total of $5.95, how much did a tie cost?
The Algebra of Logic and Sets

Consider the following compound sentences:

\[
3 < x < 5 \\
x \leq 4
\]

Each of these may be broken into two simple sentences:

\[
\begin{align*}
3 < x < 5 & \text{ means } 3 < x \text{ and } x < 5 \\
x \leq 4 & \text{ means } x < 4 \text{ or } x = 4
\end{align*}
\]

Do you see the essential difference in the nature of the limitations implied by these two different compound sentences? The one containing and is called a conjunction; for it to be true, \( x \) must satisfy both simple statements. The one containing or is called a disjunction; it is true if \( x \) satisfies at least one of the simple statements.

If you limit the domain of the variable to \{0, 1, 2, 3, 4, 5\} and consider the solution sets (truth sets) of each simple sentence and that of the compound sentence, you will discover a relationship between conjunctions of sentences and intersections of sets, and between disjunctions of sentences and unions of sets.

<table>
<thead>
<tr>
<th>Sentence</th>
<th>Truth Set</th>
<th>Sentence</th>
<th>Truth Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3 &lt; x)</td>
<td>{4, 5} = A</td>
<td>(x &lt; 4)</td>
<td>{0, 1, 2, 3} = C</td>
</tr>
<tr>
<td>(x &lt; 5)</td>
<td>{0, 1, 2, 3, 4} = B</td>
<td>(x = 4)</td>
<td>{4} = D</td>
</tr>
<tr>
<td>(3 &lt; x &lt; 5)</td>
<td>{4} = (A \cap B)</td>
<td>(x \leq 4)</td>
<td>{0, 1, 2, 3, 4} = (C \cup D)</td>
</tr>
</tbody>
</table>

A special notation is sometimes used to write compound sentences. The sign of conjunction is \(\land\) (a capital A without the bar) and is read and. The sign of disjunction is \(\lor\) and is read or. The above examples would be written thus:

\[
\begin{align*}
3 < x \land x < 5 \\
x < 4 \lor x = 4
\end{align*}
\]

The symbol for conjunction resembles that for intersection, and the truth set of a conjunction of simple sentences is the intersection of their truth sets. Further, the symbol for disjunction resembles that for union, and the truth set of a disjunction of simple sentences is the union of their truth sets.

Can you see that the truth of a compound sentence is determined by the truth of its component simple sentences? Relationships between them can
be displayed in the form of a truth table. If you will consider $p$ and $q$ as standing for simple sentences, the truth table for the disjunction and the conjunction of two simple sentences takes this form.

If $p$ represents $6 < 7$ and $q$ represents $7 < 8$, both $p$ and $q$ are true, their conjunction is true, and their disjunction is true. This fits the first line of the truth table.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
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<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

If $p$ represents $7 = 5 + 2$ and $q$ represents $6 = 3(3)$, $p$ is true, $q$ is false, their conjunction is false, and their disjunction is true. This fits the second line of the truth table. Can you illustrate the third and fourth lines?

**Questions**

Solve these compound sentences, and graph each truth set.

**SAMPLE 1.** $(x < 5) \lor (x > 7)$

Solution: \{all numbers less than 5\} $\cup$ \{all numbers greater than 7\} = \{all numbers except those between 5 and 7, inclusive\}

1. $(x = 1) \lor (x > 1)$
2. $(x = 1) \land (x > 1)$
3. $(x > 2) \lor (x < 4)$
4. $(x > 2) \land (x < 4)$
5. $(2x > 0) \lor (x \in \{\text{whole no.}\})$
6. $(2x > 0) \land (x \in \{\text{whole no.}\})$

Let $a$, $b$, and $c$ represent the sentences: $x$ is a member of set $A$; $x$ is a member of set $B$; and $x$ is a member of set $C$, respectively. By drawing Venn diagrams verify the properties of conjunction and disjunction for these sentences.

**SAMPLE 2.** Disjunction is distributive over conjunction; that is, $c \lor (b \land a)$ has the same truth set as $(c \lor b) \land (c \lor a)$.

Solution: Rewrite the expressions in terms of their truth sets:

$C \cup (B \cap A) \cong (C \cup B) \cap (C \cup A)$

Mark $C$: \[ \quad \quad B \cap A: \quad \quad C \cup B: \quad \quad C \cup A: \]
These two regions are the same.

7. Conjunction is distributive over disjunction; that is, $c \land (b \lor a)$ has the same truth set as $(c \land b) \lor (c \land a)$.

8. Disjunction is associative; that is, $c \lor (b \lor a)$ has the same truth set as $(c \lor b) \lor a$.

9. Conjunction is associative; that is, $c \land (b \land a)$ has the same truth set as $(c \land b) \land a$.

10. The truth set of $(a \lor b) \land (a \land b)$ is the same as that of $(a \land b)$.

---

**Some Musical Mathematics**

The piano keyboard (figure below) consists of eighty-eight keys. For every seven white keys, there are five black keys.

Vibrating strings produce the tones of a piano. The A next above middle C (which is the key used to tune a piano) vibrates 440 times per second; the A one octave higher (to the right) vibrates 880 times per second; and the A one octave lower vibrates 220 times per second. Notice the relation between
these numbers: $2 \times 220 = 440$ and $2 \times 440 = 880$. By making use of it, you can calculate the number of vibrations per second of any A on the keyboard.

The next note to A is the black key for A-sharp (A♯) or B-flat (Bb). It vibrates 1.06 times as rapidly as A. And the next white key, B, vibrates 1.06 times as rapidly as A♯. A similar relation exists for each key if the piano is so tuned; for example, F vibrates 1.06 times as rapidly as E. Why would this be the correct number to use? How can you now calculate the number of vibrations of any key on the piano?

You know that some combinations of notes are harmonious and that some are discordant. Harmony and discord are related to the vibrations of the notes that are sounded together. To find some pleasing combinations without using a musical instrument, first draw a circle and divide its circumference into twelve parts. Each division represents one of the twelve notes from C to B. Label the divisions to correspond to these notes. (See figure below.)

Place your pencil on point C of your “tone circle.” Go three divisions counterclockwise to A. Then go four more divisions counterclockwise to F. It is now five divisions back to C. The notes C, A, F, C make a harmonious chord when you strike them together. So do any other notes determined by going counterclockwise according to the pattern 3:4:5. Find some of these chords by using your tone circle, and then try them on a piano. Chords found by the scheme just used are major chords.

Harmonious chords may also be identified by proceeding in a clockwise direction according to the same 3:4:5 pattern. Thus C, Eb, G, C is a pleasing combination of tones, as you will find if you sound it on a piano. It is a minor chord. So is any other chord found in the same way.

Identify some minor chords; then listen to their harmony. Next sound some major chords. Do minor chords in general have a different effect on you from that of major chords? You can account for the difference if you have calculated how many times each note vibrates. Compare the numbers of vibrations of the notes in a chord. You will find that the relation of these numbers to one another is the same in every major chord you identified from your tone circle. It is the same in every minor chord you identified. But it is different in major chords from what it is in minor chords.
Hypatia was beautiful. She was also a brilliant mathematician. Her lectures at the University were attended by the most learned men of her era.

Until Hypatia's time, few Greek scholars were interested in algebra. Most of them preferred the study of geometry. But one, named Diophantus, had written on algebra. Hypatia studied his work and wrote discussions and explanations of it. Perhaps it was she who told the only story known of Diophantus' life; it dates back to her time, and clearly it is a tale told by a mathematician:

God granted him childhood for a sixth part of his life, and youth for a twelfth part. He lit him the light of wedlock after a seventh part more, and five years after his marriage he granted him a son. Alas, late-born child! After attaining the measure of half his father's life, cruel Fate overtook him [that is, he died], thus leaving to Diophantus during the last four years of his life only such consolation as mathematics can offer.

Hypatia lived in troubled times — 400 A.D. Her home was in the city of Alexandria, a part of the crumbling Roman Empire. Alexandria was inhabited by Romans, Greeks, Egyptians; by Christians, Jews, pagans; by slaves, freedmen, and freeborn. And great numbers in each group were unlearned and ignorant, easily stirred to mob violence.

One day trouble started; how, no one knows. There were riots in the streets. Hypatia was probably unaware of danger when she ventured out that day. Why should anyone hate a lecturer on algebra?

The rioting mob knew nothing of algebra and cared less. But they knew that Hypatia was different from most persons — a woman who taught men at the University. When she appeared, the rioters had found a victim for their fury. They tore her from her chariot; they slashed her to death with the sharpened oyster shells that were their weapons. So, on a day in March, 415, the first woman mathematician of history was murdered by a mob.

*The “first lady” of algebra.*
The Negative Numbers

Can you write a headline for this picture? As you analyze the picture, you may observe that it implies opposite directions. A “dip” in the stock market is the opposite of a “rise.” A temperature “below zero” is the opposite of a temperature “above zero.” A “deficit” is the opposite of a “surplus.” “Below the sea” is the opposite of “above the sea” (graphs at left). A “loss” is the opposite of a “gain.” “B.C.” is the opposite of “A.D.” Can you think of other opposites to add to this list? Mathematically, all of these concepts are treated by a single device, the subject of this chapter.

EXTENDING THE NUMBER LINE

4–1 Directed Numbers

The number line you have studied so far extends only to the right of the point labeled 0, as in Figure 4–1.

Have you ever seen a thermometer which read “below zero” or played a game in which you were “in the hole” or heard a businessman speak of being “in the red”? These examples suggest uses for a two-way scale, such as that in Figure 4–2.

Do you notice the small + and − signs written above and to the left of the numerals? Since no two points are to have the same label, you cannot write merely the numerals from 0 on, to the left, without some method of distinguishing them from those to the right. You can, no doubt, devise other schemes, but this is a convenient notation.
. The + and — signs are the symbols used to indicate addition and subtraction, but here they indicate the direction of the point from the 0-point, not an operation to be performed. In this use they are called signs of direction. Actually, you could have left the right side of the scale as it was. Attaching the small + signs to the old numerals serves to emphasize their position, but they still represent the familiar numbers of arithmetic. Each numeral with the small — sign, however, represents a new number — a negative number. You should read +2 as positive 2 and —2 as negative 2.

You speak of the direction from 0 to the positive numbers as the positive direction (to the right) and of the direction from 0 to the negative numbers as the negative direction (to the left). The distance between a number and 0 is called the magnitude of the number, regardless of its direction. Thus +2 and —2 have the same magnitude 2.

Because their numerals involve signs of direction, positive and negative numbers are known as signed numbers or directed numbers. For convenience, 0 is included in the set of directed numbers. Here are some examples illustrating the uses of directed numbers.

1. A profit of $3 could be described as $+3, and a loss of $2 as $−2.
2. If latitudes north of the equator are taken as positive, the latitude of the South Pole is −90°.
3. If counterclockwise rotations are taken as positive, the minute hand of a clock rotates through an angle of −30° in 5 minutes.
4. Suppose the thermometer reads just freezing, that is, +32° (Fahrenheit). If the temperature drops 40°, we say the change in temperature is −40°, and the resulting temperature is −8° (8° below zero).

**ORAL EXERCISES**

In Exercises 1–10, tell what letter on the number scale corresponds to each number. Take as positive directions up, north, east, right, counterclockwise.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
<th>M</th>
<th>N</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>−2</td>
<td>−1</td>
<td>0</td>
<td>+1</td>
<td>+2</td>
<td>+3</td>
<td>+4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SAMPLE.** A gain of $3

*What you say:* A gain of $3 corresponds to +3, ⊴ point M.
1. A gain of $2
2. A loss of $2
3. Three degrees above zero
4. Three degrees below zero
5. A credit of $3.50
6. A point 2 miles west of a starting point
7. The level of the Atlantic Ocean
8. A point 3 miles south of a base line
9. 2\(\frac{1}{2}\)° north latitude
10. The latitude of the equator

Express each of the following by means of a directed number.

11. Ten degrees below freezing, Centigrade scale (Freezing is 0°C.)
12. Ten degrees below freezing, Fahrenheit scale (Freezing is +32°F.)
13. A gain of six dollars
14. A loss of seven dollars
15. Nine steps to the left
16. Eight steps to the right
17. A debt of $300
18. Receipts of $17
19. A mountain height of 6200 feet
20. An ocean depth of 330 feet

For each of the following, give another number having the same magnitude.

21. A counterclockwise angle of 15°
22. A latitude of 54° north
23. 12° north latitude
24. A five-yard gain in football
25. A walk of three miles east
26. A longitude of 15° east
27. A deposit of $75
28. A withdrawal of $25

WRITTEN EXERCISES

Draw a horizontal number scale. In Exercises 1–8, mark with an arrow the point associated with the given number.

SAMPLE. +3 Solution:

1. +8
2. −7
3. +10
4. +2.5
5. −9
6. −3
7. +7.5
8. −6\(\frac{1}{2}\)

Give the coordinate of each point at which you would arrive.

9. Start at zero on the number scale; go a distance of 5 units in the positive direction, then another 5 units in the positive direction.
10. Start at zero; go a distance of 3 units in the positive direction, then 2 units in the positive direction.
11. Start at zero; go a distance of 2 units in the negative direction, then a distance of 1 unit in the negative direction.
12. Start at zero; go 2 units in the negative direction, then 3 units in the negative direction.
Comparing Numbers

The sentence $3 < 5$, and the statements, $3$ is less than $5$ and $3$ is to the left of five describe the same order relationship. Can you extend this type of thinking to negative and positive numbers?

Which is the higher temperature, $-2^\circ$ or $-10^\circ$? The answer, of course, is $-2^\circ$. On the other hand, a temperature of $+10^\circ$ is higher than a temperature of $+2^\circ$. Similarly, although $+50^\circ$ is less than $+100^\circ$ ($50 < 100$), $-50$ is greater than $-100$, ($-50 > -100$).

Notice that any number on the number line, whether positive, negative, or zero, is greater than any number to its left, and less than any number to its right. Thus, directed numbers are ordered.

**ORAL EXERCISES**

Replace each question mark with $<$ or $>$ to make a true sentence.

1. $+7 ? +9$
2. $-3 ? -5$
3. $-4 ? +3$
4. $+2 ? +1$
5. $0 ? +2$
6. $0 ? +10$
7. $-10 ? 0$
8. $-1 ? +1$
9. $-.1 ? -.2$
10. $-.5 ? -.1$
11. $+.01 ? +.1$
12. $+1.05 ? +1.6$

Which of the following sentences are true? Which are false?

13. $-1 < -\frac{1}{2}$
15. $+7 \geq -20$
16. $-\frac{2}{3} < -\frac{1}{3}$
17. $+\frac{1}{5} \leq -\frac{5}{9}$
18. $-3 \leq +2$
19. $-10 \neq +10$
20. $+8 > -10$
21. $-\pi \neq -3.1$

**WRITTEN EXERCISES**

Let the replacement set for the variable in each of the following open sentences be $\{-6, -4, -1, 0, +2, +4\}$. In each case give the solution set.

A

1. $w < -2$
2. $y \geq -5$
3. $-1 \leq z$
4. $-3 \geq t$
5. $0 \neq u$
6. $-1 < k \leq +3$
7. $-6 \leq v < -1$
8. $r > +1$
9. $x \leq x$
10. $g > q$
11. $t \geq 0$
12. $v < 0$
Given the domain of each variable as the set of directed numbers, graph the solution set of each of the following open sentences.

**SAMPLE.** \(-7 < a \leq -1\)

**Solution:**

![Graph showing the solution set of \(-7 < a \leq -1\).]

13. \(p \geq -1\)  
14. \(q < +6\)  
15. \(+4 \leq t\)  
16. \(-3 \geq m\)  
17. \(-2 \leq c \leq +2\)  
18. \(-5 < d < +5\)  
19. \(t \leq +2\)  
20. \(s < +5\)  
21. \(-5 \leq w \leq -1\)  
22. \(-6 < f < +4\)  
23. \(z \neq +1\)  
24. \(r \neq +3\)  
25. \(e = 0\)  
26. \(0 = m\)

**PROBLEMS**

Translate each of the following English sentences into an open sentence. In each case specify the replacement set for the variable.

**SAMPLE.** To the nearest penny, the price of a quart of milk has never been higher than 30 cents.

**Solution:** Let \(c = \) the price of a quart of milk. \(0 < c \leq +30;\)

\[c \in \{+1, +2, +3, \ldots, +30\}\]

1. Phil’s assets exceed $10,000.  
2. Mary has less than $50 in her bank account.  
3. Ron hit more than twice as many home runs as did Joe, who hit 5 home runs.  
4. The winter temperature at the North Pole is less than \(-10^\circ\) Fahrenheit.  
5. The altitude of the land surface of the earth varies from 1292 feet below sea level to 29,028 feet above sea level.  
6. Latitude in the equatorial zone ranges between \(23\frac{1}{2}^\circ\) above and \(23\frac{1}{2}^\circ\) below the equator.
7. Over the past 5 years Marjorie’s weight has neither increased nor decreased more than 4 pounds.

8. To the nearest eighth of a dollar, the change in the price of a stock since 1950 has never been more than $10 in either direction.

9. The temperature in Death Valley has been as high as 56.7°C above zero and as low as 9.4°C below zero.

10. In a certain game Roger never scores more than 15 points and has never “gone into the hole” for more than 10 points.

**OPERATING WITH DIRECTED NUMBERS**

4–3 Addition on the Number Line

To use directed numbers effectively, you must agree on rules for operating with them. Begin by looking at the addition of positive numbers in a new way — in terms of displacements (changes of position) on the number line.

To add +5 to +2, start at +2 and move to the right a distance of 5 units.

This displacement, indicated by the red arrow in Figure 4–3, brings you 7 units to the right of zero, to +7. The addition may be written as +2 + +5 = +7.

You know that +5 + +2 should give the same result. To use the method of displacement, you would start at +5, then move 2 units to the right, again arriving at +7 (indicated by the black arrow). What property of addition is illustrated by this? Consider the following examples.

To add −5 to −2, start at −2, then move 5 units to the left (Figure 4–4).

This brings you 7 units to the left of zero, that is, to −7. You have −2 + −5 = −7.
THE NEGATIVE NUMBERS

To add \(-2\) to \(-5\), start at \(-5\), and go 2 units to the left, again arriving at \(-7\) (Figure 4-5).

\[
\begin{array}{c}
\text{Figure 4-5}
\end{array}
\]

That is, \(-5 + (-2) = -7\), the same as \(-2 + (-5)\).

To find \(-2 + +5\), start at \(-2\) and move 5 units to the right. The result is \(+3\), shown in Figure 4-6. Thus, \(-2 + +5 = +3\). Show that \(+5 + -2\) gives the same result.

\[
\begin{array}{c}
\text{Figure 4-6}
\end{array}
\]

Similarly, as shown in Figure 4-7, \(+2 + -5 = -3\). Show that \(-5 + +2\) also gives \(-3\).

Can you visualize \(+2 + -2\) on the number line? Do you see that your final position is at zero? \(+2 + -2 = 0\). Interpreting "adding 0" as no displacement, you see also that \(+2 + 0 = 0 + +2 = +2\), and that \(-2 + 0 = 0 + -2 = -2\). Because adding 0 to any number gives the identical number as the sum, zero is called the identity element for addition.

Any two numbers on the number line, positive, negative, or zero, can be added this way, and the result will be a number on the number line. The rule may be stated:

\[
\text{To add two numbers on the number line, start at the position of the first number, then move a distance equal to the magnitude of the second number in the direction associated with the number; the number at the resulting position is the required sum.}
\]

This kind of addition is commutative and associative. (Verify this.)

Find each of the following sums.

1. \(+6 + +7\)  
2. \(+7 + +6\)  
3. \(-5 + -6\)  
4. \(-6 + -5\)  
5. \(+12\) tickets  
6. \(+5\) grams  
7. \(+22\) tickets  
8. \(+6\) grams
WRITTEN EXERCISES

(a) Add each of the following, (b) as Exercises 15–28, use the associative property to regroup the addends in Exercises 1–14, and then compute each sum again.

SAMPLE 1. Add (+9 + -7) + -3

Solution: (+9 + -7) + -3 = +2 + -3 = -1

1. (+4 + +54) + +22
2. (+3 + +22) + +16
3. -43 + (-25 + -21)
4. -34 + (-24 + -62)
5. (-7° + +8°) + 0°
6. +5° + (0° + -15°)
7. -1.4 + (-3.0 + +4.4)

Solve each of the following equations, given that the replacement set of the variable is the set of directed numbers. You may use the number line as an aid.

SAMPLE 2. +2 + y = +5

Solution: To arrive at +5 starting from +2, go a distance of 3 units to the right.

+2 + y = +5

+2 + 3 = +5.

+5 = +5 ✓

: The solution set is {+3}, Answer.
SAMPLE 3. \(-6 = r + -4\)

Solution: To arrive at \(-6\) starting from \(-4\), go 2 to the left.

\[-6 = r + -4\]
\[-6 = -2 + -4\]
\[-6 = \sqrt{\text{ }}\] : The solution set is \{-2\}, Answer.

PROBLEMS

First write the answer to each of the following problems as the indicated sum of directed numbers; then do the addition.

1. An airplane is flying east. It has a speed of 200 miles per hour in still air and is pushed by a tail wind of 45 miles per hour. Find its speed.

2. An airplane ascends to 8000 feet on its take-off. Later the pilot descends 2000 feet and levels off. At what altitude does he level off?

3. The mate on a Mississippi River boat takes soundings as follows: 7 fathoms, 3 fathoms, 5 fathoms. Represent by signed numbers (a) each sounding with respect to river level, and (b) the change between each two successive soundings.

4. In descending from the stratosphere, a pilot came down 24,000 feet, leveled off, and later descended another 7000 feet. After flying at this altitude for a time, he descended 8000 feet to land. Express the total change in altitude of the plane.

5. During a cold snap, the temperature dropped three degrees in an hour. The next hour it dropped two degrees more, and the third hour it fell another three degrees, but the fourth hour it rose one-half degree. Express the total change in temperature.

6. Mr. Jordan owed $3.56, $2.34, $34.43, $23.25. Two friends owed him $25 and $35. What was Mr. Jordan’s net financial status?
7. An elevator descended two floors, stopped for passengers, descended three floors, made another stop, and then went to the lobby, ten floors below. Express the total change in position of the elevator.

8. John Byer owned stock which rose 2 dollars per share the first day, then $\frac{1}{3}$ dollar on each of three succeeding days, and $1\frac{1}{3}$ dollars on the fifth. What was the net change in price of the stock in that time?

9. A stock increased in value during the week of March 13, as follows: Monday, $\frac{1}{8}$ dollar; Tuesday, $\frac{1}{4}$ dollar; Wednesday, $\frac{1}{6}$ dollar; Thursday, $1\frac{1}{8}$ dollars; and Friday, $\frac{3}{8}$ dollar. Express the net change in value.

10. Al Herrick's stock fell $\frac{1}{2}$ dollar per share the first day he owned it. It continued to fall $\frac{1}{3}$ dollar for each of the next two days. Express the change in value of a share of this stock during these three days.

11. One week the price of a stock changed as follows: Monday, down $1\frac{1}{8}$ dollars; Tuesday, down $\frac{3}{8}$ dollar; Wednesday, unchanged; Thursday, down $\frac{1}{2}$ dollars; Friday, up $2\frac{1}{8}$ dollars. Find the net change.

12. To test its equipment, a submarine was submerged 20 meters; then 15 meters twice; then 5 meters three times; and finally 3 meters each of 8 times. Represent the position of the submarine at its lowest depth.

4–4 The Opposite of a Directed Number

You can see that each number on the number line may be paired with another number which is the same distance from zero but in the opposite direction.

Furthermore, adding two such paired numbers on the number line gives 0. This suggests defining the opposite (additive inverse) of a directed number $a$ as the directed number whose sum with $a$ is 0. The symbol $-a$ (note the lowered position of the minus sign) denotes the opposite of $a$ or the additive inverse of $a$.

\[
- (+7) = -7 \quad \text{read the opposite of } +7 \text{ is } -7 \quad \rightarrow \quad +7 + -7 = 0
\]

\[
- (+5) = -5 \quad \text{read the opposite of } +5 \text{ is } -5 \quad \rightarrow \quad +5 + -5 = 0
\]

\[
- (0) = 0 \quad \text{read the opposite of } 0 \text{ is } 0 \quad \rightarrow \quad 0 + 0 = 0
\]

\[
- (-3) = +3 \quad \text{read the opposite of } -3 \text{ is } +3 \quad \rightarrow \quad -3 + +3 = 0
\]

The $\rightarrow$ is translated as because or meaning.

An important assumption about directed numbers is:
For every directed number $a$ there is a unique number $-a$ such that $a + (-a) = (-a) + a = 0$.

Several facts about a number and its opposite follow from this assumption:

1. If $a$ is positive, $-a$ is negative; if $a$ is negative, $-a$ is positive; if $a$ is 0, $-a$ is 0.
2. The opposite of $-a$ is $a$, that is, $-(-a) = a$.

All these relationships help in simplifying expressions.

**EXAMPLE 1.** Show on the number lines that

- $a$. $- (+3 + +4) = -1$ and
- $b$. $+3 + (- +4) = -7$; therefore,
- $c$. $- (+3 + +4) = - +3 + (- +4)$.

**Solution:**

- **a.** Add $+3$ and $+4$, and then find the opposite of this result, $-7$.

- **b.** Add the opposite of $+3$ and the opposite of $+4$; that is, add $-3$ and $-4$. This is $-7$.

- **c.** By the transitive property of equality, as both expressions equal $-7$, they equal each other.

Example 1 illustrates a very important property of opposites:

The opposite (additive inverse) of the sum of two numbers is the sum of their opposites:

$$-(a + b) = -a + (-b).$$

Try problems similar to Example 1 until you see clearly that this statement is true for any numbers $a$ and $b$, positive, negative, or zero.

Assuming that every directed number has an opposite is a device that would have enabled us to invent the negative numbers without having to think of them as partners of points on the number line. This way, $-3$ is simply the number whose sum with $+3$ is 0, $-3 = -+3$. By
agreement (page 112), \( +3 = 3 \). As a result, hereafter we will simplify notation by dropping the small \( + \) and \( - \) signs. Thus, write \( 3 \) rather than \( +3 \), and \( -3 \) for \( -3 \) or \( +(-3) \). You now read \( -3 \) either as negative \( 3 \) or as the opposite of \( 3 \).

Using addition on the number line, you may find the following sums:

**EXAMPLE 2.** \(-(-2) + (-5) = 2 + (-5) = -3\)

**EXAMPLE 3.** \([-(-2) + 3] + (-4) = [2 + 3] + (-4) = 1\)

### ORAL EXERCISES

Name the additive inverse of each of the following directed numbers.

**SAMPLE.** \(-12\)  
**What you say:** Twelve or positive twelve.

1. \(5\)  
2. \(\frac{1}{3}\)  
3. \(-2\)  
4. \(-6\)  
5. \(-\frac{3}{2}\)  
6. \(-1\frac{1}{2}\)  
7. \(2 + 9\)  
8. \(-(3 + 7)\)  
9. \(0\)  
10. \(-0\)  
11. \(7 + (-8)\)  
12. \((-9) + 4\)

### WRITTEN EXERCISES

Give two expressions for the additive inverse of each of the following.

**SAMPLE.** \((-6) + 7\)  
**What you say:** \(6 + (-7)\) or \(-1\).

17. \(-8 + (-4)\)  
18. \(-10 + (5)\)  
19. \([-6 + (-6)]\)  
20. \([-9 + (-8)]\)  
21. \(-(-8) + [-(9)]\)  
22. \((-8) + [-(7)]\)  
23. \(-[-(7)]\)  
24. \([--(3 + 2)]\)  
25. \(-[-(8)]\)  
26. \(-[-(7 + 9)]\)  
27. \(9 + (-5) + (-8)\)  
28. \(-4 + (7) + [-(9)]\)

### WRITTEN EXERCISES

Determine the value of the following expressions.

1. \(-(-4) + (-6)\)  
2. \(-(-3) + [-(6)]\)  
3. \(-(5 + 4)\)  
4. \(-[-3 + 8]\)  
5. \(-(7 + 5)\)  
6. \([-9 + (-5)]\)  
7. \(-(-5) + (-1)\)  
8. \(-[4 + (-6)]\)
9. \(-8 + (-3) + 6.6\) 
10. \(-4 + (-0) + (-2.4)\) 
11. \(15\frac{1}{2} + (-3\frac{1}{3}) + [-(6\frac{2}{3})]\) 
12. \(-5\frac{2}{3} + 7\frac{3}{5} + (-2\frac{5}{6})\)

If \((-3, -2, -1, 0, 1, 2, 3)\) is the replacement set for the variables, find the solution set of the following open sentences by substitution.

**SAMPLE.** \(-t < \frac{1}{2}\)

**Solution:** \(-3 < \frac{1}{2}; -2 < \frac{1}{2}; -1 < \frac{1}{2}; -0 < \frac{1}{2}\)

\(\therefore\) The solution set for \(-t < \frac{1}{2}\) is \(\{3, 2, 1, 0\}\).

If \(a = \frac{5}{3}, b = \frac{1}{3}, c = .25,\) and \(d = 1.5,\) evaluate the following expressions.

28. \(a + (-b) + c\) 
29. \(-a + b + (-c)\) 
30. \([-b + c + (-d)]\) 
31. \(-b + (-c) + d\) 
32. \(a + b + (-c) + (-d)\) 
33. \(-a + b + c + d\)

34. \([-a + c + (-d)]\) 
35. \(a + [-(c) + d]\) 
36. \(b + (-a) + (-d)\) 
37. \(b + [-(a + d)]\) 
38. \([a + b] + [-(c + d)]\) 
39. \([-a + b] + [c + d]\)

### 4–5 Absolute Value

Although the members of a pair of opposite numbers, such as 2 and \(-2\), \(-7\) and 7, 12 and \(-12\), are equally distant from 0, the positive member of such a pair is the greater. The greater of any directed number, other than 0, and its opposite is called the **absolute value** of either number. The absolute value of a number is denoted by placing the number between a pair of vertical bars \(|\ |\). The **absolute value of 0** is 0. It is obvious that the absolute value of any number and that of its opposite are equal. For example,

\[|-6| = |6| = 6\]

which is read: The absolute value of negative 6 equals the absolute value of positive 6, equals positive 6.
ORAL EXERCISES

Tell which of these statements are true and which are false.

1. \(|9| = 9\)  5. \(|8| > |{-8}|\)  9. \(-3 < |{-5}|\)
2. \(|{-345}| = -345\)  6. \(|-234| > |{234}|\)  10. \(|{17}| < 17\)
3. \(|{-56}| = 56\)  7. \(|{32}| > |{-56}|\)  11. \(|{-27}| < |{19}|\)
4. \(|{12\frac{1}{2}}| = -12\frac{1}{2}\)  8. \(|{-4.9}| > |{-3.4}|\)  12. \(|{18.9}| < |{-19.8}|\)

Evaluate these expressions.

13. \(|3| + |{16}|\)  17. \(-|{-7}|\)  21. \(-(2|{-5}|)\)
14. \(|{-4}| + |{-8}|\)  18. \(-|{-7}|\)  22. \(-(|-2| |-5|)\)
15. \(|{21}| + |{-19}|\)  19. \(-|{5}| + |{7.1}|\)  23. \(-|{-15}| - |{-12}|\)
16. \(|{-13}| + |{12}|\)  20. \(-|{-7}| + |{7}|\)  24. \(-(|-23| + |{15}|)\)

WRITTEN EXERCISES

Give the solution set of each of these equations.

A  1. \(|q| = 0\)  2. \(|m| = 4\)  3. \(|n| + |{-2}| = 7\)  4. \(|s| + |{4}| = 3\)

Graph the solution set of each inequality.

5. \(|x| < 3\)  6. \(|y| > 4\)  7. \(1 < |x| < 3\)  8. \(-2 < |y| < 1\)

Describe the solution sets of these open sentences.

9. \(|y| < 0\)  11. \(|y| < -y\)  13. \(y = |y|\)  15. \(|y| > 0\)
10. \(|y| \leq 0\)  12. \(|y| < y\)  14. \(-y = |y|\)  16. \(|y| \geq 0\)

Evaluate, if \(a = 2.3\), \(b = -16\), \(c = -5\frac{1}{2}\), and \(d = 3\frac{2}{5}\).

B  17. \(|a| + |b| + |c|\)  20. \(a + (-b) + |c| + d\)  23. \(a + d + |c|\)
18. \(-(a + |c|) + d\)  21. \(|a| |b| |c|\)  24. \(|b + c| + d\)
19. \(-a + b + |c| + d|\)  22. \(|b| |c| |d|\)  25. \(a + |b + c|\)

4–6 Adding Directed Numbers

To treat addition of directed numbers without the number scale, you must list the addition properties assumed for the directed numbers.
For all members \(a, b,\) and \(c\) of the set of directed numbers:

1. The closure property: the sum \(a + b\) is a unique directed number.
2. The commutative property: \(a + b = b + a.\)
3. The associative property: \((a + b) + c = a + (b + c).\)
4. The additive property of zero: \(0 + a = a + 0 = a.\)
5. The property of opposites: \(-a\) is a unique directed number such that \(a + (-a) = (-a) + a = 0.\)
6. The property of the opposite of a sum: \(- (a + b) = (-a) + (-b).\)

You will find the sixth property particularly useful in making such substitutions for negative numbers as these:

\[-7 = -(4 + 3) = -4 + (-3); \quad -6\frac{2}{3} = -(6 + \frac{2}{3}) = -6 + \left(-\frac{2}{3}\right).\]

The following examples indicate several ways in which you can use the properties of directed numbers and of equality to perform addition.

**EXAMPLE 1.** Add: \(-6 + (-5)\)

**Solution:**

\[-6 + (-5) = -(6 + 5) \quad \text{Property of the opposite of a sum}\]

\[= -11 \quad \text{Substitution principle}\]

**EXAMPLE 2.** Add: \(7 + (-3)\)

**Solution:**

\[7 + (-3) = (4 + 3) + (-3)\]

\[= 4 + [3 + (-3)]\]

\[= 4 + 0\]

\[= 4\]

**EXAMPLE 3.** Add: \(-7 + 4\)

**Solution:**

\[-7 + 4 = -(3 + 4) + 4\]

\[= -3 + (4 + 4)\]

\[= -3 + 0\]

\[= -3\]

Can you supply the properties which justify each step in Examples 2 and 3?

From the above examples and properties you can deduce the following rules for addition:

1. If \(a\) and \(b\) are each positive, \(a + b = |a| + |b|\).
   
   **EXAMPLE.** \(4 + 3 = 7\)

2. If \(a\) and \(b\) are each negative, \(a + b = -(|a| + |b|)\).
   
   **EXAMPLE.** \((-3) + (-4) = -(3 + 4) = -7\)
3. If $a$ is positive and $b$ is negative and $|a| \geq |b|$, $a + b = |a| - |b|$.

**EXAMPLE.** $7 + (-3) = 7 - 3 = 4$

4. If $a$ is positive and $b$ negative and $|b| \geq |a|$, $a + b = -(|b| - |a|)$.

**EXAMPLE.** $4 + (-7) = -(7 - 4) = -3$

**EXAMPLE 4.** Add: $5 + (-13) + 11 + (-6) + (-8)$

**Solution:**

$5 + (-13) = -8$; $-8 + 11 = 3$; $3 + (-6) = -3$; $-3 + (-8) = -11$, Answer.

$or [5 + 11] + [(-13) + (-6) + (-8)] = 16 + (-27) = -11$, Answer.

**EXAMPLE 5.** Add: $-115$

**Solution:**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Step 2</th>
<th>Step 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>171</td>
<td>-115</td>
<td>171</td>
</tr>
<tr>
<td>235</td>
<td>-313</td>
<td>235</td>
</tr>
<tr>
<td>-313</td>
<td>-428</td>
<td>406</td>
</tr>
</tbody>
</table>

$-428 + 406 = -22$, Answer.

**ORAL EXERCISES**

Add the following.

1. $4 - 3$
2. $-3 - 2$
3. $8 - 6$
4. $10 - 5$
5. $\frac{1}{5} - \frac{3}{5}$
6. $-3 + 6$
7. $-2 + 9$
8. $-4 - 4$
9. $-5 - 4$
10. $\frac{5}{7} - \frac{2}{7}$
11. $-3.8 + (-2.9)$
12. $-6.2 + (-4.8)$
13. $\frac{6}{3} + (-1.5)$
14. $(-\frac{3}{4}) + .2$
15. $3 + (-8) + 8$
16. $(-6) + (-1) + 6$
17. $(-7) + k + (-2) + (-k)$
18. $r + (-12) + (-r) + 9$

Name the properties of addition which justify these sentences.

19. $3 + (-7) = -7 + 3$
20. $-14 + (-12) = -12 + (-14)$
21. $7 + (-7) = 0$
22. $-\frac{1}{3} + \frac{1}{3} = 0$
23. $5 + (-2 + 2) = 5$
24. $[6 + (-6)] + (-7) = -7$
Add the following.

1. 6  2. -7  3. $54\frac{1}{2}$  4. $43\frac{1}{3}$  5. -18  6. -14
   10  1  -11  -4  -45  24  22
   -3  8  16  -54  -32  -11
   -9  9  -4  -12  17  35

7. -9 + 6 + 7 + (-3)  11. (-3) + 4 + 1 + (-2)
8. 7 + (-5) + (-6) + 2  12. 23 + 3 + (-6) + (-15)
9. 24 + (-19) + (-35) + (-6)  13. 7 + (-14) + (-8) + 9
10. (-55) + (-11) + 65 + 1  14. (-3) + 8 + (-5) + 35
15. $\frac{1}{2} + 2\frac{1}{2} + (-3\frac{1}{2}) + 4\frac{1}{2} + (\frac{1}{2}) + 2\frac{1}{2} + (-3\frac{1}{2}) + \frac{2}{3} + (-\frac{1}{3})$
16. $\frac{1}{2} + (-\frac{3}{2}) + \frac{3}{5} + (-\frac{5}{6}) + \frac{1}{6} + (-\frac{3}{16}) + 2 + (-3\frac{1}{2}) + (-1) + 1$
17. 2.5 + (-3.5) + 4 + (-5) + 2.5 + (-0.5) + 3.5 + 2.4 - 2.9
   - 3.6 + 2.6
18. -0.5 + (-10.1) + 0.4 + (-1.2) + (-3) + (-0.7) + (-1.4) + 5

In each case, write a chain of equations leading to the stated equation. Justify each equation.

**SAMPLE.**  $b + [a + (-b)] = a$

**Solution:**  $b + [a + (-b)] = b + [(-b) + a]$  Commutative property
   = $[b + (-b)] + a$  Associative property
   = 0 + a  Property of opposites
   = a  Property of 0

19. $[b + (-4)] + (-b) = -4$  23. $a + (b + c) = b + (c + a)$
20. $[(-60) + c] + (60) = c$  24. $(-m) + (k + m) = k$
21. $a + (b + c) = (a + c) + b$  25. $(-5) + [(-s) + (s + 5)] = 0$
22. $[a + (-b)] + [b + (-a)] = 0$  26. $[a + c] + [(-a) + (-c)] = 0$

Replace the question mark with a numeral to make a true statement.

27. (?) + 17 = 15  30. -5 + (?) = -4  33. (?) + 3 = -8
28. 10 + (?) = 7  31. 9 + (?) = -17  34. -5 + (?) = -5
29. 4 + (?) = -1  32. (?) + 6 = 5  35. $\frac{2}{3} + (?) = 1$
1. A merchant's transactions had the following results: a gain of $35, a gain of $14.75, a loss of $26.10, a gain of $18.15, a loss of $7.50. Represent his net gain or loss by means of a signed number.

2. A girls' club took in $13.00 for the semester's dues and paid out $7.50 for refreshments, $1.25 for programs, and $2.00 for a charity project. Their share in the proceeds of the class play was $6.50. Use a signed number to represent the financial condition of the club.

3. A housewife made the following entries in her household account one day: groceries $13.68, bakery $1.09, meat $4.17, return for bottles $.37, Joan's allowance $.75. Represent each item by a directed number and find the sum.

4. To buy graduation prizes, the Parent-Teacher Association needed more than the $73 in its treasury. The members presented a play, for which they paid a royalty of $25. Scenery and costumes cost $18, and the programs, $15. The sale of tickets amounted to $185. Program advertisements brought $64. Find the amount they then had.

5. A submarine submerged 375 feet below sea level fires a rocket which rises 650 feet. How far above sea level does the rocket go?

6. A football player made the following yardage on five plays: 15, −3, 8, −9, −12. What was his total net gain in yards?

7. If $G = \{-4, -1, 0, 1, 3\}$, find the set of all sums of pairs of elements of $G$. Is $G$ closed under addition?

8. Bob lost 3 pounds the first week on his 900-calorie diet, gained $1\frac{1}{2}$ pounds the second week, gained $\frac{1}{2}$ pound the third week, and lost 4 pounds the fourth week. What was his total gain or loss?

9. The temperature at noon was 49°F and at 5 p.m. it was 21.5°F. What was the net change in temperature?

4–7 Subtracting Directed Numbers

One evening the thermometer read 11° above zero. The next morning it read 5° below zero. How much had the temperature changed? Do you get a 16° drop? Do you realize that you just subtracted 11 from −5?
Another situation illustrates something else you know about subtraction. If you buy 85 cents' worth of goods and give the clerk a dollar, he may count your change saying, \( 85, 95 \) (handing you a dime), one dollar (handing you a nickel). The clerk did a subtraction problem (100–85) by adding. Another way of saying this is that \( x \) has the same value in both of these equations:

\[
100 - 85 = x \quad \text{and} \quad 85 + x = 100.
\]

Similarly, \( x \) has the same value in both of these equations:

\[
(-5) - 11 = x \quad \text{and} \quad 11 + x = -5.
\]

Guided by these results, we make this definition: *For all directed numbers \( a \) and \( b \), any directed number satisfying the equation \( b + x = a \) is called the difference of \( a \) and \( b \), that is, \( a - b \).*

Using only this definition, you do a subtraction problem by asking yourself, "What number added to \( b \) gives \( a \)?" You can find a simple expression for \( a - b \) by transforming the equation \( b + x = a \):

\[
\begin{align*}
\quad & b + x = a \hfill \\
\quad & x + b = a \hfill \\
\quad & x + b + (-b) = a + (-b) \hfill \\
\quad & x + 0 = a + (-b) \hfill \\
\therefore & \quad x = a + (-b)
\end{align*}
\]

The last equation evidently has just one root, \( a + (-b) \). Checking this root in the original equation, you have:

\[
\begin{align*}
\quad & b + x = a \hfill \\
\quad & b + a + (-b) = a \hfill \\
\quad & b + (-b) + a = a \hfill \\
\quad & 0 + a = a \hfill \\
\therefore & \quad a = a \quad \checkmark
\end{align*}
\]

Since the one and only root of \( b + x = a \) is \( a + (-b) \), it follows that:

\[
a - b = a + (-b).
\]

To perform a subtraction, replace the subtrahend by its opposite, and add.
Does this rule give a meaningful expression for \( a - b \)? As every number has an opposite, if you know \( b \) then you know \( -b \). Also, since \( a + (-b) \) is a sum, it represents a definite number. Hence, the rule shows that the set of directed numbers is closed under subtraction.

Using this rule, you always can replace a subtraction by an addition:

<table>
<thead>
<tr>
<th>Subtraction</th>
<th>Addition</th>
<th>Value</th>
<th>Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 - 2</td>
<td>6 + (-2)</td>
<td>4</td>
<td>4 + 2 = 6</td>
</tr>
<tr>
<td>6 - (-2)</td>
<td>6 + 2</td>
<td>8</td>
<td>8 + (-2) = 6</td>
</tr>
<tr>
<td>-6 - 2</td>
<td>-6 + (-2)</td>
<td>-8</td>
<td>-8 + 2 = -6</td>
</tr>
<tr>
<td>-6 - (-2)</td>
<td>-6 + 2</td>
<td>-4</td>
<td>-4 + (-2) = -6</td>
</tr>
<tr>
<td>( a - (a + 7) )</td>
<td>( a + (-a) + (-7) )</td>
<td>-7</td>
<td>( -7 + (a + 7) = a )</td>
</tr>
</tbody>
</table>

**ORAL EXERCISES**

In each case, subtract the lower number from the number above it.

1. 13 - 8 = 5
2. 17 - 9 = 8
3. 9 - 17 = -8
4. 8 - 13 = -5
5. 7 - 3 = 4
6. 11 - 5 = 6
7. (-18) - 17 = -35
8. (-22) - 46 = -68
9. -28 - (-3) = -25
10. -157 - (-108) = -49
11. -12 - 17 = -29
12. 15 - 15 = 0
13. \( a + 4 - a \) = 4
14. 5 + t - t = 5
15. -2 + m - 8 + m = -10
16. \( r + (-4) + 7 \) = r

Perform each of the indicated subtractions.

17. 27 - 5 = 22
18. 25 - 29 = -4
19. 1.3 - (-2.0) = 3.3
20. 2.3 - (-.7) = 3.0
21. (-.12) - (-.2) = .08
22. (-.20) - (-.30) = .10
23. (-.3) - .5 = -.8
24. (-.16) - .4 = -.56
25. (-.5) - (-.3) = -.2
26. (-.2) - (-.7) = .5
27. 0 - .4 = -.4
28. 0 - (-.8) = .8
29. -3.9 - 0 = -3.9
30. -2.4 - 0 = -2.4
31. 8\( \frac{1}{2} \) - (-8\( \frac{1}{2} \)) = 16
32. -3\( \frac{1}{3} \) - 3\( \frac{1}{3} \) = -6
33. \(-2\frac{3}{5} \) - \(-2\frac{3}{5} \) = 0
34. 25\( \frac{1}{5} \) - 25\( \frac{1}{5} \) = 0
35. 100 - 100 = 0
36. (-7) - (-7) = 0
37. (x + 3) - (x + 2) = 1
38. (a + 5) - (a + 4) = 1
39. h - (h + 1) = -1
40. p - (p + 3) = -3
Rewrite these subtraction exercises as additions, and then find the sums.

**SAMPLE 1.**

89
99

Solution: 
89 + (—99) = —10

Check: 
—10 + 99 = 89

**SAMPLE 2.**

[r + 5] — [r — 4]

Solution: 
[r + 5] — [r — 4] = r + 5 — [r + (—4)]
= r + 5 + (—r) + 4
= 9

Check: 
9 + [r — 4] = 9 — 4 + r = r + 5

**A.**

1. 29
   —30
2. —69
   —72
3. —25
   —25
4. —80
   —20
5. t + 2
   t + 5
6. s + (—7)
   s + (—3)
7. a + b — 12
   a + b — 4
8. a — k + 14
   a — k — 9
9. 1.5 — .5
10. 1.7 — 1.9
11. .6 — (—.3)
12. —5.7 — (—.7)
13. 0 — (—3/4)
14. 0 — 3/3
15. [—300 + 450] — [230 — 1066]
16. [1492 — 1678] — [—44 + 12]
17. x + y — (x — y) — y
18. u — v — (u + v) + v
19. (h + 9) — (—7 + h)
20. (—10 — k) — (12 + k)

Write the following phrases in algebraic symbols, and simplify.

**B.**

21. —2 decreased by 7
22. —7 decreased by 18
23. b less (b + 4)
24. c less (8 + c)
25. Take 23π + 1 from 23π — 1.
26. Take 8π — 5 from —8π + 5.
27. From the sum of 37 and —12 subtract 49.
28. From the sum of 42 and —51 subtract 27.
29. Subtract —97 + a from —32 + a.
30. Subtract 18 — r from 19 — r.
31. Prove: $-(a - b) = -a + b$.

32. Prove: $-(a + b + c) = (-a) + (-b) + (-c)$.

Which of these sets are closed under subtraction? Explain.

33. {odd integers}

34. {even integers}

35. {odd integers, 0}

36. {0, $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{3}{2}$, $-\frac{3}{2}$}

Solve these problems by using directed numbers.

1. Find the difference in altitude between Salton Sea, California, 244 feet below sea level, and a spot in Death Valley, 276 feet below sea level.

2. Find the change in temperature on a winter day when the temperature dropped from 3° below zero to a low of 11° below zero.

3. At 6 P.M. the thermometer read 8° above zero. At midnight it read 5° below zero. Find the change in temperature.

4. In the New York City Rapid Transit System a high point is 161 feet above sea level, and a low point is 113 feet below sea level. Find the change in altitude in going from the high point to the low.

5. The Peloponnesian War began in 431 B.C. Peace was finally made in 404 B.C. How long did the war last?

6. The Greek mathematician Archimedes was born in 287 B.C. and died in 212 B.C. How long did he live?

7. John owes his father $4.38. How much must John pay to acquire a credit of $1.25 with his father?

8. In playing a game Ellen was 175 points “in the hole.” How many points must she make to have a score of 250 points?

9. Carthage was destroyed in 146 B.C. How many years ago was that event? (Assume no year 0.)

10. If the sea is 37,800 feet deep and the highest mountain is 29,012 feet high, find the difference in elevation between them.

11. Mr. Lescaire had a bank balance of $317.25 on Monday. On Friday the bank said he was overdrawn by $9.47. How much had Mr. Lescaire spent during that period?

12. If New York’s latitude is 41°N and Rio de Janeiro’s is 23°S, find the difference in latitude between the two cities.
4-8 Multiplying Directed Numbers

Probably you were first introduced to multiplication in arithmetic by some explanation such as this: “When we write 3 \times 2, we mean take 2 three times. If you have 3 boxes each with 2 apples, you have 3 \times 2 apples. You also have 2 + 2 + 2 apples.”

If you try to give meaning to 3 \times (−2), you cannot talk about apples in boxes, but you can talk about (−2) + (−2) + (−2), so you can say that that is what 3 \times (−2) means. But, when you try to talk about (−3) \times 2, you have trouble. You cannot take 2 “minus three” times conveniently. To help solve this dilemma, consider a different kind of example.

Suppose water is flowing into a tank at the rate of 3 gallons per minute (3). You can make the following statements.

1. Two minutes hence (2), there will be 6 gallons more (6) in the tank.

   \[(2)(3) = 6\]

   positive number \times positive number gives positive number

2. Two minutes ago (−2), there were 6 gallons less (−6) in the tank.

   \[(-2)(3) = -6\]

   negative number \times positive number gives negative number

Suppose that water is flowing out of the tank at the rate of 3 gallons per minute (−3). You can make these statements.

3. Two minutes hence (2), there will be 6 gallons less (−6) in the tank.

   \[(2)(-3) = -6\]

   positive number \times negative number gives negative number

4. Two minutes ago (−2), there were 6 gallons more (6) in the tank.

   \[(-2)(-3) = 6\]

   negative number \times negative number gives positive number
The rules suggested by these examples can be developed from the following assumptions for multiplication.

For \(a, b,\) and \(c,\) members of the set of directed numbers:

1. The closure property: for every \(a\) and \(b,\) the product \(ab\) is a unique directed number.
2. The commutative property: \(ab = ba.\)
3. The associative property: \(a(bc) = (ab)c.\)
4. The distributive property: \(a(b + c) = ab + ac.\)
5. The multiplicative property of 0: \(a \cdot 0 = 0 \cdot a = 0.\)
6. The multiplicative property of 1: \(a \cdot 1 = 1 \cdot a = a.\)

This last property might make you curious about the product \(a \cdot (-1).\) Would you guess that it would be \(-a?\) To verify this guess, show that the sum of \(a(-1)\) and \(a\) is zero:

\[
\begin{align*}
  a(-1) + a & \neq 0 \\
  a(-1) + a(1) & \neq 0 & \text{Multiplicative property of 1} \\
  a(-1) + 1 & \neq 0 & \text{Distributive property} \\
  a(0) & \neq 0 & \text{Property of opposites} \\
  0 & = 0 \checkmark & \text{Multiplicative property of zero}
\end{align*}
\]

Therefore, the *multiplicative property of \(-1\)* is:

Multiplying any number by \(-1\) gives its opposite.

For any \(a,\) \(a(-1) = (-1)a = -a.\)

A special case of this property occurs when \(a = -1;\) this gives \((-1)(-1) = 1.\) You now can justify the products you obtained in the four cases of multiplication illustrated by the water tank problem by writing the following chains of equalities:

1. \((2)(3) = 6\
2. \((-2)(3) = [(-1)(2)](3) = (-1)[2(3)] = (-1)(6) = -6\
3. \((2)(-3) = (2)\{(-1)(3)\} = [(-1)(2)(3) = (-1)[2(3)] = (1)(6) = -6\
4. \((-2)(-3) = [(-1)(2)][(-1)(3)] = [(-1)(-1)][(2)(3)] = 1(6) = 6\

Similarly, for all numbers \(a\) and \(b:\)

\[
\begin{align*}
  b(-a) &= (-a)(b) = [-1(a)b = (-1)(ab) = -ab \\
  (-a)(-b) &= [-1(a)[-1(b)] = [(-1)(-1)(ab) = 1(ab) = ab
\end{align*}
\]
Do you see that the following statements are true?

1. The absolute value of the product of two directed numbers is the product of the absolute values of the numbers.
2. The product of a positive and negative number is a negative number.
3. The product of two positive numbers or of two negative numbers is a positive number.

By pairing \((-1)(-1) = 1\), you can extend these rules to any number of factors.

1. The absolute value of an indicated product of numbers is the product of the absolute values of the numbers.
2. An indicated product containing an odd number of negative factors is a negative number.
3. An indicated product containing an even number of negative factors is a positive number.

Since the distributive property is assumed to hold for directed numbers, variables in the terms of an expression are treated as they have been previously.

**Example.** Simplify: \(6x - 4y - 5x + 8y + 7x - 9y\)

**Solution:** \(6x - 4y - 5x + 8y + 7x - 9y = (6 - 5 + 7)x + (-4 + 8 - 9)y = 8x - 5y\)

**Oral Exercises**

Find each of the indicated products.

1. \((4)(5)\)  
2. \((-6)(-2)\)  
3. \((7)(-3)\)  
4. \((2)(9)\)  
5. \((-5)(-4)\)  
6. \((-15)(-2)\)  
7. \((-3)(5)\)  
8. \(-4(7)(-1)\)  
9. \(-1(1)(-1)\)  
10. \(\frac{1}{2}(-2)\)  
11. \((-3)(-\frac{1}{3})\)  
12. \(4a(-5a)(10)\)  
13. \((-3)(2a)(a)\)  
14. \((7a)(0)(-6b)\)  
15. \(4(-\frac{1}{2})(-2b)b^2\)  
16. \(7(-\frac{1}{3})(-3d)d^3\)  
17. \((-4)(-2)(0)(-1)\)  
18. \((-x)(-xy^2)(-y)(0)\)  
19. \((-2)^3\)  
20. \((-3)^3\)  
21. \(2(-3)^4\)
Name the opposite of each of the following.

22. $7x$ 
23. $-3r$ 
24. $-4a + 3b$ 
25. $3t - 2s$ 
26. $-2x - 7$ 
27. $5k - 6$

Subtract the lower expression from the one above it.

28. $-7x + 7y$ 
   5x + (-5y)
29. $-3a + 8b$ 
   9a + (-9b)
30. $8m + 12n$ 
   -m + (-3n)
31. $-20b^2$ 
   -33b²
32. $49x + 7y$ 
   76x - 8y
33. $-29m^2 + 15m$ 
   -31m² + 7m

The following proofs are valid for all a, b, and c. Justify each step.

**SAMPLE.** To prove, $a(b - c) = ab - ac$

<table>
<thead>
<tr>
<th>Steps:</th>
<th>What you say:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(b - c) = a[b + (-c)]$</td>
<td>Meaning of subtraction</td>
</tr>
<tr>
<td>$= a(b) + a(-c)$</td>
<td>Distributive property</td>
</tr>
<tr>
<td>$= ab + a[c(-1)]$</td>
<td>Multiplicative property of $-1$</td>
</tr>
<tr>
<td>$= ab + (ac)(-1)$</td>
<td>Associative property</td>
</tr>
<tr>
<td>$= ab + [-(-ac)]$</td>
<td>Multiplicative property of $-1$</td>
</tr>
<tr>
<td>$= ab - ac$</td>
<td>Meaning of subtraction</td>
</tr>
</tbody>
</table>

This exercise shows that multiplication of directed numbers is distributive with respect to subtraction.

34. To prove, $-(-b) = b$

$-(-b) = (-1)(-b)$
$= (-1)[(-1)(b)]$
$= [-1(-1)]b$
$= 1 \cdot b$
$= b$

35. To prove, $-(a + b) = -a - b$

$-(a + b) = (-1)(a + b)$
$= (-1)a + (-1)b$
$= -a + (-b)$
$= -a - b$

36. To prove, $-(a - b) = b - a$

$-(a - b) = (-1)[a + (-b)]$
$= (-1)a + (-1)(-b)$
$= -a + b$
$= b + (-a)$
$= b - a$
Evaluate each of the following numerical expressions (a) by combining terms, and (b) by using the distributive property.

**SAMPLE.** \((-3)[7 - (-2)]\)

**Solution:**

a. \((-3)[7 - (-2)]\)
   \[= (-3)[7 + 2]\]
   \[= (-3)(9)\]
   \[= -27\]

b. \((-3)[7 - (-2)]\)
   \[= (-3)(7) - (-3)(-2)\]
   \[= -21 - 6\]
   \[= -27\]

---

**A**

1. \((-4)(10 + 11)\)
2. \((-3)[(-2) + (-5)]\)
3. \((20 - 21)(4)\)
4. \((5)(-6 - 3)\)
5. \(0(-8 + 5)\)
6. \([-3 - (-7)]0\)
7. \((4)(3) - 4(13)\)
8. \((-2)(-2)\)
9. \((99)(12) + (-12)\)
10. \((40)(99) + (40)(-99)\)

11. \((-13)(-7) + (-7)(13)\)
12. \((-12)(4 + \frac{1}{3})\)
13. \(14(-5 - \frac{1}{7})\)
14. \((-48)(-3\frac{1}{2})\)
15. \(-(-7 + 12)\)
16. \(-(-34 - 20)\)
17. \(0.8 - (2.5 - 0.32)\)
18. \(6 - (-1.9 + 0.27)\)
19. \(-3 - 0.2(-7 - 3)\)
20. \(-7 - 0.2(-1 - 0.4)\)

Combine the similar terms in each of the following expressions.

21. \(2a - 3 + 5a - 10a + 8\)
22. \(4r - 7 + 3r + 9 - 8r\)
23. \(-6x + 5 - 2x + x - 12\)
24. \(-8y + 8 - 6y - y + 14\)
25. \(-4nt + 8 - 6 - nt + 3nt\)
26. \(-9hk + 5hk - 8 + hk + 5\)
27. \(4r + 5s - r + s - 6r\)
28. \(16a - 9b - a + b - 7b\)
29. \(1.5n - 8 - 3.5n + 5\)
30. \(\frac{1}{2}y - \frac{1}{2} - \frac{2}{3}y + 1 + y\)

31. \(-3a + b + 4a + a - b\)
32. \(9u - u + 5 - 6u - 6 + 2u\)
33. \(33k^3 - k^3 + 4k^3 - 40k^3\)
34. \(xyz - 8xyz + 5xyz - 2xyz\)
35. \(-d^2 - 1d^2 + 0.5d^2 + 1.8d^2\)
36. \(5(r + s) - 6(r + s) - 8(r + s)\)
37. \(4x^2 - 5x - 6x^2 + 7x\)
38. \(-13y + 2y^2 - 4y^2 + 6y\)
39. \(m^2 - 2m - 3 - m^2 + 4m\)
40. \(u^3 - 2u^2 + 4u + 3u^2 - 4u\)

**B**

41. \(-2(r + 3s) + 5(-r - s)\)
42. \(-7(a - 3b) + 9(-b + a)\)
43. \(3(p - 2q) - (5q - 2p)\)
44. \(-4(-7v + t) - (-t - 3v)\)
Evaluate each of the following algebraic expressions, using \( a = -1, b = .2, c = -2, x = .3, \) and \( y = -3. \)

49. \( .7xy - .5b \)
50. \( .3c - .2ab \)
51. \( -\frac{1}{2}b + .4y - \frac{1}{3}x \)

52. \( .2(a + 1)^2 \)
53. \( a^2 + .2a + .1 \)
54. \( 1.3(y + 1)^3 \)

55. \( y^3 - 3y^2 + 3y - 1 \)
56. \( .04x^3 + .05y^3 \)
57. \( .01c^4 - .01a^4 \)

58. \( a(b + 3c)^3 \)
59. \( c(2b - y)^2 \)

60. \( x^2(y - a^3) \)
61. \( y^3(b^2 + a^5) \)
62. \( 1.1(2a^2 - b)(2a^2 + b) \)
63. \( 1.01(3y + x^2)(3y - x^2) \)

4–9 Dividing Directed Numbers

You know the following two facts for arithmetic numbers: (1) division by zero is meaningless; (2) division is the inverse operation of multiplication, that is, \( 6 \div 2 = 3 \) because \( 3 \cdot 2 = 6 \); and division may be written in the form of a fraction, that is, \( 6 \div 2 = \frac{6}{2} \). To give the division of directed numbers the same properties, this definition is made: for all directed numbers \( a \) and \( b \), the directed number \( x \) satisfying the equation \( xb = a \) is called the quotient of \( a \) and \( b \left( a \div b \text{ or } \frac{a}{b} \right) \).

You have been studying opposites, or additive inverses. Now you shall need to know about multiplicative inverses or reciprocals. Two numbers whose product is \( 1 \), the identity element for multiplication, are called reciprocals. For example,

\[
6 \cdot \frac{1}{6} = 1, \quad .2 \cdot 5 = 1, \quad \frac{m}{n} \cdot \frac{n}{m} = 1, \quad 1 \cdot 1 = 1
\]

\[
-\frac{1}{3}(-5) = 1, \quad -.25(-4) = 1, \quad -1 \cdot (-1) = 1.
\]

That each number except zero has a reciprocal is a basic assumption.

For every directed number \( a \) other than 0, there is a unique number, denoted by \( \frac{1}{a} \), such that \( a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1. \)

Do you see that a number and its reciprocal are either both positive or both negative numbers?
A statement about reciprocals that corresponds to the property of opposites on page 121 is:

The reciprocal of a product of two numbers, each different from 0, is the product of the reciprocals of the numbers.

\[
\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}
\]

Reciprocals enable you to express a quotient as a product. If \( b \neq 0 \), the root of \( xb = a \) is \( a \cdot \frac{1}{b} \), as you can show.

\[
xb = a \\
xb \cdot \frac{1}{b} = a \cdot \frac{1}{b} \\
x \cdot 1 = a \cdot \frac{1}{b} \\
x = a \cdot \frac{1}{b}
\]

This last equation has just one root, \( a \cdot \frac{1}{b} \). Checking in the original equation:

\[
xb = a \\
\left( a \cdot \frac{1}{b} \right) b = a \\
a \left( \frac{1}{b} \cdot b \right) = a \\
a \cdot 1 = a \\
a = a \, \checkmark
\]

Since the one and only root of \( xb = a \) is \( a \cdot \frac{1}{b} \), it follows that:

\[
a \div b = a \cdot \frac{1}{b}, \quad b \neq 0
\]

To perform a division, replace the divisor by its reciprocal, and multiply.
Does this rule give a meaningful expression for \( a \div b \)? As every number except 0 has a reciprocal, and \( b \neq 0 \), there is a number \( \frac{1}{b} \). As \( a \cdot \frac{1}{b} \) is a product, it represents a definite number. Thus, the set of directed numbers is closed under division, not including division by 0.

**EXAMPLE 1.** \( \frac{-12}{-3} = -12 \left( -\frac{1}{3} \right) = 4 \)

**EXAMPLE 2.** \( \frac{-2}{5} = -2 \left( \frac{1}{5} \right) = -\frac{2}{5} \)

**EXAMPLE 3.** \( \frac{2}{-5} = 2 \left( -\frac{1}{5} \right) = -\frac{2}{5} \)

**EXAMPLE 4.** \( \frac{2}{3} \div \left( -\frac{4}{5} \right) = \frac{2}{3} \left( -\frac{5}{4} \right) = -\left( \frac{2}{3} \cdot \frac{5}{4} \right) = -\frac{5}{6} \)

**ORAL EXERCISES**

Give each of the following quotients.

1. \( \frac{12}{3} \)  
6. \( \frac{0}{-8} \)  
11. \( \frac{-3}{15} \)  
16. \( (-12b) \div 4 \)

2. \( \frac{12}{-3} \)  
7. \( \frac{3}{3} \)  
12. \( \frac{-15}{15} \)  
17. \( (-50a) \div (-10) \)

3. \( \frac{-12}{3} \)  
8. \( \frac{-5}{-5} \)  
13. \( a \div 1 \)  
18. \( (-60y) \div (-12) \)

4. \( \frac{-12}{-3} \)  
9. \( \frac{5}{15} \)  
14. \( a \div (-1) \)  
19. \( \frac{a}{a} \text{ for } a \neq 0 \)

5. \( \frac{0}{3} \)  
10. \( \frac{4}{-4} \)  
15. \( 6a \div (-2) \)  
20. \( -\frac{a}{a} \text{ for } a \neq 0 \)

Solve each of the following equations.

21. \( 5r = -10 \)  
22. \( 6s = -18 \)  
23. \( -2t = 8 \)  
24. \( -3t = 6 \)  
25. \( -4t = -8 \)  
26. \( -3t = -9 \)  
27. \( -5z = 0 \)  
28. \( 0 = -3z \)  
29. \( -4 = 3w \)  
30. \( 3 = -5w \)  
31. \( -7 = -7x \)  
32. \( -5x = -5 \)
Give the multiplicative inverse of each number.

33. \( \frac{1}{33} \)  
34. \( \frac{1}{-1} \)  
35. \( \frac{3}{4} \)  
36. \( \frac{10}{36} \)  
37. \( -\frac{9}{8} \)  
38. \( -\frac{3}{10} \)

In Exercises 43–46, find the value of \( a \).

43. \( \frac{1}{a} = -2 \)  
44. \( \frac{1}{a} = -\frac{1}{3} \)  
45. \( \frac{1}{a} = -\frac{4}{3} \)  
46. \( \frac{1}{a} = -1 \)

Determine which of the following sets of numbers are closed under (a) division, and (b) multiplication.

47. \{1\}  
48. \{-\frac{1}{2}, -1, \frac{1}{2}, 1\}  
49. \{-1, 0, 1\}  
50. \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}
51. \{3, 1, \frac{1}{3}\}  
52. \{0\}  
53. \{\text{positive numbers}\}  
54. \{\text{negative numbers}\}  
55. \{-1, -2, -3, -4, \ldots\}  
56. \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{8}, \ldots\}

**WRITTEN EXERCISES**

Find each of the indicated quotients.

**A**

1. \((-\frac{3}{2}) \div 3\)  
2. \(-\frac{.25}{-5}\)  
3. \((-12) \div (-\frac{1}{3})\)  
4. \((-\frac{4}{3}) \div 4\)  
5. \(-\frac{.36}{6}\)  
6. \((-16) \div (-\frac{1}{4})\)  
7. \((36) \div (-\frac{1}{2})\)  
8. \(24.5 \div -5\)  
9. \((3b) \div (-\frac{1}{3})\)  
10. \((8) \div (-\frac{1}{3})\)  
11. \(2.52 \div -7\)  
12. \((7c) \div (-\frac{1}{3})\)  
13. \((-15.6) \div 12\)  
14. \(1.82 \div (-13)\)  
15. \((-1.195) \div (-13)\)  
16. \((-39.1) \div 17\)  
17. \(4.56 \div (-24)\)

Evaluate each of the following expressions. Use \( r = -2, s = -1, t = 3, u = -6, v = 10 \).

**B**

19. \(\frac{vu}{rst}\)  
20. \(\frac{rt^2}{su}\)  
21. \(\frac{r^2}{s^3}\)  
22. \(\frac{rt^2}{u}\)  
23. \(\frac{uv^2}{r^3}\)  
24. \(\frac{su^2}{r^2}\)  
25. \(\frac{s^6}{rt}\)  
26. \(\frac{s^6}{uv}\)  
27. \(\frac{t^3}{r^2u}\)  
28. \(\frac{r^3}{t^2v}\)  
29. \(\frac{(r + 2)^4}{u}\)  
30. \(\frac{(u + 6)^3}{r}\)
Averages and Directed Numbers (Optional)

The arithmetic average, or mean, of a set of \( n \) numbers is their sum divided by \( n \). Sometimes you can reduce the labor by using directed numbers as follows:

1. Assume an average; that is, make a guess.
2. Express as a directed number the difference (or deviation) between each number and your guess.
3. Find the average of these differences (the average deviation).
4. Add the average deviation to the assumed average (your guess), and you have the correct average.

**EXAMPLE**

Excluding the ends, the linemen on the State University football team weighed as follows: 227 lb., 194 lb., 200 lb., 189 lb., 230 lb. Find the average weight.

**Solution:**

<table>
<thead>
<tr>
<th>1. Assume an average of 200 pounds.</th>
<th>Assumed Average: 200 lb.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Subtract the assumed average from each weight to get the deviations.</td>
<td>Weight</td>
</tr>
<tr>
<td></td>
<td>227</td>
</tr>
<tr>
<td></td>
<td>194</td>
</tr>
<tr>
<td></td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>230</td>
</tr>
</tbody>
</table>

3. Divide the sum of the deviations by the number of cases to find the average deviation.

\[
\text{Sum of deviations: } 40
\]

\[
\text{Average deviation: } \frac{40}{5} = 8
\]

4. Add the average deviation to the assumed average to find the true average.

\[
\text{True average } = 200 + 8 = 208 \text{ lb., Answer.}
\]
The better your guess, the smaller the average deviation, but what you guess really doesn't matter. Suppose you assumed 210 pounds as the average.

<table>
<thead>
<tr>
<th>Weight</th>
<th>Deviation</th>
<th>Sum of deviations:</th>
<th>Average deviation:</th>
<th>True average =</th>
<th>Answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>227</td>
<td>17</td>
<td></td>
<td></td>
<td>-10</td>
<td>208 lb.</td>
</tr>
<tr>
<td>194</td>
<td>-16</td>
<td></td>
<td></td>
<td>-10/5 = -2</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>-10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>189</td>
<td>-21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>230</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To see why this method works, suppose \( M \) represents the true average of numbers represented by \( a, b, c, d, e, \) and \( f \) and assume an average \( m \). The deviations are \( a - m, b - m, c - m, d - m, e - m, \) and \( f - m \), where \( m \) is the assumed average.

\[
\text{Sum of deviations} = a + b + c + d + e + f - 6m
\]

\[
\text{Average deviation} = \frac{a + b + c + d + e + f - 6m}{6}
\]

\[
\therefore \text{Average deviation} = \frac{M}{6} - m
\]

or, \( M = m + \text{average deviation} \).

Of course, the same method would have worked for 7, 8, or any number of members, as well as for 6.

**Problems**

Do each problem by assuming an average and finding the deviations from it.

1. Find the average of the following tuition charges: $2200, $1980, $2050, $1880, $2100, and $1930.

2. Weather balloons released on successive days reached these altitudes (in meters): 7650, 8630, 5600, 9550, 8550, 7550, 8520. Find the average altitude reached by these balloons.

3. In test dives, a diving bell reached these depths: \(-5200 \text{ ft.}, -5600 \text{ ft.}, -5900 \text{ ft.}, -6100 \text{ ft.}, -6500 \text{ ft.}\). Find the average depth.

4. Find the average of the following weights: 17.4 g., 16.6 g., 16.7 g., 17.9 g., 15.9 g., 17.3 g., and 17.5 g.
5. Fahrenheit temperature readings taken at 8 A.M. each morning during one week were: 13°, 5°, —6°, —9°, —14°, —4°, 2°. Find the average 8 A.M. temperature for that week.

6. The following daily net changes in the price of one stock were observed in the course of two weeks: $-1\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{1}{2}$, $2\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{1}{2}$, $3\frac{1}{2}$, 0, $-\frac{1}{4}$, $4\frac{1}{2}$. Find the average daily net change in price.

7. At different times, eight scientists determined experimentally the value of a constant, as follows: 4.177, 4.188, 4.196, 4.196, 4.186, 4.188, 4.184, 4.181. Find the average value.

8. A high school physics class was trying to verify a rule. They obtained the following values for a constant: 1654.8, 1655.6, 1654.8, 1643.5, 1645.1, 1646.0, 1649.6, 1645.8. Find the average value.

Chapter Summary

Inventory of Structure and Method

1. The number scale can be extended to the left as well as to the right of the zero-point. Points to the left correspond to negative numbers, whose numerals include minus signs. Points to the right correspond to positive numbers, whose numerals may include plus signs. For example, $-3$ marks the point 3 units to the left of 0; $+3$, or 3, the point 3 units to the right of 0.

A number $a$ is greater than every number to its left and less than every number to its right on the number line.

2. The sum $a + b$, for any numbers $a$ and $b$ on the number line is found by moving from $a$ a distance $|b|$ units in the direction associated with $b$. Every number $a$ on the number line has an opposite or additive inverse such that $a + (-a) = 0$. Thus, $+(3) = -3$ and $-(3) = +3$. $-0 = 0$.

The addition of directed numbers can be developed without reference to the number line if you assume closure, commutative, and associative properties, and properties of zero and of opposites.

3. The difference $a - b$ is defined as that number which added to $b$ gives $a$. Subtracting $b$ is the same as adding the opposite of $b$. The set of directed numbers is closed under subtraction.

4. You can deduce other properties of multiplication of directed numbers from the closure, commutative, associative, and distributive properties and properties of 1 and 0. You have $(−1) \cdot a = −a$. For $a$ and $b$, any
directed numbers, \( |ab| = |a| \cdot |b| \), \( ab > 0 \) if \( a \) and \( b \) are both positive or both negative, and \( ab < 0 \) if one factor is positive and the other, negative. If \( a = 0 \) or \( b = 0 \), \( ab = 0 \).

5. If \( b \neq 0 \), the quotient \( a \div b \), or \( \frac{a}{b} \), is defined as that number which multiplied by \( b \) gives \( a \); \( \frac{a}{0} \) is not defined. Every number \( a, a \neq 0 \), has a reciprocal or multiplicative inverse, \( \frac{1}{a} \); \( a \) and \( \frac{1}{a} \) are both positive or both negative. Zero has no reciprocal. \( \frac{a}{b} = a \left( \frac{1}{b} \right), b \neq 0 \).

6. (Optional) A relatively easy way of finding an average \( A \) of a set of \( n \) numbers is to use a guessed average \( G \). Subtract \( G \) from each number to find its deviation. \( A = G + \) the average deviation.

**Vocabulary and Spelling**

- two-way scale (p. 111)
- signs of direction (p. 112)
- positive number (p. 112)
- negative number (p. 112)
- magnitude (p. 112)
- directed number (p. 112)
- signed number (p. 112)
- displacement (p. 116)
- identity element for addition (p. 117)

- opposite of a number (p. 120)
- additive inverse (p. 120)
- absolute value (p. 123)
- reciprocal (p. 138)
- identity element for multiplication (p. 138)
- multiplicative inverse (p. 138)
- average (p. 142)
- deviation (p. 142)

**Chapter Test**

4-1 Use a number to express the following.

1. a. 3 seconds before blast-off
   b. the longitude of the Greenwich meridian

4-2 2. The greater of the two numbers, 3 and \(-5\), is \( \frac{3}{2} \).

3. Graph the solution set for the variable on the number line. Consider the set of directed numbers as the replacement set.
   a. \(-2 \leq r \leq 0 \)
   b. \( p < -2 \)
4. Translate into an open sentence and give the replacement set for the variable: In making a tool, a company specifies that the diameter should be 3 inches, but may be as much as 1% off.

5. Add: \((47) + [(−71) + (−14)]\)

6. A submarine submerged to a depth of 125 feet rises 80 feet. What is its new position with respect to the surface?

7. Using \(\{-3, -1, 0, 3, 4\}\) as the replacement set for \(s\), solve
   \[
   \text{a. } −s = 3 \\
   \text{b. } −s > 0
   \]

8. If \(|x| = 2\), then \(x\) may equal \(\pm 2\) or \(\pm 2\).

9. Add: \((-92) + (−36) + (153) + (−87)\)

10. Perform the indicated operations: \(15 − 44 − 1 + 7 − 12\)

11. Subtract the sum of 19 and \(-5\) from their difference.

12. A family moves from a city whose average temperature is 61.3°F to a city whose average temperature is 47.6°F. Find the change in average temperature.

13. Perform the indicated operations: \(59(−18) − 32(−59)\)

14. Simplify: \(3(a − b) − 7a + 3b\)

15. Evaluate: \(-5m^4 \cdot \frac{1}{3}n^{17}\), when \(m = −3\) and \(n = −1\)

16. Multiply and simplify: \(5(ab^4 − 8b^5) − 3(a^3b − 3a^2b^2)\)

17. Solve:
   \[
   \text{a. } −1.56 = 13t \\
   \text{b. } −16h = −12
   \]

18. Write the multiplicative inverse of:
   \[
   \text{a. } −\frac{1}{8} \\
   \text{b. } 1\frac{3}{4}
   \]

19. Find the quotient: \(36x^9y^8 ÷ (−24)\)

20. (Optional) Using \(-3°\) as the assumed average, find the true average of these hourly temperatures:
   \(-5°, −5°, −3°, −1°, 0, 3°, 10°, 13°\).

**Chapter Review**

4-1 Directed Numbers

1. Numerals with plus signs designate points to the ___ of the zero-point.
2. Numerals with ___ signs designate points to the left of zero.
3. Zero is neither ___ nor ___.
4. The number corresponding to the point 4.5 units to the left of the origin is written ___.
5. Positive and negative numbers are known as ___ or ___ numbers because their numerals contain signs of direction.
Questions 6–9 refer to the number line below.

Identify each of the following quantities by a lettered point.

6. A 3-point penalty
7. A discharge of 2.5 amperes
8. A rise of 1 1/2 points
9. The ground floor of a house

4–2 Comparing Numbers

In the following, consider to the right and up as positive directions.

10. A number on the horizontal number line is \( ? \) than any number to the left of it.
11. Zero is less than any number to the \( ? \) of it on the horizontal scale.
12. Saying that \( n < 0 \) is equivalent to saying that \( n \) is a \( ? \) number.
13. A number on the vertical scale is \( ? \) than any number above it.

State which number in each pair is the greater.

14. \(-3, -2\)
15. \(-3, 2\)
16. \(-3, 0\)
17. \(-3, -5\)

In Exercises 18–20, consider the replacement set of the given variable to be \( \{-5, -4, -2, 0, 1, 4\} \). Find the solution set in each case.

18. \( n > 0 \)
19. \( s \leq -2 \)
20. \(-3 < w < 1\)

4–3 Addition on the Number Line

21. Addition of 5 means a displacement of 5 units to the \( ? \).
22. Addition of \(-3\) means a displacement of 3 units to the \( ? \).
23. A displacement of \(-2\) from \(-5\) brings you to \( ? \).

Perform the indicated additions.

24. \((-24) + (-37)\) + 45
25. \(48 + (-71) + (-19 + 42)\)

Using the set of directed numbers as the replacement set for the variable, solve each of the following.

26. \( m + (-21) = -12 \)
27. \( r + (-17) = -17 \)
4-4 The Opposite of a Directed Number

28. The opposite of a positive number is the ___ number having the ___ absolute value.
29. The opposite of zero is ___.
30. The identity element for addition is ___.
31. When the sum of two numbers is ___, each is the opposite or the ___ ___ of the other.
32. The opposite of the sum of two numbers is the sum of the ___ of the numbers.

If \( \{ -4, -2, -1, 0, 1, 4 \} \) is the replacement set for \( x \), find the solution set.

33. \(-x = 1\)  
34. \(-x < 1\)  
35. \(-x > -2\)

4-5 Absolute Value

36. The absolute value of a number corresponds to its ___ from zero, regardless of the direction.
37. The absolute value of a number is also called its ___.
38. The absolute value of \(-3\) is written ___ and equals ___.

For each number, give another having the same absolute value.

39. 3.2  
40. \(-12\)  
41. \(3\frac{1}{2}\)  
42. \(-0.085\)

For Exercises 43-46, find the solution set. The replacement set is \( \{ -5, -4, -2, 0, 1, 4 \} \).

43. \(4 = |u|\)  
44. \(-3 = |t|\)  
45. \(-4 < |v|\)  
46. \(2 \leq |-s|\)

4-6 Adding Directed Numbers

47. The sum of positive numbers is a ___ number.
48. The sum of negative numbers is a ___ number.
49. When you add a positive and a negative number,
   a. The numerical value of the sum is the ___ between the greater absolute value and the smaller.
   b. The sum is positive if the smaller absolute value belongs to the ___ number.
   c. The sum is negative if the greater absolute value belongs to the ___ number.

In Exercises 50-51, find the sum.

50. \((17) + (-32) + (-28) + (13)\)
51. \((-48) + (73) + (21) + (-46)\)
Subtracting Directed Numbers

In Exercises 52–53, (a) rewrite the expression as simply as possible, and (b) find the sum.

52. \((382) + (-425) + (-36) + (291)\)
53. \((-78) + (112) + (259) + (-407)\)
54. \(n - 58 - (-91)\)
55. \(-39 - (-77 + t)\)
56. Subtraction is the _?_ of addition.
57. Subtracting _-5_ is the same as adding _?_.

In each of Exercises 58–59, (a) rewrite the expression in terms of addition only, (b) find the result.

58. \(26 - 59 - (-12)\)
59. \((-17) - (83) - (-62)\)
60. From the sum of 38 and \(-83\) subtract \(-29\).
61. Subtract the sum of \(-47\) and 74 from \(-12\).
62. \(r + 28 - (-81)\)
63. \(-32 - s + 47\)
64. Find the change in temperature when it dropped from 1° below zero to 8° below zero.
65. An ancient Greek lived from 421 B.C. to 373 B.C. How old was he when he died?

Multiplying Directed Numbers

66. The product of two positive numbers or of two negative numbers is a _?_ number.
67. The product of a positive number and a negative number is a _?_ number.
68. The multiplicative property of \(-1\) is that multiplying a number by \(-1\) gives the _?_ of the number.

In Exercises 69–73, simplify the expression.

69. \(-38(7)\)
70. \(-14(31 - 12)\)
71. \((-6\frac{1}{2})(17) - 6\frac{1}{2}(-15)\)
72. \(\frac{h}{4} - (-8)\)
73. \(-3 - (-\frac{1}{8}k)\)

In each of the following expressions, combine similar terms.

74. \(13a - 20b - 3a - 11b\)
75. \(32m - 48mn + mn - m\)
76. \(-83(27)(-7)(0) = \_\)
77. \((-43)(-52)(10)(-1)^{32} = \_\)

Evaluate each expression using \(m = -1, r = 2, s = -2, t = 3\).
78. \(-5st^2\)
79. \(-5m(r^3 - s^3)\)

### 4-9 Dividing Directed Numbers

Pages 138–142

80. The quotient of two positive or two negative numbers is \_\_\_.
81. The quotient of a positive and a negative number is \_\_\_.
82. Zero may not be used as a \_\_\_.
83. When the dividend is 0 and the divisor is not 0, the quotient is \_\_\_.

In Exercises 84–85, find the quotient.
84. \((-30) ÷ (-45)\)
85. \((-18) ÷ [3 ÷ (-\frac{1}{6})]\)

Solve for the indicated variable.
86. \(-14v = 126\)
87. \(-36w = \frac{9}{4}\)
88. \(1 = -\frac{3}{2}x\)
89. The product of a number and its reciprocal is \_\_\_.
90. Every number, except \_\_\_, has one, and only one, reciprocal or \_\_\_.
91. The quotient of two directed numbers may be expressed as the product of the dividend and the \_\_\_ of the divisor.
92. The identity element for multiplication is \_\_\_.

Give the reciprocal of each of the following.
93. \(-2\)
94. \(2\frac{1}{3}\)
95. \(\frac{2}{3}\)
96. \(\frac{2}{a - b}, a \neq b\)

State whether each of the following sets is closed under division.
97. \(\{\frac{1}{2}, 1, 2\}\)
98. \(\{\text{the reciprocals of the directed numbers}\}\)

### 4-10 Averages and Directed Numbers (Optional)

Pages 142–144

99. The difference between the value being considered and the assumed average is called the \_\_\_ and is expressed as a \_\_\_ number.
100. Using 5 as an assumed average, find the average of \(-12, -3, 0, 2, 7, 13, 14\).
In planning the vast and intricate networks of modern communications, the engineer must have a sound mathematical background. One particularly significant achievement of electronic engineering, the successful operation of undersea cables, for example, would not be possible if compensations were not made for the loss of power (due to the cable’s resistance to the electric current) over the long distance to be covered. The engineer must calculate the power loss and design the devices, called repeaters, which amplify the current at regular intervals and prevent the signal from dying out. The photograph shows a laboratory model of a cable repeater which is being tested under a variety of extreme conditions. More than 100 such repeaters are used in an Atlantic cable; they must function perfectly for years, without needing any maintenance.

Illustrated on the work pad is a more general type of problem in electrical engineering. In order to adapt a motor for the available source of current, an engineer wishes to reduce the voltage from 90 volts to 30 volts. He installs a resistor network consisting of a 100-ohm resistor, $R_1$, and another resistor, $R_2$, whose strength he must determine.

The measurement of the resistance in ohms ($R$), is the ratio of the voltage across the resistor to the current through the resistor, or $R = \frac{E}{I}$. By applying the formula

$$\frac{R_1}{R_1 + R_2} \cdot E_1 = E_2,$$

the engineer calculates that the second resistor, $R_2$, must be 200 ohms in strength.
Clock Arithmetic

A number system is specified by giving a set of numbers and telling how to add and multiply the numbers. However, the names addition and multiplication may not mean the familiar operations of adding and multiplying directed numbers.

Consider a system consisting of just five numbers \{0, 1, 2, 3, 4\}. To compute a sum in this system, imagine a number line which forms a closed circle like a five-hour clock; then start at the first addend, and count clockwise the number of spaces named by the second addend. For example, \(3 + 4 = 2\) would be computed as shown in the figure. By similar means, verify each sum in the addition table.

To multiply in this set of clock numbers, take \(\times\) to mean \(b + b + b + \ldots + b\), to \(a\) terms. Thus, \(2 \times 3 = 3 + 3 = 1\), and \(3 \times 4 = 4 + 4 + 4 = 2\). By counting, check each product in the multiplication table.

### Questions

Compute in the system of clock numbers \{0, 1, 2, 3, 4\} for Questions 1-3.

1. **a.** \((3 + 3) + 4\) and \(3 + (3 + 4)\)  
   **b.** \((2 + 3) + 1\) and \(2 + (3 + \)  
   **c.** What property of numbers do these results illustrate?

2. Does the system of clock numbers have the commutative property for addition? multiplication? Give two examples to illustrate each answer.

3. Is the system closed under addition? subtraction? multiplication?
4. What is the identity element for addition? for multiplication?

5. Solve the following equations in the system.
   a. $x + 3 = 2$
   b. $2 + y = 0$
   c. $s + 4 = 3$

6. Solve in the system.
   a. $3x = 2$
   b. $3 = 2w$
   c. $2t + 1 = 4$

7. Why are negative numbers unnecessary in clock arithmetic?

8. In the clock arithmetic of a twelve-hour clock dial:
   a. What number is the identity element for addition? Give four examples.
   b. Does this number also have the multiplicative property of zero? Give four examples.

---

**Magic Squares**

A magic square is divided into a number of smaller squares, called **cells**, with a different number written in each cell, so that the sum of each row, of each column, and of each diagonal is the same. A magic square of the third order has three rows and three columns. Illustrated is a pure magic square of the third order.

A pure magic square is a magic square containing consecutive integers. There are eight ways to arrange the numbers in a pure magic square of the third order. The number 5 is always in the center, and even numbers are in the four corners. You can easily make a magic square of the third order which is not pure by multiplying the number in each cell by $-1$ or 2 or any other number. How does the distributive property assure you that such a square will still be magic?

You may construct a magic square of 9, 25, 49, or any other number of cells (so long as the number is the square of an odd number) by following these rules.

1. Write the digits in order, putting each in a separate cell. Begin by writing 1 in the middle cell of the top row.

2. Move from cell to cell by going up and to the right one step at a time. If you find yourself going out of the square, or getting into a cell that is already filled, make the move and then proceed as follows:
   a. When a move takes you out at the top of the square, drop down to the lowest cell in that column. When a move takes you out at the right of the square, shift to the cell farthest left in that row.

---

![Magic Square Example](8|1|6)
3|5|7
4|9|2
b. When a move takes you to a cell already filled, drop down one row, instead. When a move takes you out the upper right-hand corner, drop down one row, instead.

Using these rules to construct a pure magic square of the third order, as the figure shows, your very first move takes you out at the top of the square, so you drop down to the bottom of that column, and put 2 in the lowest cell. Your second move takes you out at the right of the square, so you shift to the left of that row, and put 3 in the left-hand cell.

Your next move also leads to difficulty; it takes you to a cell already filled. So you drop down one row, and put 4 directly beneath 3. Your next two moves are unobstructed.

You are now at the upper right-hand corner of the square. Therefore, you drop down one row, and put 7 beneath 6. The next move takes you out at the right again, so you shift left. And the final move takes you out at the top, so you drop to the bottom!

![Image of a magic square]

Of course, the reason your moves take you out so often is that the square is small. Try building a magic square of the fifth order, and one of the seventh order.

In a pure magic square of the third order, as you surely have discovered, the sum of each row, column, and diagonal is 15. In a pure magic square of the fifth order the sum is 65, and in one of the seventh order it is 175. You can find this sum for any pure magic square if you know how many cells are on each side. If \( n \) represents this number, the sum is \( \frac{3}{2}(n^3 + n) \).

<table>
<thead>
<tr>
<th>Third order: ( \frac{3}{2}(3^3 + 3) = \frac{3}{2} \cdot 30 = 15 )</th>
<th>Fifth order: ( \frac{3}{2}(5^3 + 5) = \frac{3}{2} \cdot 130 = 65 )</th>
<th>Seventh order: ( \frac{3}{2}(7^3 + 7) = \frac{3}{2} \cdot 350 = 175 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 3 2 13</td>
<td>5 10 11 8</td>
<td>9 6 7 12</td>
</tr>
<tr>
<td>4 15 14 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Of course, if the square is not pure you multiply this result by the smallest number in the square.

So far, nothing has been said about magic squares containing an even number of cells. But there are such squares, and you can find the sum of their columns, rows, and diagonals by the same formula. You cannot construct them by the same rules, however. They are based on an entirely different principle.
Aladdin, Sindbad, and Harun al-Rashid — these names are familiar to you from *The Arabian Nights*. Have you supposed them to be the names of mythical characters? Harun al-Rashid, at least, was real. He reigned in Bagdad from 796 to 808. He not only went about among his subjects, as recorded in *The Book of a Thousand Nights and a Night*; he also encouraged his nobles to study science and mathematics.

During his reign the Mohammedans were invading and conquering the non-Moslem lands to the west of them, including northern Africa. Thus the learning that centered at the University of Alexandria became a prize of war: books in Greek were brought to Bagdad from Alexandria. It was Harun al-Rashid who caused them to be translated into Arabic.

Soon Mohammedan scholars were engaged in the serious study of mathematics. Among these scholars was the caliph's son, Al-Mamun. By the time he succeeded his father as caliph, there was, in the words of *The Arabian Nights* (for Al-Mamun also figures in those tales), "none more accomplished in all branches of knowledge than he." It was only natural that his court should include many learned men.

One of Al-Mamun's scholarly courtiers — the greatest mathematician in all Arabia — was Al-Khowarazmi. He wrote a book with this title: *ilm al-jabr wa'l muqabalah*. The book was called *al-jabr* for short. This shortened title has been used for many books from Al-Khowarazmi's day to the present. Of course, the spelling has changed. Now it is usually spelled *algebra*.

*Arabian scholars at work. When this picture was made, Bagdad was the center of learning. Interest in astronomy made knowledge of algebra necessary, and the development of algebra made possible increased knowledge of astronomy.*
CHAPTER 5

MOON ORBIT

INSUFFICIENT INITIAL VELOCITY

23.700 MPH OR LESS

4 DAYS TRANSIT TIME

23.730 MPH INITIAL VELOCITY

2 1/2 DAYS TRANSIT TIME

23.860 MPH INITIAL VELOCITY

1 1/2 DAY TRANSIT TIME

24.200 MPH INITIAL VELOCITY

EARTH ROTATION
As the picture indicates, missiles differ. So do numbers. Although most of your concern in this course has been with equal expressions, it is necessary to consider conditions which may exist among numbers that are unequal. A grasp of such situations is necessary for a full understanding of numbers. Also, complex problems which are solvable only in terms of inequalities arise in applied mathematics. Here is an example:

The time of flight of a ballistic missile fired at the moon is extremely sensitive to the initial velocity of the final stage. An initial velocity of 23,700 m.p.h. will be insufficient to reach the moon. However, increasing this value by only 30 m.p.h. would make a four-day earth-moon mission possible. Increasing this velocity an additional 130 m.p.h. would reduce the transit time to 2.5 days.

OPEN SENTENCES IN THE SET OF DIRECTED NUMBERS

5-1 Transforming Equations

Transformations used in the set of arithmetic numbers to replace one equation by an equivalent equation are valid also in the set of directed numbers. For example, the same directed number may be added to or subtracted from the value of each member of an equation without changing the solution set of the equation. Also, multiplying or dividing each member by a nonzero directed number produces an equivalent equation.
EXAMPLE 1. Solve:  \( x + 36 = 1 - 4(x - 5) \)

Solution:

1. Copy the equation; use the distributive property; combine similar terms.

2. Subtract 36 from each member, or add \(-36\) to each member.

3. Add \(4x\) to each member.

4. Divide each member by 5.

Check is left to you.

EXAMPLE 2. Solve for \(t\):

\[ I = prt \]

Solution:

\[ I = \frac{pr}{r} \]

\[ I = \frac{pr}{pr} \]

\[ I = t \]

EXAMPLE 3. Solve for \(g\):

\[ d = \frac{1}{2}gt^2 \]

Solution:

\[ 2d = 2\left(\frac{1}{2}gt^2\right) \]

\[ 2d = gt^2 \]

\[ \frac{2d}{t^2} = \frac{gt^2}{t^2} \]

\[ \frac{2d}{t^2} = g \]

ORAL EXERCISES

Solve for the variable in red, or for the indicated variable.

1. \( x + 5 = -7 \)  
2. \( x + 3 = -8 \)  
3. \( y - 8 = -12 \)  
4. \( y - 1 = -29 \)  
5. \(-2z = +8 \)  
6. \(-3z = +15 \)  
7. \( \frac{t}{3} = -11 \)  
8. \( \frac{t}{-5} = -9 \)  
9. \( 5k = 4k - 1 \)  
10. \(-2 = -2 + 5k \)  
11. \(-12m + 8 = -13m \)  
12. \(-6m + 2 = -7m \)  
13. \( 5r + 5 = 0 \)  
14. \(-7r + 7 = 0 \)  
15. \( 3x = a \)  
16. \( \frac{x}{2} = b \)  
17. \( \frac{x}{a} = bc \)  
18. \( \frac{1}{2}y = a - \frac{1}{2}y \)
Solve for the variable in red, or for the indicated variable.

A

1. \(18y = 203 - 11y\)
2. \(15x = 144 - 9x\)
3. \(7 = \frac{n}{2} - 1\)
4. \(9 = \frac{x}{5} - 1\)
5. \(19 = 11 - \frac{n}{5}\)
6. \(20 = 1 - \frac{7s}{2}\)
7. \(360 + 36z = 30z\)
8. \(714 + 38r = 21r\)
9. \(39b = 171 + b\)
10. \(106x = 540 + 34x\)
11. \(-20y = 221 + 6y\)
12. \(-47w = 40 + 13w\)
13. \(rsx = 4rs\)
14. \(mnx = mn\)
15. \(x + 6a = 7a\)
16. \(x - 4a = a\)
17. \(\frac{2y}{3} + 7 = 5\)
18. \(\frac{3w}{4} + 13 = 7\)
19. \(\frac{5y}{2} = .3\)
20. \(\frac{3x}{.5} = .6\)
21. \(w + l = 3w - 2l\)
22. \(t - 2h = 5t + 3h\)
23. \(y - a = 2b\)
24. \(3a = p + 2c\)
25. \(p = a - \text{prt}\)
26. \(p = a - \text{prt}\)
27. \(3x - 7b = x\)
28. \(3m + 4y = 2y\)
29. \(p = a + b + c\)
30. \(s = c + o + p\)
31. \(P = \frac{W}{t}\)
32. \(p = \frac{w}{A}\)
33. \(A = \frac{bh}{2}\)
34. \(V = \frac{Bh}{3}\)
35. \(A = 4\pi r^2\)
36. \(C = 2\pi r\)
37. \(V = \frac{1}{3}\pi r^2h\)
38. \(V = \frac{2}{3}\pi r^3\)
39. \(V = \frac{1}{3} Bh\)
40. \(9 - \frac{1}{2}(3n - 6) = n + 2\)
41. \(2n - \frac{3}{8}(4n + 12) = n + 7\)
42. \(5(x + 1) = 4(x + 2)\)
43. \(3(8x - 2) = 3(4 + 2x)\)
44. \(4(3n + 2) = 6(3 - n) + 8\)
45. \(5(x + 1) = 3(x + 2)\)
46. \(5r + (r + 15) - 3(r + 2) = 0\)
47. \(69 = 4(h + 5) - (h - 1)\)
48. \(5(y + 2) = 13 + 4(2y - 1)\)
49. \(5(y + 2) = 6 + 3(2y - 1)\)
50. \(3(x - 9) = 74 - 2(61 + 2x)\)
51. \(2w - 4(w + 2) = 5(w + 4)\)

5-2 The Properties of Inequality

When you consider two different numbers, you know that one is larger than the other. The order property of numbers is usually stated:

For each pair of directed numbers \(a\) and \(b\), one and only one of the following statements is true: \(a < b\); \(a = b\); \(a > b\).
The meaning of this fact on the number line is shown in these figures.

\[ a < b \]
\[ a = b \]
\[ a > b \]

You can see from the following number line that if point \(a\) lies to the left of point \(b\) and if point \(b\) lies to the left of point \(c\), then point \(a\) is to the left of point \(c\).

This illustrates the **transitive property of inequality** in the set of directed numbers:

For any directed numbers \(a, b,\) and \(c\):

1. if \(a < b\) and \(b < c\), then \(a < c\); similarly,
2. if \(a > b\) and \(b > c\), then \(a > c\).

On the number line, \(-4\) lies to the left of \(3\). If you move 5 units from \(-4\) and also 5 units *in the same direction* from \(3\), you arrive at points in the same order as \(-4\) and \(3\).

\[ 5 \quad 5 \]
\[ -4 \quad 0 \quad 1 \quad 3 \quad 8 \]
\[ -9 \quad -4 \quad -2 \quad 0 \quad 3 \]

From \(-4 < 3\)

it follows that \(-4 + (5) < 3 + (5)\) or \(1 < 8\)

and \(-4 + (-5) < 3 + (-5)\) or \(-9 < -2\).

This illustrates the **additive property of inequality**:

For any directed numbers \(a, b,\) and \(c\):

1. if \(a < b\), then \(a + c < b + c\) and \(a - c < b - c\); similarly,
2. if \(a > b\), then \(a + c > b + c\) and \(a - c > b - c\).
Notice what happens if you multiply each member of the inequality
\(-4 < 3\) by 2. Since \((-4)(2) = -8\) and \((3)(2) = 6\), and also \(-8 < 6\), it follows that
\((-4)(2) < (3)(2)\)

Thus, multiplying each member by 2 preserves the order, direction, or sense of the inequality.

On the other hand, multiply each member of \(-4 < 3\) by \(-2\). Since \(8 > -6\), it follows that
\((-4)(-2) > (3)(-2)\).

Thus, multiplying each member of the inequality by \(-2\) reverses the sense of the inequality.

When you multiply an inequality by a directed number, you must take into account the direction associated with the multiplier. The multiplicative property of inequality states:

For any directed numbers \(a, b,\) and \(c,\) if \(a < b,\) then

1. \(ac < bc,\) when \(c > 0,\)

2. \(ac = bc,\) when \(c = 0,\)

3. \(ac > bc,\) when \(c < 0;\) similarly,

4. \(\frac{a}{c} < \frac{b}{c},\) when \(c > 0,\) and

5. \(\frac{a}{c} > \frac{b}{c},\) when \(c < 0.\)

You can find the solution set of an inequality by transforming it into an equivalent inequality, that is, one with the same solution set, by use of the properties of inequality.
EXAMPLE

Solve the inequality $7x - 13 < 3x - 1$, and graph its solution set.

Solution:

1. Copy the inequality.  
   $7x - 13 < 3x - 1$

2. Add 13 to each member.  
   $7x - 13 + 13 < 3x - 1 + 13$  
   $7x < 3x + 12$

3. Subtract $3x$ from each member.  
   $7x - 3x < 3x + 12 - 3x$  
   $4x < 12$

4. Divide each member by 4.  
   $\frac{4x}{4} < \frac{12}{4}$  
   $x < 3$

   $\therefore$ The solution set is  
   \{all directed numbers less than 3\}

5. Graph the solution set.

-2 -1 0 1 2 3

If you transform an inequality such as

$12 \geq -6r$

by dividing by negative six, remember to reverse the order of the inequality:

$-2 \leq r$.

ORAL EXERCISES

State the transformation you use to solve each of these inequalities.

SAMPLE.  

$- \frac{h}{5} > 12$

What you say: Multiply each member by $-5$ and reverse the order of the inequality; $h < -60$.  

The solution set is \{all directed numbers less than $-60$\}.
Solve each inequality. In Exercises 1–22, show also the graph of the solution set.

**A**

1. \(5a - 1 \geq 9\)  
2. \(6b - 11 \leq 13\)  
3. \(2 - 3s < 11\)  
4. \(5 - 2t > 17\)  
5. \(7 + \frac{z}{4} \leq 0\)  
6. \(8 + \frac{w}{3} \geq 0\)  
7. \(-14 \leq 3t - 2\)  
8. \(26 \geq -10 + 4n\)  
9. \(\frac{3p}{2} - 1 > -3\)  
10. \(\frac{3q}{2} - 2 < -5\)  
11. \(3x + 5 < x - 5\)  
12. \(y - 1 > 9 - 4y\)  
13. \(-6w + 5 + w > -(13 + w)\)  
14. \(-6v - 2 + v < 13 + v\)  
15. \(6x - 3(4 - 2x) \geq 0\)  
16. \(-y + 2(9 - y) \leq 0\)  
17. \(-12 \left(\frac{t}{6} - \frac{1}{3}\right) < 2t\)  
18. \(15 \left(\frac{1}{5} - \frac{w}{3}\right) > -2w\)  
19. \(0 \geq -17(6r - 2)\)  
20. \(0 \leq -51(4v + 3)\)  
21. \(-2(3m - 6) < 6(2 + m)\)  
22. \(15(-4 - z) > -5(12 - 3z)\)

**B**

23. \(\frac{3}{10}(2t - 20) + 5 \leq \frac{t - 60}{10}\)  
24. \(\frac{5}{6}(18 - g) - 1 \geq \frac{g + 90}{6}\)  
25. \(2(h - 1) < 2h\)  
26. \(3(1 + y) > 3y\)  
27. \(3(|x| + 1) \leq 3 - |x|\)  
28. \(-2(2|n| - 1) \geq \frac{1}{2}(4 - 6|n|)\)  
29. \(-2(1 - 7b) < 7(2b - 5) - 17\)  
30. \(-3(2 + 5c) > 1 - 5(3c - 1)\)  
31. \(7(d + 3) - 5(d - 3) \geq 2(d + 20) - 4\)  
32. \(3(k - 4) - (k + 8) \leq -2 + 2(k - 9)\)
5-3 Pairs of Inequalities (Optional)

Often you are interested in variables whose values must satisfy two inequalities at the same time. For example, the solution set of the sentence $-1 < x < 4$ consists of those numbers for which both of the inequalities $-1 < x$ and $x < 4$ hold.

Similarly, to solve the inequalities

1. $-2 < x + 4 \leq 5$

you must find the values of $x$ for which

$$-2 < x + 4 \quad \text{and} \quad x + 4 \leq 5.$$ 

Simplifying each of these inequalities by subtracting 4 from each member, you find

$$-2 - 4 < x + 4 - 4 \quad \text{and} \quad x + 4 - 4 \leq 5 - 4$$

$$-6 < x \quad \text{and} \quad x \leq 1.$$ 

Thus, the pair of inequalities 1 is equivalent to the pair of inequalities 2. $-6 < x \leq 1$

The graph appears as:

Sometimes you want to find the values of a variable for which at least one of a pair of inequalities is true. The inequality

$$|2y - 1| > 5$$

will be satisfied provided either

$$(2y - 1) > 5 \quad \text{or} \quad -(2y - 1) > 5$$

To simplify these inequalities, first multiply each member of the second inequality by $-1$ and obtain this pair:

$$2y - 1 > 5 \quad \text{or} \quad 2y - 1 < -5$$

Add 1 to each member: $2y > 6 \quad \text{or} \quad 2y < -4$

Divide each member by 2: $y > 3 \quad \text{or} \quad y < -2$
Thus the graph of the solution set of \(|2y - 1| > 5\) consists of the two sets of points indicated below.

In dealing with two inequalities it is essential to decide whether you want the set of numbers satisfying both of them or the set satisfying at least one of them. For example, the set satisfying both of the inequalities \(x \leq 3\) and \(x > 3\) is the empty set, whereas the set of numbers satisfying at least one of them is the set of directed numbers!

**WRITTEN EXERCISES**

Solve each pair of inequalities, and graph the solution set.

**SAMPLE 1.** \(t + 2 \geq -4\) and \(t - 3 < 4\)

**Solution:**

\[
\begin{align*}
    t + 2 & \geq -4 \\
    t + 2 - 2 & \geq -4 - 2 \\
    t & \geq -6 \\
    t - 3 & < 4 \\
    t - 3 + 3 & < 4 + 3 \\
    t & < 7 \\
\end{align*}
\]

\(\therefore\) \(-6 \leq t < 7\)

\(\text{Answer.}\)

**SAMPLE 2.** \(t + 2 \geq -4\) or \(t - 3 < 4\)

**Solution:**

\[
\begin{align*}
    t + 2 & \geq -4 \\
    t + 2 & \geq -4 \quad \text{or} \quad t - 3 < 4 \\
\end{align*}
\]

\(\therefore\) \(t \geq -6\) or \(t < 7\)

The solution set is the set of directed numbers, the graph being the entire number line:

\(\text{Answer.}\)

1. \(-3 \leq x + 4 \leq 0\)
2. \(1 < 5 + y \leq 7\)
3. \(-8 \leq -1 + 3a < 11\)
4. \(-7 \leq 4b - 5 \leq 19\)
5. \(|x| \geq 0\)
6. \(|b - 1| > 0\)
7. \(m - 1 > -1\) and \(m - 2 \leq 0\)
8. \(4 + n < -3\) and \(-2 + n \geq -9\)
5–4 A Plan for Solving Problems

You have already solved some problems using methods outlined in Chapter 2 (page 57). The first step helps you to determine what you are to find. To take the second step, you must know what facts the problem gives you; therefore, ask yourself these questions: What does the problem ask? What facts does the problem give? It is helpful, when the problem permits, to make a sketch that illustrates it.

Example 1

A certain roll has 10 fewer calories than twice the number of calories in a slice of white bread. Together they contain at least 185 calories. Find the smallest possible number of calories in the slice of white bread.

Solution:

1. What does the problem ask? The smallest possible number of calories in a slice of white bread.
   
   Let \( x \) = this number of calories.
   
   The roll has \( 10 \text{ fewer than twice} \) the number of calories in the slice of bread.
   
   Therefore, \( 2x - 10 \) will represent the number of calories in the roll.

2. What other facts does the problem give?

<table>
<thead>
<tr>
<th>calories in roll</th>
<th>together with</th>
<th>calories in bread</th>
<th>are at least</th>
<th>185</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x - 10 )</td>
<td>+</td>
<td>( x )</td>
<td>( \geq )</td>
<td>185</td>
</tr>
</tbody>
</table>
3 Solve the inequality: \[ 2x - 10 + x \geq 185 \]
\[ 3x \geq 195 \]
\[ x \geq 65 \]

4 Check in the words of the original problem.

If the slice of bread contains at least 65 calories, then the roll has at least \[ 2x - 10 = 2(65) - 10 = 120 \] calories.

Is the sum of 65 + 120 at least 185? \[ 185 \geq 185 \, \checkmark \]

The slice of white bread contains at least 65 calories, Answer.

**EXAMPLE 2**

Ben’s age is four years less than three times that of his younger sister Amy. Half of Ben’s age increased by Amy’s age is 2 years more than twice Amy’s age. Find their ages.

*Solution:*

1 Let \( n \) = Amy’s age in years.

Then \( 3n - 4 = \) Ben’s age.

2 \[
\begin{array}{c}
\text{Half of Ben’s age} \\
\frac{1}{2}(3n - 4)
\end{array} 
\quad \text{increased by} \quad \begin{array}{c}
\text{Amy’s age} \\
n
\end{array} 
\quad \text{is} \quad \begin{array}{c}
2 \text{ years} \\
2
\end{array} \quad \text{more than} \quad \begin{array}{c}
\text{twice} \\
2n
\end{array}
\]

3 \[
\frac{3}{2}n - 2 + n = 2 + 2n
\]
\[
\frac{1}{2}n = 4
\]
\[
n = 8
\]
\[
3n - 4 = 3(8) - 4 = 20
\]

4 Is Amy’s age 8 years? Is Ben’s age 20 years?

a. Half of Ben’s age added to Amy’s age = \[ \frac{20}{2} + 8 = 18 \]
b. Two years more than twice Amy’s age = \[ 2 + 2(8) \frac{2}{2} = 18 \]

\[ 18 = 18 \, \checkmark \]

Amy’s age = 8 years 
Ben’s age = 20 years }

Answer.
Take the four steps given on page 57 in solving each problem, making a sketch when possible.

1. Michael and Robert are going fishing. Michael owns the boat; therefore the boys have agreed that he is to get 5 more fish than Robert. If the total catch is 19 fish, how many will each receive?

2. The Red Cross knitted 50 sweaters within ten days. The Junior Red Cross assisted, contributing 2 dozen fewer than the senior organization. How many did the Junior Red Cross knit?

3. The Jowett family budgets part of its weekly income of $150 for food. Half the remainder of the income exceeds the amount spent on food by from $15 to $30. How much do they spend on food per week?

4. Tom and Otto picked 36 quarts of berries. Tom picked 3 more than half the number Otto picked. How many did each pick?

5. Mrs. Abbott decided on Christmas Day to save $150 for next Christmas. She started saving $5 a week, but at the end of 12 weeks saw she could reduce that amount. By how much could she reduce her weekly saving and still have $150 or more at the end of the year?

6. The length of a playground exceeds twice its width by 25 feet, and 650 feet of fencing are needed to enclose it. Find its dimensions.

7. The length of a rectangle exceeds three times the width by 6 feet, and the perimeter is 188 feet. Find the dimensions of the rectangle.

8. In a certain puzzle, the larger of two numbers must exceed three times the smaller by 5, and their difference must be at least 31. Find the least possible value of the smaller number.

9. To be called “Limited” a train’s average speed must be 5 miles an hour more than twice the average speed of a “Local.” If the Limited travels at 63 miles an hour, what is the speed of the Local?

10. A child’s bank contained twice as many nickels as pennies and two-thirds as many dimes as nickels, the total value being at least $3.65. Find the smallest possible number of coins in the bank.

11. Bill Jones wanted Sally Smith’s telephone number. Sally said that ninety added to her age equaled six times her telephone number, minus 6060. Bill knew that Sally was eighteen years old, but he didn’t know enough algebra to call her. Find Sally’s telephone number.

12. The area of a twelve-foot square equals the area of a rectangle 9 feet wide. Find the length of the rectangle.
13. A rectangle is 9 feet by 8 feet. Its area is three times the area of a rectangle 12 feet long. Find the width of the second rectangle.

14. In a new school building, 270 cubic feet of air are to be allowed for each pupil. To meet this requirement, what should be the height of the ceiling of a classroom, 30 feet by 24 feet, seating 36 pupils?

15. A coal bin is 15 feet long, 10 feet wide, and 9 feet high. If $10\frac{1}{2}$ tons of egg coal which runs 28 pounds to the cubic foot are put in, to what height will the coal reach?

16. A bicycle wheel has a diameter of 2 feet. How many revolutions will it make in going 5500 feet? (Hint: The rule for finding the circumference of a circle is given by the equation $c = \pi d$; use $\pi = \frac{35}{11}$.)

17. David is making a model of a rectangular solid from a piece of wire 52 inches long. The length is to be twice the width, and the height is to be 1 inch more than the width. What are the dimensions of the solid?

18. An isosceles triangle is a triangle having two sides equal in length. The base of an isosceles triangle is a whole number and is 4 feet less than the sum of the two equal sides. The perimeter is a whole number between 0 and 75 feet. Find the possible lengths of each side.

19. Mrs. Fry weighs 50 pounds less than her husband. Their combined weight is at least 220 pounds more than that of their daughter, who weighs half as much as Mr. Fry. What is Mrs. Fry's minimum weight?

20. Mr. Martin earns three times as much in his regular job as he does as a writer. His total income is at least $14,000 more than that of his sister, who earns only half as much as Mr. Martin does in his regular job. What is the least amount he earns in his regular job?

21. Farmer Brown needs .03 acre of land to grow 1 bushel of corn and .06 acre to grow 1 bushel of wheat. He has at most 480 acres of land for planting and wants to use at least half of that acreage. If he decides to grow twice as much corn as wheat, find (a) the maximum and (b) the minimum number of bushels of corn he can grow.

22. In a factory the time required to assemble a table is 20 minutes and a chair, 30 minutes. The factory has at least 126 hours of labor available each day, but can provide as much as 140 hours daily. If four times as many chairs as tables are produced, find the minimum and the maximum number of chairs the factory can produce.
Problems about Consecutive Integers

Integer is another name for any whole number, positive, negative, or zero. The integers have many interesting properties, and to talk about them you need a few descriptive terms. An integer which is twice some integer is called even; all others are called odd. For example, 2, 126, 0, −10 are even integers; 3, −15, 77 are odd.

The word consecutive is used here to mean following in order, just exactly as in ordinary language when you say, “I got A in algebra for three consecutive weeks.” Counting by ones gives consecutive integers: 1, 2, 3, 4, . . . . The three largest consecutive two-digit integers are 97, 98, 99. Likewise, −6, −5, −4, −3, is a set of consecutive integers. Counting by twos from an even integer gives consecutive even integers: 2, 4, 6, 8, or −4, −2, 0. Counting by twos from an odd integer gives consecutive odd integers: 15, 17, 19, or −5, −3, −1, 1. Some consecutive multiples of five are 5, 10, 15, 20.

Two consecutive integers differ by 1; two consecutive even integers differ by 2. Two consecutive odd integers also differ by 2. If \( x \) represents any integer, then \( x + 1 \) is the next larger integer and \( x - 1 \) is the next smaller integer. If \( x \) represents an even integer, then \( x + 2 \) is the next larger even integer. What is the next smaller even integer? If \( x \) is an odd integer, then \( x + 2 \) is the next larger odd integer. What is the next smaller odd integer?

**EXAMPLE**

Find three consecutive integers whose sum is 48.

*Solution:*

1. Let \( n \) = the first integer.

Then \( (n + 1) \) = the second integer and \( (n + 2) \) = the third integer.

2. The sum of the integers is 48: \( n + (n + 1) + (n + 2) = 48 \)

3. Solve the equation: \( 3n + 3 = 48 \)

\[ 3n = 45 \]
\[ n = 15 \]

\[ \therefore n + 1 = 16 \]
and \( n + 2 = 17 \)
4. Is the sum of these integers 48?  \[ 15 + 16 + 17 = 48 \]
\[ 48 = 48 \checkmark \]
The three consecutive integers are 15, 16, 17, Answer.

**ORAL EXERCISES**

1. Represent 4, 6, 7, and 8 in terms of \( k \), if \( k \) equals 5.
2. Let \( m \) represent 14. How can you represent the numbers 13, 15, 16, and 17 in terms of \( m \)?
3. Let \( n \) be the first number in the series 10, 12, 14, and 16. Express the other numbers in terms of \( n \).
4. Represent 5, 7, 9, and 11 in terms of \( h \), letting \( h = 5 \).
5. Let \( n \) represent any even integer. What is the next even integer? the preceding even integer?
6. Let \( x \) represent any odd integer. What is the next odd integer? the preceding odd integer?
7. Let \( x \) represent any even integer. Is \( (x + 1) \) an even integer or an odd integer? \( (x - 1) \)?
8. Let \( n \) be any integer. Is \( 2n \) an even integer or an odd integer? Is \( (2n + 1) \) an even integer or an odd integer? \( (2n - 1) \)?
9. Represent 9, 8, and 7, in that order, letting \( n = 9 \).
10. Represent the numbers 46, 47, and 48, letting \( k = 48 \).
11. If \( (x + 7) \) is an integer, what is the next smaller integer?
12. If \( (z - 1) \) is an integer, what is the next larger integer?

**PROBLEMS**

1. The sum of two consecutive integers is 57. Find the numbers.
2. The sum of two consecutive integers is 75. Find the numbers.
3. Find three consecutive odd integers whose sum is 57.
4. Find three consecutive odd integers whose sum is 111.
5. Find four consecutive even numbers whose sum is 100.
6. Find four consecutive even numbers whose sum is 164.
7. Find three consecutive integers, if the sum of the first and third is 128.
8. Find four consecutive integers, if the sum of the third and fourth is 63.
9. George Dean plans to use 60 inches of lumber for four shelves whose lengths are to be a series of consecutive even numbers. How long shall he make each shelf?

10. The sides of a triangle are consecutive numbers. If the perimeter of this triangle is 240 feet, find the length of each side.

11. The smaller of two consecutive even integers is 2 more than twice the larger. Find the numbers.

12. The larger of two consecutive odd integers is 4 less than \( \frac{1}{3} \) the smaller. Find the numbers.

13. Find four consecutive integers such that five times the fourth diminished by twice the second is 7.

14. Find four consecutive even integers such that four times the fourth decreased by one-half the second is 9.

15. Three times the smaller of two consecutive odd integers is less than twice the larger. What are the largest possible values for the integers?

16. Three consecutive even integers are such that their sum is more than 24 decreased by twice the third integer. What are the smallest possible values for the integers?

17. The larger of two consecutive integers is greater than 4 more than half the smaller. What are the smallest possible values for the integers?

18. Three consecutive integers are such that the sum of the first and third is less than 18 increased by half the second. What are the largest possible values for the integers?

5–6 Problems about Angles

Think of the figure composed of two rays \( p \) and \( q \) drawn from a point \( O \). Then think of the ray \( q \) as having turned or rotated about \( O \), starting at \( p \) and going to its indicated position. As shown, the rotation may be clockwise or counterclockwise.
The figure composed of two rays drawn from a point, together with the rotation that sends one ray into the other is called a **directed angle**. Counterclockwise rotation yields a **positive directed angle**; clockwise rotation yields a **negative directed angle**. Ray $p$ is the **initial side** of the angle and ray $q$ is the **terminal side**. The point $O$ is the **vertex** of the angle.

A common unit of measure of an angle is a **degree**, written as $1^\circ$. A **degree** is $\frac{1}{360}$ of a complete rotation of a ray about a point. The directed angles whose measures are $1^\circ$, $30^\circ$ (read “30 degrees”), $90^\circ$, $180^\circ$, $-45^\circ$, $-180^\circ$, and $-360^\circ$ are shown:

Two angles are **complementary angles** if the sum of their measures is $90^\circ$. Each is the **complement** of the other. If an angle contains $n$ degrees, its complement contains $(90 - n)$ degrees.

Two angles are **supplementary angles** if the sum of their measures is $180^\circ$. Each is the **supplement** of the other. If an angle contains $n$ degrees, its supplement contains $(180 - n)$ degrees. The diagrams on the next page show complementary and supplementary angles.
EXAMPLE. How large is an angle whose supplement contains $21^\circ$ less than four times its complement?

Solution:

1. Let $n =$ the number of degrees in the angle.
   
   Then $(90 - n) =$ the number of degrees in its complement,
   
   and $(180 - n) =$ the number of degrees in its supplement.

2. $\begin{align*}
   \text{Supplement} & \quad = \quad \text{four times the complement} \quad \text{less} \quad 21 \\
   (180 - n) & \quad = \quad 4(90 - n) \quad - \quad 21
\end{align*}$

Steps 3 and 4 are left to you.

The three line segments that compose a triangle intersect by pairs and so form three angles. If you tear off the corners from any paper triangle and fit them together as shown in Figure 5-1, you will notice...
that the three angles fit together to form a straight angle. This suggests a property of all triangles which is proved in geometry.

The sum of the measures of the angles of any triangle is $180^\circ$.

\[ \text{Figure 5–1} \]

**ORAL EXERCISES**

Give the complement of each angle.

1. $20^\circ$  
2. $70^\circ$  
3. $25^\circ$  
4. $15^\circ$  
5. $60^\circ$  
6. $80^\circ$  
7. $-12^\circ$  
8. $-22^\circ$  
9. $33^\circ$  
10. $73^\circ$  
11. $3x$ degrees  
12. $2n$ degrees  
13. $-110^\circ$  
14. $-120^\circ$  
15. $-90^\circ$  
16. $-0^\circ$

State the number of degrees in the supplement of each angle.

17. $50^\circ$  
18. $70^\circ$  
19. $105^\circ$  
20. $115^\circ$  
21. $-94^\circ$  
22. $-84^\circ$  
23. $a$ degrees  
24. $2a$ degrees  
25. $-150^\circ$  
26. $-135^\circ$  
27. $0.75^\circ$  
28. $\frac{1}{3}^\circ$  
29. $(n - 3)$ degrees  
30. $(x - 10)$ degrees  
31. $(2n + 5)^\circ$  
32. $(3x - 10)^\circ$

**WRITTEN EXERCISES**

In each exercise, two angles of a triangle are given. Find the number of degrees in the third angle.

1. $20^\circ$, $70^\circ$  
2. $30^\circ$, $80^\circ$  
3. $60^\circ$, $60^\circ$  
4. $110^\circ$, $40^\circ$  
5. $100^\circ$, $60^\circ$  
6. $13^\circ$, $139^\circ$  
7. $n^\circ$, $2n^\circ$  
8. $\frac{3}{4}n^\circ$, $\frac{1}{3}n^\circ$  
9. $m^\circ$, $n^\circ$  
10. $x^\circ$, $90^\circ$  
11. $n^\circ$, $(n + 30)^\circ$  
12. $2n^\circ$, $(n - 20)^\circ$
In Exercises 13–18, find the number of degrees in \( a + b \), if the measures of \( a \) and \( b \) are as indicated.

13. \( a = 30^\circ \), \( b = \frac{1}{2} \) of a complete rotation clockwise
14. \( b = 15^\circ \), \( a = \frac{1}{4} \) of a complete rotation clockwise
15. \( a = \) a positive straight angle, \( b = \) a negative straight angle
16. \( a = \frac{3}{4} \) of a complete rotation clockwise
   \( b = \frac{1}{2} \) of a complete rotation counterclockwise
17. \( a = \frac{1}{6} \) of a complete rotation counterclockwise
   \( b = \frac{1}{3} \) of a complete rotation clockwise
18. \( a = \frac{1}{4} \) of a complete rotation counterclockwise
   \( b = \frac{1}{4} \) of a complete rotation clockwise

Exercises 19 and 20 refer to the Law of Reflection: \( i = r \).

19. \[
\begin{align*}
i &= (2n + 30)^\circ \\
r &= (4n - 10)^\circ 
\end{align*}
\]
Find \( n \).

20. \[
\begin{align*}
a &= 2m^\circ \\
b &= (m + 10)^\circ 
\end{align*}
\]
Find \( m \).

Exercises 21–26 refer to the science of navigation in which a compass direction is expressed as a bearing. The bearing of a line of motion is the angle it makes with the north line, measured clockwise from north, through a point at which the observations are made. Find each bearing.

21. \[
\begin{align*}
&\text{N} \\
&\text{W} \\
&\text{E} \\
&\text{S}
\end{align*}
\]

22. \[
\begin{align*}
&\text{N} \\
&\text{W} \\
&\text{E} \\
&\text{S}
\end{align*}
\]

23. \[
\begin{align*}
&\text{N} \\
&\text{W} \\
&\text{E} \\
&\text{S}
\end{align*}
\]
24. An angle is 12° more than its complement. Find the number of degrees in the complement.

25. Find two complementary angles if one is 28° less than the other.

26. An angle is 15° less than twice its complement. Find the angle.

2. Find two complementary angles if one is 18° less than 3 times the other.

5. Find two supplementary angles if one is four times the other.

6. Find two supplementary angles if one is five times the other.

7. One angle of a triangle is twice as large as another. The third angle contains 5° more than the larger of these. Find each angle.

8. One angle of a triangle is three times as large as another. The third angle is 20° less than the sum of the first two angles. Find the number of degrees in each angle.

9. In any isosceles triangle two angles are equal to each other. The third angle of one isosceles triangle is 36° less than the sum of the other two. Find each angle of the triangle.

10. Two angles of a triangle are equal, but the third angle is 23° less than 2\(\frac{1}{3}\) times the sum of the first two. How many degrees are in each angle?

11. One angle of a triangle exceeds another by 23°. The third angle is 6° less than the sum of the other two. Find the angles.

12. A triangle is to be drawn in which one angle is 18° larger than another, and the third, 12° less than the sum of the others. Find the angles.

13. How large is an angle whose complement contains 5° more than half its supplement?

14. How large is an angle whose supplement contains 12° less than twice its complement?
CHAPTER FIVE

5-7 Uniform Motion Problems

An object which moves without changing its speed is said to be in uniform motion. Often, charts can help you in organizing the given facts in problems involving uniform motion. The basic principle you will need in such cases is:

\[ d = r \times t \]

**EXAMPLE 1.** (Motion in Opposite Directions) Mr. Rush and Mr. Slow arrange to meet at an airport that is between, and in a straight line with, their home airports. Mr. Rush’s jet travels at 600 miles per hour; Mr. Slow’s plane travels at 320 miles per hour. They leave their home airports, which are 1380 miles apart, at the same time. If each plane is scheduled for a nonstop flight, in how many hours will they meet?

**Solution:**

1. Let \( n \) = the number of hours before the men meet.

2. Make a sketch illustrating the given facts.
   - Mr. Rush’s jet rate is 600 m.p.h.
   - Mr. Slow’s plane rate is 320 m.p.h.
   - Total distance is 1380 miles.
   - Each travels the same number of hours.

   \[
   600n + 320n = 1380
   \]

<table>
<thead>
<tr>
<th>Rule</th>
<th>( r \times t = d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mr. Rush</td>
<td>600</td>
</tr>
<tr>
<td>Mr. Slow</td>
<td>320</td>
</tr>
</tbody>
</table>

Mr. Rush’s jet distance + Mr. Slow’s plane distance = Total distance

\[
600n + 320n = 1380
\]
Solve the equation:

\[ 600n + 320n = 1380 \]
\[ 920n = 1380 \]
\[ n = 1 \frac{1}{2} \]

To check whether the men met in 1 hour and a half, you have to answer this question: How far did each man fly?

Mr. Rush flew 600 miles: \( 600 \cdot 1 \frac{1}{2} = 900 \) miles

Mr. Slow flew 320 miles: \( 320 \cdot 1 \frac{1}{2} = 480 \) miles

The sum of these distances is 1380 miles

\[ 1380 = 1380 \checkmark \]

The men will meet in 1 hour and a half, Answer.

**EXAMPLE 2.** (Motion in the Same Direction) An airplane which maintains an average speed of 350 miles per hour passed an airport at 8 A.M. A jet following that course, at a different altitude, passed the same airport at 10 A.M. and overtook the airplane at noon. At what rate was the jet flying?

Solution:

Let \( x \) = the rate of the jet in m.p.h.

Make a chart of the facts given in the problem.

<table>
<thead>
<tr>
<th>Rule</th>
<th>( r \cdot t = d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airplane</td>
<td>350</td>
</tr>
<tr>
<td>Jet</td>
<td>( x )</td>
</tr>
</tbody>
</table>

Rate of airplane is 350 m.p.h.

Periods of time under consideration

Airplane: 8 A.M. to noon, or 4 hours

Jet: 10 A.M. to noon, or 2 hours

Each plane covered the same distance.

Make a sketch illustrating the facts given in the problem.

Distance of jet = Distance of airplane

\[ 2x = 1400 \]

Steps 3 and 4 are left to you.
EXAMPLE 3. (Round Trip) A man leaves his home and drives to a convention at an average rate of 50 miles per hour. Upon arrival, he finds a telegram advising him to return at once. He catches a plane that takes him back at an average rate of 300 miles per hour. If the total traveling time was $1\frac{3}{4}$ hours, how long did it take him to fly back? How far from his home was the convention?

Solution:

Let $h =$ number of hours flown.
Then $\frac{7}{4} - h =$ number of hours driven.

The given facts are these: a. The total time is $1\frac{3}{4}$ hours. b. The driving rate is 50 miles per hour. c. The flying rate is 300 miles per hour. d. The number of miles driven is the same as the number of miles flown.

<table>
<thead>
<tr>
<th>Rule</th>
<th>$r \cdot t = d$</th>
<th>Distance driven</th>
<th>Distance flown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driving</td>
<td>$50 \frac{7}{4} - h$</td>
<td>$50(\frac{7}{4} - h)$</td>
<td>$50(\frac{7}{4} - h) = 300h$</td>
</tr>
<tr>
<td>Flying</td>
<td>$300h$</td>
<td>$50(\frac{7}{4} - h)$</td>
<td>$300h$</td>
</tr>
</tbody>
</table>

Steps 3 and 4 are left to you.

PROBLEMS

Make a drawing and a chart for each problem. Then form the equation, solve it, and check your answer in the words of the problem.

1. A westbound jet leaves Central Airport traveling 625 miles an hour. At the same time, an eastbound plane departs at 325 miles an hour. In how many hours will the planes be 1900 miles apart?

(continued on page 181)
Number System Structure

Faced with such problems as counting the animals in his flock or comparing the size of his warrior band with that of his enemy, man eventually conceived the natural or counting numbers. To count you need a first number, and you have to know what number comes next after any given number. Hence, two important properties of the set of natural numbers are: (i) there is a first natural number, namely, 1; (ii) every natural number has an immediate successor in the set, such that between a natural number and its immediate successor no other natural numbers can be inserted. Consequently, the set of natural numbers is usually pictured as a succession of equally spaced points extending without end in one direction along a line.

The operations of addition and multiplication arise when you seek to count the members in the set formed by combining the elements in two or more sets having no members in common. For example, if you have 3 pennies in one row and 2 in another row, you have, in all, 5 pennies, a fact expressed by the symbols: \(3 + 2 = 5\). On the other hand, if you have 3 pennies in each of 2 rows, you have 6 pennies in all: \(3 \times 2 = 6\). This suggests the property of closure for addition and multiplication:

For any natural numbers \(a\) and \(b\) the sum \(a + b\) and the product \(ab\) are both definite natural numbers.

This assumption implies not only that sums and products exist in the set of natural numbers, but also that they are unique. If you replace \(a\) and \(b\) by natural numbers \(c\) and \(d\) such that \(a = c\) and \(b = d\), then \(a + b = c + d\) and \(ab = cd\).

The invention of a number for the empty set was a fairly late but significant advance in algebra. When we adjoin 0 to the set of natural numbers, we call the enlarged set, the system of whole numbers. Zero, the first whole number, is the unique number such that for every number \(a\), \(0 + a = a + 0 = a\).

Order in the set of whole numbers can be defined in terms of addition: \(a < b\) or \(b > a\) provided that the equation \(a + x = b\) has a solution in the set of positive whole numbers. For example, \(2 < 3\) because \(2 + 1 = 3\). Whole numbers are also called integers. The positive integers are the whole numbers greater than 0. If \(a \leq b\), \(b - a\) is the whole number whose sum with \(a\) is \(b\). For example, \(3 - 2 = 1\), but \(2 - 3\) does not exist in the set. The quotient \(a\) exists in this set only if \(b \neq 0\) and \(a\) is a multiple of \(b\); that is, \(a = qb\) for a whole number \(q\). In this case, \(\frac{a}{b} = q\).

Operations with the whole numbers are assumed to have the basic properties of arithmetic stated on page B. Representing the structure of the system of whole numbers by the incomplete diamond pictured on that page suggests that a number system containing only whole numbers is also incomplete.

Once man advanced from the counting process to the problem of measuring such quantities as length and weight, he found that whatever standard measuring units he used, he met lengths and weights which did not contain the standard unit a whole number of times. Therefore, he divided his standard units into halves, thirds, quarters, and so on, thus introducing such common fractions as \(\frac{1}{2}\), \(\frac{1}{3}\), and \(\frac{1}{4}\).
WHOLE NUMBERS

COMMUTATIVE PROPERTIES

\[ a + b = b + a; \quad ab = ba \]

ASSOCIATIVE PROPERTIES

\[ (a + b) + c = a + (b + c); \quad (ab)c = a(bc) \]

DISTRIBUTIVE PROPERTIES

\[ a(b + c) = ab + ac; \quad a(b - c) = ab - ac \]

PROPERTIES OF EQUALITY

- \[ a = a. \]
- If \( a = b \), then \( b = a. \)
- If \( a = b \) and \( b = c \), then \( a = c. \)
- If \( a = b \), then:
  1) \[ a + c = b + c \quad \text{and} \quad ac = bc; \]
  2) \[ a - c = b - c, \quad \text{if} \ a \geq c; \]
  3) \[ \frac{a}{c} = \frac{b}{c}, \quad \text{if} \ c \neq 0 \quad \text{and} \quad a \text{ is a multiple of } c. \]

PROPERTIES OF ONE AND ZERO

- \[ 1 \cdot a = a; \quad 1 + a = a; \quad 0 \cdot a = a; \quad 0 + a = 0 \]

PROPERTIES OF INEQUALITY

One, and only one, of these statements is true for each \( a \) and each \( b \): \( a = b, a < b, b < a. \)

- If \( a < b \) and \( b < c \), then \( a < c. \)
- If \( a < b, \) then:
  1) \[ a + c < b + c, \quad \text{and, if} \ 0 < c, \ ac < bc; \]
  2) \[ a - c < b - c, \quad \text{if} \ a \geq c; \]
  3) \[ \frac{a}{c} < \frac{b}{c}, \quad \text{if} \ 0 < c \quad \text{and} \quad a \text{ and } b \text{ are multiples of } c. \]
Later appeared fractions like \( \frac{2}{3}, \frac{3}{2} \), and, in general, \( \frac{a}{b} \), where \( a \) is any integer and \( b \) is any positive integer. Notice that \( \frac{1}{b} \) is defined to be the unique number having the property that \( 2 \cdot \frac{1}{b} = 1 \), while \( \frac{1}{b} \) is the number for which \( b \cdot \frac{1}{b} = 1 \). Then, just as \( \frac{2}{3} = 2 \cdot \frac{1}{3} \) or \( \text{two-thirds} \), so also does \( \frac{a}{b} = a \cdot \frac{1}{b} \).

The extended number system contains the whole numbers; for example, \( 1, \frac{1}{2}, \frac{2}{3} = \frac{2}{2} \), and \( 0 = \frac{0}{1} \). Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the positive rational numbers. A rational number, but not a positive rational number, contains 0 and the positive rational numbers, you use the following rules. Assume that \( b \neq 0 \) and \( d \neq 0 \).

1. \( \frac{a}{b} = \frac{c}{d} \), if and only if, \( ad = bc \)
2. \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \)
3. \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \)
4. \( \frac{a}{b} > \frac{c}{d} \), if, and only if, \( ad > bc \)
5. \( \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \), if \( a > c \)
6. \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \), if \( c \neq 0 \)

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the property of reciprocals:

For every nonzero number \( a \) there is a unique number, denoted \( \frac{1}{a} \) and called the reciprocal or multiplicative inverse of \( a \), such that

\( a \cdot \frac{1}{a} = 1 \).
POSITIVE RATIONAL NUMBERS AND ZERO

The set of positive rational numbers and zero satisfies all the basic laws of arithmetic stated on page B. In addition, it has the following properties:

PROPERTIES OF CLOSURE: For all numbers $a$ and $b$ and nonzero number $c$ in the set $a + b$, $ab$ and $\frac{a}{c}$ are definite numbers in the set.

PROPERTIES OF RECIPROCALS: 1) For every nonzero number $a$ in the set there is in the set a unique number, denoted by $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

2) $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$

PROPERTY OF DENSITY: Between any two different numbers in the set is another number in the set.

Since the set is closed under division, excluding division by zero, the last of the properties of equality and of inequality on page B can be stated more simply.

If $a = b$ and $0 \neq c$, then $\frac{a}{c} = \frac{b}{c}$.

If $a < b$ and $0 < c$, then $\frac{a}{c} < \frac{b}{c}$. 
Later appeared fractions like \( \frac{2}{3}, \frac{3}{4} \), and, in general, \( \frac{a}{b} \), where \( a \) is any integer and \( b \) is any positive integer. Notice that \( \frac{1}{2} \) is defined to be the unique number having the property that \( 2 \cdot \frac{1}{2} = 1 \), while \( \frac{1}{b} \) is the number for which \( b \cdot \frac{1}{b} = 1 \). Then, just as \( \frac{2}{3} = 2 \cdot \frac{1}{3} \) or two-thirds, so also does \( \frac{a}{b} = a \cdot \frac{1}{b} \).

The extended number system contains the whole numbers; for example, \( 1 = \frac{1}{1} \), \( 2 = \frac{2}{1} \), and \( 0 = \frac{0}{1} \). Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the *positive rational numbers*. Zero is a rational number, but not a positive rational number.

To conform with the basic properties of arithmetic in operating with 0 and the positive rational numbers, you use the following rules. Assume that \( b \neq 0 \) and \( d \neq 0 \).

1. \( \frac{a}{b} = \frac{c}{d} \), if, and only if, \( ad = bc \)
2. \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \)
3. \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \)
4. \( \frac{a}{b} > \frac{c}{d} \), if, and only if, \( ad > bc \)
5. \( \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \), if \( \frac{a}{b} \geq \frac{c}{d} \)
6. \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \), if \( c \neq 0 \)

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the property of reciprocals:

For every nonzero number \( a \) there is a unique number, denoted \( \frac{1}{a} \) and called the *reciprocal* or *multiplicative inverse* of \( a \), such that

\[
\frac{1}{a} \cdot a = 1
\]
RATIONAL NUMBERS

Besides having all the properties stated on pages B and C, the set of rational numbers has the following properties.

PROPERTIES OF OPPOSITES: 1) For every rational number \( a \) there is a unique rational number, denoted by \(-a\), such that \( a + (-a) = (-a) + a = 0\).

2) \( -(a + b) = (-a) + (-b) \)

3) \( -a = (-1)a \).

PROPERTY OF CLOSURE: For all rational numbers \( a \) and \( b \), \( a - b \) is a definite rational number.

Since the set is closed under subtraction, you may state an equality and an inequality property more simply than on page B. Furthermore, the set has an additional property of inequality.

If \( a = b \), then \( a - c = b - c \),

If \( a < b \), then \( a - c < b - c \).

If \( a < b \) and \( 0 > c \), then \( ac > bc \) and \( \frac{a}{c} > \frac{b}{c} \).
Later appeared fractions like \( \frac{2}{3}, \frac{3}{4}, \) and, in general, \( \frac{a}{b} \), where \( a \) is any integer and \( b \) is any positive integer. Notice that \( \frac{1}{2} \) is defined to be the unique number having the property that \( 2 \cdot \frac{1}{2} = 1 \), while \( \frac{1}{b} \) is the number for which \( b \cdot \frac{1}{b} = 1 \). Then, just as \( \frac{2}{3} = 2 \cdot \frac{1}{3} \) or two-thirds, so also does \( \frac{a}{b} = a \cdot \frac{1}{b} \).

The extended number system contains the whole numbers; for example, \( 1 = \frac{1}{1} \), \( 2 = \frac{2}{1} \), and \( 0 = \frac{0}{1} \). Because each common fraction except zero is a ratio of positive integers, you call the nonzero common fractions the positive rational numbers. Zero is a rational number, but not a positive rational number.

To conform with the basic properties of arithmetic in operating with 0 and the positive rational numbers, you use the following rules. Assume that \( b \neq 0 \) and \( d \neq 0 \).

1. \( \frac{a}{b} = \frac{c}{d} \), if, and only if, \( ad = bc \)
2. \( \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \)
3. \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \)
4. \( \frac{a}{b} > \frac{c}{d} \), if, and only if, \( ad > bc \)
5. \( \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \), if \( a > c \)
6. \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \), if \( c \neq 0 \)

If you know how to operate with whole numbers, these rules enable you to calculate sums, products, differences, and quotients and to compare numbers in the extended system. Moreover, you can show that this system is closed under addition, multiplication, and division, except division by 0. In particular, you can find the quotient of 1 and any positive rational number. This is the property of reciprocals:

For every nonzero number \( a \) there is a unique number, denoted \( \frac{1}{a} \) and called the reciprocal or multiplicative inverse of \( a \), such that \( a \cdot \frac{1}{a} = 1 \cdot a = 1 \).

For example, the reciprocals of \( 3, \frac{1}{3}, \) and \( \frac{7}{4} \) are \( \frac{1}{3}, 5, \frac{4}{7} \).

To picture the system of positive rational numbers and 0 on a line, start with the representation of the set of whole numbers. Then divide the unit intervals into halves, thirds, and so on, thus locating points to be labeled \( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \ldots \); and so on. By continuing this procedure, you pair every positive rational number \( r \) with a point whose distance from the point 0 is given by \( r \). This geometric picture of the set of positive rational numbers and 0 suggests that this set has an order pattern completely different from that of the set of whole numbers. It is not true in the extended system that every number has an immediate successor; for example, you can find a rational number as close to \( \frac{1}{2} \) as you like. The system of positive rational numbers and 0 has the property of density:

Between every pair of rational numbers is another rational number.
In extending the number system, you have gained not only more numbers, but also more number properties. The growing structure of the number system is suggested by the development of the diamond and by the additional properties stated on page C.

Although the system of positive rational numbers and zero is adequate for measuring many quantities, still other quantities can be specified only by giving a direction as well as a magnitude. Thus, a bank balance includes debits as well as credits, and a displacement involves direction as well as distance. Moreover, subtraction is not always possible in the system of common fractions. To remedy this situation, as well as to answer the need for directed numbers, we extend the number system again by inventing negative rational numbers: \(-1\), \(-\frac{3}{4}\), \(-\frac{1}{5}\), and so on. Order in the new system is shown on the number line with negative numbers paired with points to the left of 0.

![Number line with negative numbers](image)

Numbers like 1 and \(-1\) or \(\frac{4}{3}\) and \(-\frac{4}{3}\) are called opposites or negatives of one another. In general, \(-a\) stands for the opposite of \(a\), so that \(-3\) is negative 3, while \(-(-3)\) is 3. The positive and negative rational numbers and 0 form the system of rational numbers. Fundamental in this system is the property of opposites:

For every number \(a\) there is a unique number \(-a\), called the opposite or additive inverse of \(a\), such that \(a + (-a) = (-a) + a = 0\).

This property together with the basic laws of arithmetic and the rules for calculating with 0 and the positive numbers enables you to discover the rules for operating with directed numbers. First define \(|a|\), the absolute value of \(a\), to be the greater of a nonzero number \(a\) and its opposite \(-a\). Define \(|0|\) to be 0. The following are the rules of operation for directed numbers.

\[
\begin{align*}
\text{Addition: } & a + b = |a| + |b|, \text{ if } a \geq 0 \text{ and } b \geq 0 \\
& = -(|a| + |b|), \text{ if } a \leq 0 \text{ and } b \leq 0 \\
& = |a| - |b|, \text{ if } a \geq |b| \text{ and } b < 0 \\
& = -(|b| - |a|), \text{ if } |b| \geq a \geq 0 \text{ and } b < 0
\end{align*}
\]

\[
\begin{align*}
\text{Subtraction: } & a - b = a + (-b) \\
& a = |a| \cdot |b|, \text{ if } a \geq 0 \text{ and } b \geq 0 \\
& = -(|a| \cdot |b|), \text{ if } a \geq 0 \text{ and } b \leq 0 \\
& = -(|a| \cdot |b|), \text{ if } a \leq 0 \text{ and } b \geq 0 \\
& = -(|a| \cdot |b|), \text{ if } a \leq 0 \text{ and } b \leq 0
\end{align*}
\]

Since you can add, subtract, multiply, and divide (except by 0) in the system of rational numbers and since these operations satisfy the basic laws of arithmetic, you might believe that we now have the complete number system. Certainly, as the many faceted diamond shown on page D suggests, the set of rational numbers has a highly developed structure. However, even the ancient Greeks realized that the rational numbers were inadequate for making certain measurements. The third transparency will consider the question of whether every distance can be measured by a rational number, and will extend the number system still further.
2. At a certain time two airplanes start from the same airport and travel in opposite directions at 350 miles an hour and 325 miles an hour, respectively. In how many hours will they be 2025 miles apart?

3. A train starts from Grand City and travels toward Belleville 388 miles away. At the same time a train starts from Belleville and runs at the rate of 47 miles an hour toward Grand City. They pass each other 4 hours later. Find the rate of the train from Grand City.

4. At 12 noon, two river steamers are 120 miles apart. They pass St. Louis at 6 P.M. headed in opposite directions. If the northbound boat steams at 9 miles per hour, find the rate of the southbound boat.

5. A train left Omaha at 9 A.M. traveling at 50 miles per hour. At 1 P.M. a plane also left Omaha and traveled in the same direction at 300 miles per hour. At what time did the plane overtake the train?

6. A freight train left Beeville at 5 A.M. traveling 30 miles per hour. At 7 A.M. an express train traveling 50 miles per hour left the same station. When did the express overtake the freight?

7. An airplane traveling 280 miles per hour leaves San Francisco 13 1/2 hours after a steamship has sailed. If the plane overtakes the ship in 1 1/2 hours, find the rate of the steamship.

8. Mr. Gomez is 213 miles on his way when his secretary starts out to overtake him in a plane going 320 miles an hour. How fast is Mr. Gomez traveling if the plane overtakes him in three-quarters of an hour?

9. A troop of scouts hiked to the county scout cabin at the rate of 2 miles per hour. They rode back to headquarters at 18 miles per hour. If the round trip took 10 hours, how far is headquarters from the cabin?

10. While waiting for a connecting train, a traveler takes a bus ride at 10 miles per hour to a certain point and then walks back at 2 miles an hour. If he returns 3 hours after he left, how far did he walk?

11. Tom sets out on a hike. After walking for a while at 5 miles an hour, he discovers that he has forgotten his lunch. A passing truck takes him home at 20 miles an hour. When he gets home, he finds that he has lost exactly one hour. How far had he walked?

12. Dick's motorboat can make an average of 8 miles an hour. One day he sets out for a trip, only to have the motor break down. Dick rows back at 2 miles an hour. When he reaches his dock, he finds that he has been gone 5 hours. How far has he rowed?

13. Jim took a trip of 1020 miles. He traveled by train at 55 miles an hour and the same number of hours by plane at 285 miles an hour. How many hours did the trip take?
14. Mrs. Asbury traveled south, half the time by automobile and half the time by train. She averaged 45 miles per hour by automobile and 50 miles per hour by train. The total trip was 665 miles. How long was Mrs. Asbury traveling?

15. A bus and a train start for the same destination at the same time. The highway runs along the railroad track. The bus averages 31 miles an hour, and the train averages 39 miles an hour. In how many hours will they be 24 miles apart?

16. A jet plane traveling 600 miles per hour can make a certain trip in 33 hours less time than a train traveling at 50 miles an hour. How long does the train trip take?

17. In a run around a 150-yard track, Stan and Walter ran at 325 yards a minute and 300 yards a minute respectively. In how many minutes will Stan have run a track length farther than Walter?

18. In a race, Ned is 50 feet in front of Jed after 10 seconds. How fast can Ned run if Jed can run 20 feet per second?

19. A private airplane had been flying for 1 hour when a change of wind direction doubled the effective rate. If the entire trip of 240 miles took 2 1/2 hours, how far did the plane go in the first hour?

20. A ship must average 22 miles per hour to make its ten-hour run on schedule. During the first four hours, bad weather caused it to reduce its speed to 16 miles per hour. What should its speed be for the rest of the trip to keep the ship to its schedule?

21. Mr. Hardy's car breaks down at a village, and he is told that it will take 1 1/2 hours to repair. He takes a bus into the country and intends to walk back over an alternate route which is no shorter than the bus route. If he walks at 4 miles per hour and the bus averages 20 miles per hour, what is the greatest distance from the village he could ride?

22. A salesman starts at 9 A.M. to deliver certain parts to Mr. Rector, install them, and return. Although he could go 12 miles per hour on his bicycle, the salesman does the 6 miles to the Rector place at 8 miles per hour. When he arrives, he finds a message saying that he must be back by 11 A.M. at the latest. Find the maximum time he can spend in installing the parts.

5-8 Mixture Problems

Often a chemist mixes solutions of different strengths of a chemical to obtain a solution of a desired strength. Similarly, merchants mix goods of two or more qualities in order to sell the blend at a given price. All these problems are solved in the same way. The sum of the
values or units of weight of the original ingredients must equal the value or weight of the final mixture.

**EXAMPLE.** At a Book Fair, 600 books were sold, some pocket editions at 35 cents each and the rest hard-covered books at 50 cents each. The total receipts were equivalent to the last year’s intake when the same number of books were sold at an average price of 40 cents per book. How many of each kind of book were sold?

**Solution:**

Let \( n \) = number of pocket editions sold.

Then \( 600 - n \) = number of hard-covered books sold.

<table>
<thead>
<tr>
<th>Kind</th>
<th>Number</th>
<th>Unit price in cents</th>
<th>Total receipts in cents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pocket editions</td>
<td>( n )</td>
<td>35</td>
<td>( 35n )</td>
</tr>
<tr>
<td>Hard-covered books</td>
<td>( 600 - n )</td>
<td>50</td>
<td>( 50(600 - n) )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>600</td>
<td>40</td>
<td>( 40 \cdot 600 )</td>
</tr>
</tbody>
</table>

Total receipts from pocket editions this year + Total receipts from hard-covered books this year = Total receipts from all books last year

\[
35n + 50(600 - n) = 40 \cdot 600
\]

Steps 3 and 4 are left to you.

**PROBLEMS**

1. At its opening, the Burnside Public Market decides to distribute 1000 souvenirs of two kinds. Some souvenirs cost 20 cents each, and the others cost 25 cents each. If a total of $220 is allowed for the souvenirs, how many of each kind are needed?

2. The school store sold 348 notebooks the first day of school, some at 25 cents each, the rest at 38 cents each. The total receipts for notebooks was $100.91. How many of each kind were sold?
3. A confectioner makes 100 pounds of candy to sell at $1.75 a pound. He mixes candy worth $1.65 a pound with some worth $1.90 a pound. How many pounds of each does he use?

4. A merchant mixes tea worth 90 cents a pound with some worth $1.50 a pound to make 20 pounds of a blend which he can sell at $1.20 a pound. How many pounds of each kind of tea does he use?

5. How much 84-cent coffee must be blended with 24 pounds of 60-cent coffee if the blend is to be sold at 68 cents a pound?

6. Tickets to the Saturday movie cost 18 cents for children and 42 cents for adults. A total of $69.12 was collected for 328 tickets. As a check on the ticket seller, find how many adults attended.

7. Billy Little’s bank opened when it registered $1.00. Billy counted the coins and said to his brother, “Altogether there are 48 pennies and nickels. Bet you can’t figure out how many of each kind there are!” Show how you would answer the challenge.

8. A man purchases some three-cent stamps and some one-cent stamps for $3.05. There are 19 more three-cent stamps than one-cent stamps. How many of each kind does he buy?

9. A nut shop sells almonds for $1.80 a pound, walnuts for $1.30 a pound, and peanuts for 65 cents a pound. The shopkeeper makes a mixture of 21 pounds of these nuts to sell for $1.00 a pound. He uses twice as many pounds of peanuts as walnuts. How much of each does he use?

10. Fred paid $3.00 for stamps. He bought three times as many 3-cent stamps as 6-cent stamps, twice as many 1 1/2-cent stamps as 3-cent stamps, and the same number of 1-cent stamps as of 1 1/2-cent stamps. How many stamps of each sort did Fred buy?

11. Your boss asks you to change a $20 bill so that there will be twice as many quarters as half dollars, and half as many single dollar bills as half dollars. Show him politely how smart you are by explaining how this is impossible.

12. Cathy purchased 100 items in a stationery store for one dollar. She bought pencils at 10 cents each, 9 times as many erasers at 5 cents each, and clips at two-for-a-penny. How many of each did Cathy buy?

13. How much of an alloy which contains 6 grams of gold in 10 grams must be melted with another alloy which contains 3 grams of gold in 10 grams, to give an alloy containing 5 grams of gold in 10?

14. A chemist has two solutions of sulfuric acid. The first is half sulfuric acid and half water; the second is three-fourths sulfuric acid and one-fourth water. He wishes to make 10 liters of a solution which is two-thirds sulfuric acid and one-third water. How many liters of each available solution should he use?
THE HUMAN EQUATION

A Legendary Hero and a Resourceful Archbishop

If you study Spanish or read Spanish folk tales in translation, you are almost sure to encounter the legendary hero El Cid. He lived about 1100, and was not a mathematician. He played a colorful part in the fighting whereby the Spanish Christians wrested Toledo from the Mohammedans, who had held this Spanish city for four centuries. This victory led directly to the spread of algebra throughout Europe. It did so because the newly appointed Archbishop of Toledo (who was not a mathematician) was a resourceful man.

When Toledo returned to Christian rule, it was a great center of learning. Here were books (in Arabic) which contained knowledge long since forgotten in the rest of Europe. Here, too, were Arabic books which recorded new knowledge.

The Archbishop was determined that this learning should be made available to the scholars of Europe. The books should be translated into Latin. But who was to do the translation? Few men knew both Arabic and Latin.

The answer was teamwork. The people of Toledo spoke Spanish as well as Arabic. And most of the scholars of Europe knew Spanish, or could learn it quickly. So the Archbishop set up teams, each consisting of a scholar and a Toledo citizen. The citizen would read an Arabic book aloud, translating into Spanish as he went along, and the scholar would write the Latin of what he heard, translating from Spanish as he went along.

One of the learned men who came to Spain in those days was an Englishman, called Robert of Chester. He it was who translated Al Khowarizmi’s masterpiece, *ilm al-jabr wa’l muqabalah*, and thus made algebra available to Europeans.

Page from a copy of Robert of Chester’s Latin translation of Al Khowarizmi’s algebra. This manuscript was painstakingly copied from Robert’s original by Johann Scheubel, a mathematician.
Chapter Summary

Inventory of Structure and Method

1. You can obtain (or derive) equivalent equations in the set of directed numbers by applying the same properties of equality as in the set of arithmetic numbers.

2. If $a$ and $b$ represent directed numbers, exactly one of the following holds: $a < b$, $a = b$, or $a > b$. Graphically, $a < b$, or $b > a$ means that $a$ is to the left of $b$ on the number line.

3. The transitive property of inequality is:
   
   If $a < b$ and $b < c$, then $a < c$ and similarly, if $a > b$ and $b > c$, then $a > c$.

   The additive property (this includes subtraction) of inequality is:
   
   For any number $c$, if $a < b$, then $a + c < b + c$; if $a > b$, then $a + c > b + c$.

   The multiplicative property (this includes division) of inequality:
   
   For $c$ positive
   
   if $a < b$, then $ac < bc$, and if $a > b$, then $ac > bc$;

   For $c$ negative
   
   if $a < b$, then $ac > bc$, and if $a > b$, then $ac < bc$.

4. You may obtain equivalent inequalities by applying the additive and multiplicative properties in much the same way as for an equation, except that you reverse the sense of the inequality if you multiply or divide its members by a negative number.

5. The sentence $b < x + a < c$ represents the two inequalities $b < x + a$ and $x + a < c$ whose separate solution sets are $b - a < x$ and $x < c - a$. Their common solution set $b - a < x < c - a$ is therefore the solution set of $b < x + a < c$. This solution set may be obtained from the given sentence by subtracting $a$ from each member.

   The inequality $|x| \leq a$ will be satisfied only when $x \geq -a$ and $x \leq a$.

   This pair of inequalities will in turn be satisfied at the same time by the elements of the solution set: $-a \leq x \leq a$.

   On the other hand, the inequality $|x| \geq a$ will be satisfied when $x \geq a$ or $x \leq -a$. 

   ![Number Line Diagram]
6. In solving problems, start by asking yourself two questions: What does the problem require? What facts are given or available? A sketch may help identify the available facts.

7. The rule \( d = r \cdot t \) can be applied to problems involving moving objects, whether they move in the same or in different directions on the same straight line.

**Vocabulary and Spelling**

order property of numbers (p. 159)  
order, direction or sense, of an inequality (p. 161)  
equivalent inequality (p. 161)  
integer (p. 170)  
consecutive integers (p. 170)  
even integer (p. 170)  
odd integer (p. 170)  
directed angles (p. 173)  
positive angle (p. 173)  
negative angle (p. 173)  
initial side (of an angle) (p. 173)  
terminal side (of an angle) (p. 173)  
vertex (of an angle) (p. 173)  
degree (angle measure) (p. 173)  
complementary angles (p. 173)  
supplementary angles (p. 173)

### Chapter Test

5-1 Solve and check each equation.

1. \( \frac{3y}{5} + 63 = 27 \)  
2. \( 5s - 3(4s + 7) = 9 - s \)  
3. Solve for \( x \): \( \frac{x}{a} - c = b \)  
4. Solve for \( l \): \( p = 2(l + w) \)

5-2 Perform each indicated operation, and write the resulting sentence.

5. a. Add \(-4\) to each member of \( x + 4 < -1 \).  
b. Divide \( 21 > -7v \) by \(-7\).

Solve each inequality, and graph its solution set.

6. \( 8x + 2 > 5x - 4 \)  
7. \( 47 \leq 7 - 5(3t - 2) \)

5-3  
8. (Optional) a. \( |1 - y| \geq 2 \)  
b. \( |3m - 6| > 9 \)

5-4  
9. In a rectangle, the length and width, combined, total 36 yards. Twice the length is 8 yards less than three times the width. Find the dimensions.
10. Typist $A$ can address 50 fewer than twice the number of envelopes addressed by typist $B$ in an hour. Together they address at least 175 envelopes hourly. Find the least number of envelopes typist $A$ addresses in an hour.

5-5 11. Find three consecutive odd integers such that the sum of the first and third exceeds the second by 27.

5-6 12. The complement of an angle is 18 degrees less than twice the angle. Find the angle and its complement.

13. A triangle is to be drawn with one angle $12^\circ$ greater than another, and the third $12^\circ$ less than the sum of the other two. Find the angles.

5-7 14. An airplane left Seattle at 1 p.m. and flew directly east at 340 miles per hour. At 2:30 p.m., a jet plane traveling at 595 miles per hour started east from the same airport. In how many hours did the jet overtake the first plane?

5-8 15. Adult tickets for a lecture at the school auditorium were 50 cents each, but students were admitted for 20 cents. The auditorium, seating 700, was filled; a total of $305 was realized. How many adults and how many students attended the lecture?

Chapter Review

5-1 Transforming Equations  

1. In solving an equation, transformations used in the set of arithmetic numbers are valid also in the set of __?__ numbers.

2. If the same number is added to or subtracted from each member of an equation, the resulting equation is __?__ to the original equation.

3. Multiplying or dividing each member of an equation by a non-zero number produces an equation having the same __?__ __?__.

Of what transformation operation is each of the following an example?

4. $a = b$; so $a + b = 2b$  
6. $m = r$; so $1 = \frac{r}{m} (m \neq 0)$

5. $c = d$; so $c - 3 = d - 3$  
7. $b = c$; so bat = cat
In each of the following, the variables represent directed numbers which make the first sentence true. Give a reason that justifies each lettered statement.

8. \( 5 - 5k = 0 \)  
   \(-5k = -5 \) (a)  
   \( k = 1 \) (b)  
   \( 5 - 5(1) = 0 \) (c)

9. \(-4 = \frac{3}{2}p - 4 \)  
   \( 0 = \frac{3}{2}p \) (a)  
   \( 0 = 3p \) (b)  
   \( 0 = p \) (c)

Solve each equation.

10. \( 15n = 2n - 1.3 \)
12. \( 7 + 4(t - 3) = t - 5 \)
11. \( 10 = 8 - \frac{3}{3}w \)
13. \( 2 - (r - 3) = 3r + 3(r + 4) \)

Solve each of the following equations for the variable in red.

14. \( cz = d \)
18. Solve for \( i: p = a - i \)
15. \( 2y - 3p = 5y \)
19. Solve for \( C: F = \frac{8}{3}C + 32 \)
16. Solve for \( r: t = \frac{d}{r} \)
20. \( \frac{n}{a} = c \)
17. Solve for \( a: m = \frac{1}{2}(a + b) \)
21. \( x + 4a = -x \)

**5-2 The Properties of Inequality**

22. If \( a \neq b \), then \( a > b \) or \( a \ ? b \).

In Exercises 23–33, \( a \), \( b \), and \( c \) are any directed numbers.

23. On a number line, if \( a \) is to the left of \( b \) and \( b \) is to the left of \( c \), \( a \) is to the \( ? \) of \( c \).
24. If \( a > b \) and \( b > c \), then \( a \ ? c \).
25. If \( a < b \), then \( a + c \ ? b + c \).
26. If \( a < b \), then \( a - c \ ? b - c \).
27. If \( a < b \) and \( c > 0 \), then \( ac \ ? bc \).
28. If \( a < b \) and \( c < 0 \), then \( ac \ ? bc \).
29. If \( a < b \) and \( c = 0 \), then \( ac \ ? bc \).
30. If \( a < b \) and \( c > 0 \), then \( \frac{a}{c} \ ? \frac{b}{c} \).
31. If \( a < b \) and \( \frac{a}{c} > \frac{b}{c} \), then \( c \ ? 0 \).
32. If \( a \neq 0 \), then \( a^2 \ ? 0 \), and \( -a^2 \ ? 0 \).
33. If each member of \(-5a < 25\) is divided by \(-5\), then \( a \ ? -5 \).
34. You can transform inequality $4x \geq 9x + 10$ into $-5x \geq 10$ by adding ? to each member.

35. If you transform an inequality by multiplying or dividing by any negative number, you must ? the sense of the inequality.

36. If $-5x \geq 10$, then $x$ ? $-2$.

Solve each inequality, and graph its solution set.

37. $4n - 1 < n + 2$

38. $4 - 3m < 8 + 5m$

5-3 Pairs of Inequalities (Optional) Pages 164-166

39. The solution set of the sentence $-3 < x < 2$ consists of those numbers for which ? and ? both are true.

40. The inequality $|3x - 2| \geq 4$ will be satisfied if $3x - 2 \geq ?$ or $3x - 2 \leq ?$.

41. Solve the inequality: $|3x - 2| \geq 4$.

42. The inequality $|3x - 2| < 4$ is equivalent to $3x - 2 > ?$ and $3x - 2 < ?$.

43. Solve the inequality: $|3x - 1| > 4$.

44. The set satisfying both $x < 1$ and $x > 1$ is the ? set.

45. The set satisfying $x \leq 1$ or $x > 1$ is the set of ? numbers.

5-4 A Plan for Solving Problems Pages 166-169

46. In an election, A received half as many votes as B, and C’s votes exceeded B’s by 110. If 1000 votes were cast, find the results of the election.

47. One side of a triangle must exceed twice a second side by 5 inches, while the third side must be twice the second side. If the perimeter is to be at least 5 inches, find the smallest possible value of the longest side.

5-5 Problems about Consecutive Integers Pages 170-172

48. Write the five consecutive even integers that follow $-3$.

49. Find three consecutive integers whose sum is 2274.

50. Find five consecutive odd integers whose sum is 85.

5-6 Problems about Angles Pages 172-177

51. If two angles of a triangle are complementary, the third angle must contain ? degrees.
52. One of two complementary angles is 24 degrees less than the other. Find the angles.

53. Find two supplementary angles, one of which is five times as large as the other.

54. A triangle is to be drawn in which two angles are equal and the third is at most 12 degrees less than twice their sum. Find the smallest possible value of one of the equal angles.

5–7 Uniform Motion Problems  

55. Amy walks to the repair shop at 2 miles an hour, picks up her bicycle, and rides home at 10 miles an hour. If the round trip took 1 1/2 hours, how far is the shop from Amy’s house?

56. Mr. Ford traveled from Axton to Trent via Grenby, which is midway between them. On the first leg of his trip, he was able to ride at 45 miles an hour, but from Grenby to Trent, poorer roads caused him to go at 36 miles an hour. Still, the whole trip took 1 hour. How far is Grenby from Axton?

57. A bus and a train leave the same station at the same time and travel in the same direction along parallel routes. The train runs twice as fast as the bus. They are from 12 1/2 to 15 miles apart at the end of half an hour. Find the smallest and largest possible value of each rate.

58. Jim took an 80-mile trip, part by train and the rest by automobile. If he had traveled an extra 10 miles of the trip by automobile, he would have gone just twice as far by train as by automobile. How many miles did he actually travel by train?

5–8 Mixture Problems  

59. How many pints of oil worth 12 cents a pint must be mixed with 100 pints worth 5 cents a pint to produce an oil which can be sold at 7 cents a pint?

60. A confectioner has 10 pounds of nut-and-fruit candy which isn’t selling well at $1.25 a pound. He buys cream at 65 cents a pound and mixes them with his candy. He puts out an assortment which he can sell at 85 cents a pound. How many pounds of creams does he buy?

61. An after-school performance was attended by 115 persons. Students were charged 15 cents, but all others paid 35 cents. In all, $25.65 was collected. How many students attended?
Psychometrists and Mathematics

Psychometrists measure mathematically the speed and accuracy of mental processes. By interpreting statistically the results of large numbers of tests on learning, motivation, memory, and other psychological variables, psychometrists gain information about the range and distribution of human behavior.

In 1905 a French psychologist named Alfred Binet perfected a scale of normal (average) intelligence at various age levels. The Binet scale uses the fact that children in each age group normally possess a certain kind of knowledge. The Binet scale groups together in a series of tests the items of knowledge common to each age level. The most difficult test passed by an individual determines his mental age.

Many of you may know your I.Q., but do you know how this figure was derived? An intelligence quotient (I.Q.) is computed on the work pad. This is a simple but much used application of mathematics in the field of psychometrics. The mental age (MA), as determined by a Binet test, is divided by the chronological age (CA), and the quotient is multiplied by 100. You can see that an individual whose mental and chronological ages are the same has an I.Q. of 100 (or 100%), whereas someone whose mental age is less than his chronological age has an I.Q. of less than 100. On the other hand, the boy in the photograph was found to have a mental age of 9 yrs. 10 mos. at a chronological age of 7 yrs. 3 mos. and his I.Q., shown on the work pad, is 136.
The Day and the Date

Do you know on what day of the week you were born? You can find out without asking your family and without consulting a calendar. In fact, you can find out the day of the week for any date in history.

To find Valentine’s day, February 14, 1896, proceed as follows:

1. Write these numbers, one beneath the other:
   - The last two digits of the year . . . . . . . . 96
   - One fourth of these two digits. (There is no remainder; 1896 was a leap year.) . . . . 24
   - The key number for February . . . . . . . . 3
   - The number of the day of the month . . . . . 14
   - The key number for the century . . . . . . . . 2

2. Find the sum of these numbers . . . . . . . . . 139

3. Divide the sum by 7. . . . . . . . . . . . . . 139 ÷ 7 = 19\(\frac{6}{7}\)

The remainder indicates the day of the week. A remainder of 1 means Sunday; 2, Monday, and so on through Friday; and 0, or no remainder, means Saturday. The remainder is 6; so you know that Valentine’s Day was a Friday in 1896.

Table B shows two key numbers for the eighteenth century, because in 1752 the calendar was reformed, or brought back in line with the motions of the heavenly bodies.

<table>
<thead>
<tr>
<th>Table A</th>
<th>Table B</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>Twentieth century . . . . . . . . . 0</td>
</tr>
<tr>
<td>(leap years, 0)</td>
<td>Nineteenth century . . . . . . . . . 2</td>
</tr>
<tr>
<td>February</td>
<td>Eighteenth century</td>
</tr>
<tr>
<td>(leap years, 3)</td>
<td>Sept. 14, 1752 to Jan. 1, 1800 . . 4</td>
</tr>
<tr>
<td>March</td>
<td>Jan. 1, 1700 to Sept. 2, 1752 . . 1</td>
</tr>
<tr>
<td>April</td>
<td>Seventeenth century . . . . . . . . . 2</td>
</tr>
<tr>
<td>May</td>
<td>Sixteenth century . . . . . . . . . . 3</td>
</tr>
<tr>
<td>June</td>
<td>Fifteenth century . . . . . . . . . . 4</td>
</tr>
<tr>
<td>July</td>
<td>Fourteenth century . . . . . . . . . . 5</td>
</tr>
<tr>
<td>August</td>
<td>Thirteenth century . . . . . . . . . . 6</td>
</tr>
<tr>
<td>September</td>
<td>Twelfth century . . . . . . . . . . . . 7</td>
</tr>
<tr>
<td>October</td>
<td>Eleventh century . . . . . . . . . . . . 8</td>
</tr>
<tr>
<td>November</td>
<td>Tenth century . . . . . . . . . . . . . 9</td>
</tr>
<tr>
<td>December</td>
<td>(Add 1 for each century you go back.)</td>
</tr>
</tbody>
</table>
CHAPTER FIVE

Extra for Experts

Systems of Numeration

In the decimal system 1234 means $1 \times 10^3 + 2 \times 10^2 + 3 \times 10 + 4$. The value of the same sequence of digits in numeral systems whose bases are 7 and 5 are shown in this table. Note that in the 7 system each place has a value 7 times the place at its right, and in the 5 system each place has a value 5 times the place at its right.

<table>
<thead>
<tr>
<th>BASE 7</th>
<th>BASE 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 7^3 + 2 \times 7^2 + 3 \times 7 + 4$</td>
<td>$1 \times 5^3 + 2 \times 5^2 + 3 \times 5 + 4$</td>
</tr>
<tr>
<td>$1 \times 343 + 2 \times 49 + 3 \times 7 + 4$</td>
<td>$1 \times 125 + 2 \times 25 + 3 \times 5 + 4$</td>
</tr>
<tr>
<td>343 + 98 + 21 + 4</td>
<td>125 + 50 + 15 + 4</td>
</tr>
<tr>
<td>466</td>
<td>194</td>
</tr>
</tbody>
</table>

Write: $1234_{VII} = 466_X$

Read: 1–2–3–4, base 7

= 4–6–6, base 10.

Write: $1234_V = 194_X$

Read: 1–2–3–4, base 5

= 1–9–4, base 10.

To convert $466_X$ to base 7, divide repeatedly by 7, noting the remainders on the right. Continue the process until the quotient is 0.

\[
\begin{align*}
\frac{466}{7} &= 66 \text{ with remainder 4} \\
\frac{66}{7} &= 9 \text{ with remainder 3} \\
\frac{9}{7} &= 1 \text{ with remainder 2} \\
\frac{1}{7} &= 0 \text{ with remainder 1}
\end{align*}
\]

What do the numerals in red signify? The 1 signifies that 466 can be divided by $7^3$ once. The 2 signifies that the remainder of $466 - 1 \cdot 7^3$ can be
divided by $7^2$ twice. The 3 signifies that the remainder of 466 $-$ 1 $\cdot 7^3$ $-$ 2 $\cdot 7^2$ can be divided by 7 three times. The 4 signifies that 466 $-$ 1 $\cdot 7^3$ $-$ 2 $\cdot 7^2$ $-$ 3 $\cdot 7$ = 4, which is less than 7. Together, these are the coefficients of $1 \cdot 7^3 + 2 \cdot 7^2 + 3 \cdot 7^1 + 4$ or 1234\text{VII}.

Similarly, to convert 194 to base 5, divide by 5 until the quotient is 0.

Numerals in the decimal system are successions of basic digits 0, 1, 2, \ldots, 9, each less than 10. Since each basic numeral in the 7 system is a remainder in a division by 7, the basic numerals in this system are 0, 1, 2, 3, 4, 5, 6. Similarly, in the 5 system the basic numerals are 0, 1, 2, 3, 4, and in a base $n$ system they are 0, 1, 2, \ldots $(n - 1)$.

Do you know that many modern digital computers use the binary (base 2) system? Since numerals in the binary system are successions of the digits 0 and 1, the machine can denote numbers by a sequence of electronic devices, each either on or off. A 1 is registered by an \text{ON} state, while a 0 is recorded by an \text{OFF} state. As the table shows, it takes four digits, 1010\text{II}, to write 10\text{X}.

<table>
<thead>
<tr>
<th>BASE 2 EQUIVALENCE TABLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^3$</td>
</tr>
<tr>
<td>0 •</td>
</tr>
<tr>
<td>0 •</td>
</tr>
<tr>
<td>0 •</td>
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<td>0 •</td>
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<tr>
<td>1 •</td>
</tr>
<tr>
<td>1 •</td>
</tr>
</tbody>
</table>

Questions

1. Find the largest digit in a numeral system with the given base.
   a. 6  b. 4  c. 2  d. h

2. Convert the given numbers to the decimal system.
   a. $1357_{\text{IX}}$  b. $1021_{\text{III}}$  c. $10101_{\text{II}}$  d. $246_{\text{VII}}$

3. Convert the given numbers to the system with the base indicated.
   a. $345_{\text{X}}$; \text{VII}  b. $20_{\text{X}}$; \text{II}  c. $500_{\text{X}}$; \text{V}  d. $701_{\text{X}}$; \text{VI}

4. Extend the binary equivalency table above to 40\text{X}, using 1 and 0.

5. a. How can you tell the difference between odd and even numbers in the binary system?  b. Why does $2_{\text{X}} = 10_{\text{II}}$?

6. In any numeral system with base $n$, why does $n = 10_{\text{n}}$?
Working with Polynomials

In your study of arithmetic, you learned basic combinations, and rules for finding answers using the combinations in appropriate patterns. In studying polynomials, you will learn similar operations and patterns and the reasons for them.

Patterns occur in many situations. For example, the odometer (lower photo), which records an automobile's mileage, and the abacus (upper photo) illustrate the same pattern of "carrying in addition." As a figure on the odometer changes from 9 to 0, the figure to its left moves up one numeral. On an abacus, when all ten beads on one wire are counted and moved to the zero position, one bead on the wire to the left is counted.

**ADDITION AND SUBTRACTION OF POLYNOMIALS**

6–1 Adding Polynomials

Each of the terms 7, \( x \), \( 5y^2 \), and \(-2xy\) is called a monomial (mo-no-me-ell). A monomial is a term which is either a numeral (7), a variable \((x)\), or a product of a numeral and one or more variables.

A monomial or the indicated sum of monomials is called a polynomial (pol'e-no-me-ell). For example, \( 5y^2 + 7 \) is a polynomial of two terms, called a binomial (by-no-me-ell), and \( a^2 - 2ab + b^2 \) is a polynomial of three terms, called a trinomial (try-no-me-ell).

The degree of a monomial in a variable is the number of times that the variable occurs as a factor in the monomial. For example, \( 3x^2y^5z \) is of degree 2 in \( x \), 5 in \( y \), and 1 in \( z \). The degree of a monomial is the sum of the degrees in each of its variables. The degree of \( 3x^2y^5z \) is 8, because \( 2 + 5 + 1 = 8 \). If a monomial other than 0 contains no variables, its degree is zero. The monomial 0 has no degree.

The degree of a polynomial is the same as the greatest of the degrees of its terms. The degrees of the terms of \( 15x^4 - 7x^3y^2 + 2xy \) are, in order, 4, 5, and 2. Thus, the degree of this polynomial is 5.
To add two polynomials, you use the commutative, associative, and distributive properties to combine similar terms. For example:

\[(5x^2y + 15) + (7x^2y - 3x + 2) = (5 + 7)x^2y - 3x + (15 + 2) = 12x^2y - 3x + 17\]

To add two polynomials, combine their similar terms.

You may also do the addition vertically:

\[
\begin{array}{c}
5x^2y \\
7x^2y - 3x + 2 \\
\hline
12x^2y - 3x + 17
\end{array}
\]

You can check the addition of polynomials by assigning a numerical value to each variable and evaluating the expressions. For example,

Check: Let \(x = 2\) and \(y = 3\).

\[
\begin{align*}
5x^2y + 15 &= 5 \cdot 4 \cdot 3 + 15 = 75 \\
7x^2y - 3x + 2 &= 7 \cdot 4 \cdot 3 - (3 \cdot 2) + 2 = 80 \\
12x^2y - 3x + 17 &= 12 \cdot 4 \cdot 3 - (3 \cdot 2) + 17 = 155
\end{align*}
\]

\[155 = 155 \checkmark\]

Though you choose small numbers for checking, why should you not select 1 or 0?

In working with polynomials, arrange the terms in order of either decreasing degree or increasing degree in a particular variable.

In order of decreasing degree in \(a\):

\[a^3 + 6a^2 + 12a + 8\]

In order of increasing degree in \(b\):

\[9 - 6b^2 + b^4\]

In order of decreasing degree in \(r\):

\[32r^5 + 7r^4d - 2r^2d^2 - 18d^3\]

**ORAL EXERCISES**

Give each of the following indicated sums. What is the degree of each sum?

1. \[\frac{2x + 3}{3x - 1}\]

2. \[\frac{6a - 7}{4a + 3}\]

3. \[\frac{a^2 + 1}{a^2 - 1}\]

4. \[\frac{x - y^2}{x + y^2}\]

5. \[\frac{x^2 - y^2 + z^2}{2x^2 - 3y^2 - 2z^2}\]

6. \[\frac{2n^2 - 5k + 6}{n^2 - k - 8}\]

7. \[\frac{a^2 + b - c}{a^2 - c}\]

8. \[\frac{3b^3 + 4d}{2c^2 + 3d}\]

9. \[\frac{-x^3 - y^2 + 8}{x^3 + y^2 - 6 - x^3 + y^2 + 2}\]
10. \((-2a + 3) + (-2a - 3)\)
11. \((-5x - 6y) + (-5x + 6y)\)
12. \((2x^3 + 9) + (y^2 + 1)\)
13. \((3x^3 - 5y^2 + 8) + (-x - 10)\)
14. \((3x^2 + b) + (3x^2 - b)\)
15. \((5x - 4y^2) + (7x - 3y^2)\)
16. \((x^2 - 2x) + (-3x^3 + 6)\)
17. \((y^2 + 4y) + (-y - 4)\)
18. \((4a^4 - 10a^2) + (-10a^2 + 25)\)
19. \((b^4 - 6b^2) + (5b^2 - 30)\)
20. \((x^2 - 3x + 7) + (2x^2 + 5x - 9)\)
21. \((a^2 + 5a - 9) + (3a^2 - 10a - 10)\)
22. \((a^2 - 2ab^2 + b^2) + (a^2 + 2ab^2 + b^2)\)
23. \((-2z^3 + 5z) + (z^4 - 5z^2 + 4)\)

Read each of the following polynomials, giving the terms in order of decreasing degree in a variable.

**SAMPLE.** \(3r + 2r^2 + 4 + r^3\)

**What you say:** \(r^3 + 2r^2 + 3r + 4\)

Find each of the sums indicated.

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (-2.5n - 1.6h)</td>
<td>6. (-\frac{1}{3}x - \frac{1}{2}y - \frac{1}{4})</td>
<td>11. (x + 5)</td>
<td>14. (-.5rs + 4)</td>
</tr>
<tr>
<td>(-.5n - 2.4h)</td>
<td>(x - y + 1)</td>
<td>(4x - 7)</td>
<td>(2rs + .3)</td>
</tr>
<tr>
<td>2. (-4.1r + 2.1s)</td>
<td>7. (\frac{3}{4}x - y)</td>
<td>12. (z - 7)</td>
<td>(-.6rs + 9)</td>
</tr>
<tr>
<td>(-2.9r - 4.2s)</td>
<td>(\frac{1}{4}x - y)</td>
<td>(3z - 5)</td>
<td></td>
</tr>
<tr>
<td>3. (x^3 - y^3)</td>
<td>8. (-z + \frac{1}{2}y)</td>
<td>13. (6xy - .9)</td>
<td></td>
</tr>
<tr>
<td>(\frac{3}{2}x^3 + \frac{1}{2}y^3)</td>
<td>(z - \frac{1}{2}y)</td>
<td>(-.3xy + 8)</td>
<td></td>
</tr>
<tr>
<td>4. (m^2 + n^2)</td>
<td>9. (4yz - 6)</td>
<td>(-5xy + 1.7)</td>
<td></td>
</tr>
<tr>
<td>(\frac{1}{4}m^2 - \frac{1}{8}n^2)</td>
<td>(-7yz + 16)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. (-\frac{3}{2}m + \frac{3}{4}n + \frac{4}{6})</td>
<td>10. (10ab - 13)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m - n - 1)</td>
<td>(-23ab + 17)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Check each of the following additions. When values are given for the variables, use them. When an addition is incorrect, do it correctly.

21. \(5x - 6\)
    \(2x + 1\)
    \(7x - 5\)
    Let \(x = 3\).
    \(5x - y + 2\)
    \(-x - y - 3\)
    \(7x + 2y - 8\)
    Let \(x = 5, y = 2\).

22. \(3y + 7\)
    \(2y - 80\)
    \(5y - 87\)
    Let \(y = 2\).
    \(x^2 + x + 6\)
    \(3x^2 - x - 8\)
    \(4x^2 - 2\)
    Let \(x = 2\).

23. \(3r - 8\)
    \(-2r + 1\)
    \(-r + 3\)
    \(4r - 4\)
    Let \(r = 2\).

24. \(-x^2 - xy - 2.3y^2\)
    \(x^2 - xy + 2.2y^2\)

25. \(8x^3 - 40x^2 + 50x\)
    \(-20x^2 + 100x - 125\)

26. \(2y^3 - 6y^2 + 2y\)
    \(-y^2 + 3y - 1\)

6–2 Subtracting Polynomials

To subtract polynomials, use the same procedure as with binomials: add the opposite of each term of the subtrahend to the minuend. For example,

\[(3a + 4) - (a - 2) = 3a + 4 + (-a) + 2 = 2a + 6\]

**Example 1.** \((15x^2 - 3x + 9) - (2x^2 + x - 3)\)

**Solution 1:**
\[
(15x^2 - 3x + 9) - (2x^2 + x - 3)
= 15x^2 - 3x + 9 - 2x^2 - x + 3
= 13x^2 - 4x + 12
\]
Solution 2: \[
\begin{array}{c}
15x^2 - 3x + 9 \\
2x^2 + x - 3 \\
13x^2 - 4x + 12
\end{array}
\]

To subtract one polynomial from another, add to the minuend the opposite of each term of the subtrahend.

Symbols of inclusion may appear in an equation to indicate addition or subtraction of a polynomial.

**EXAMPLE 2.** Solve for \(x\):

\[
x^2 + 4x + (x - 16 + 2x) = 5x - (2 + 5x - x^2)
\]

**Solution:**

\[
x^2 + 4x + (x - 16 + 2x) = 5x - 2 - 5x + x^2
\]

Simplify parentheses \(\rightarrow\)

\[
x^2 + 4x + 3x - 16 = -2 + 0 + x^2
\]

Add like terms \(\rightarrow\)

\[
x^2 + 7x - 16 = x^2 - 2
\]

Subtract \(x^2\) from each member \(\rightarrow\)

\[
7x - 16 = -2
\]

Add 16 to each member \(\rightarrow\)

\[
7x = 14
\]

Divide each member by 7 \(\rightarrow\)

\[
x = 2
\]

**Check:**

\[
\begin{align*}
2^2 + 4 \cdot 2 + (2 - 16 + 2 \cdot 2) &= 5 \cdot 2 - (2 + 5 \cdot 2 - 2^2) \\
4 + 8 + (2 - 16 + 4) &= 10 - (2 + 10 - 4) \\
12 + (-10) &= 10 - (8)
\end{align*}
\]

\[
2 = 2 \checkmark
\]

\[\therefore\] The solution set is \(\{2\}\), Answer.

**Note:** In the solution use was made of the fact that adding the same polynomial to each member of an open sentence produces an equivalent sentence.

**ORAL EXERCISES**

Subtract the lower polynomial from the one above it.

1. \[
\begin{array}{c}
2a + b \\
a - b
\end{array}
\]

4. \[
\begin{array}{c}
3x + y \\
x - y
\end{array}
\]

7. \[
\begin{array}{c}
3.7b + 4a \\
-1.5b + 3a
\end{array}
\]

2. \[
\begin{array}{c}
7a + 8b \\
-2a + b
\end{array}
\]

5. \[
\begin{array}{c}
3a + 2b - 4 \\
-3a + 4b
\end{array}
\]

8. \[
\begin{array}{c}
5.4r^2 - 8rd \\
3.1r^2 + 5rd - 5
\end{array}
\]

3. \[
\begin{array}{c}
6x - 3y \\
-3x + 2y
\end{array}
\]

6. \[
\begin{array}{c}
2x - y + 3 \\
-2y + 5
\end{array}
\]
Perform the following subtractions.

**SAMPLE.**

\[(5z + 6) - (2z - 8)\]

*What you say:* 
\[5z + 6 - 2z + 8 = 3z + 14\]

9. \[(5a - 6) - (3a - 2)\]

10. \[(6b + 6) - (b - 1)\]

11. \[(x + y) - (x + y)\]

12. \[(x - y) - (x + y)\]

13. \[(a - b) - (2a - b)\]

14. \[(x + y) - (x - y)\]

15. \[(5a - 6b + c) - (3a + b - c)\]

16. \[(3x - 2y - 4) - (2x + 4y - 2)\]

17. \[(22n - k) - (n - k)\]

18. \[(7x - y) - (x - y)\]

**WRITTEN EXERCISES**

Subtract. Check by evaluation, using \(x = 3, y = 2, a = 5, b = 4\).

**A**

1. \[
\begin{array}{c}
21 - 14x \\
12 - 6x
\end{array}
\]

2. \[
\begin{array}{c}
3x + 2y \\
x - y
\end{array}
\]

3. \[
\begin{array}{c}
2x^2 + 2x + 7 \\
x^2 + 3x - 8
\end{array}
\]

4. \[
\begin{array}{c}
3b^2 + b - 1 \\
2b^2 + 6b + 5
\end{array}
\]

5. \[
\begin{array}{c}
.4x^2 - .7 \\
.5x^2 - 3x - .8
\end{array}
\]

6. \[
\begin{array}{c}
2 - 3a + 2a^2 \\
5 - 6a + a^2
\end{array}
\]

**Solve each of the equations.**

10. \[2y + (5 - y) = 18\]

11. \[5w - (2w + 3) = 12\]

12. \[13z - (z + 21) = 39\]

13. \[3x - (5 - 2x) = 35\]

14. \[7x + (8x + 54) = 9\]

15. \[9y + (7y + 67) = 3\]

16. \[5x + (2 - 7x) = 30\]

17. \[y + (9 - 8y) = 65\]

18. \[9n - (3n - 6) = 42\]

19. \[6z - (4z - 8) = 24\]

20. \[(3x + 4) - (x + 8) = 18\]

21. \[(5y + 7) - (y + 2) = 17\]

22. \[7n - (n^2 - n - 9) = 17 - n^2\]

23. \[19x - (1 - 2x - x^2) = 22 + x^2\]

24. \[(2x + 4) - (x - 7) = (4 - 3x) + (3x + 1)\]

25. \[(3y + 6) - (2y - 2) = (17 - 2y) + (2y - 13)\]

26. \[10 - (y + 6) + (3y - 2) = -5y + 9 - (-5y + 3)\]

27. \[12 - (z + 8) - (5z - 7) = -6z + 16 - (-6z + 7)\]

28. \[x - (.5x + 1) - (.3x + 2) = .5x - (.5x - 3)\]

29. \[.5n + (n - 1) - (.2n + 10) = .2n - (.2n - 28)\]

**B**

30. \[2a - [5a - (6a + 2)] = 10 - a\]

31. \[7x - [x - (2x + 8)] = 3x - 2\]
32. \[20x - [2x - (x + 2) + x^2 - 6] = 28 - x - x^2\]
33. \[5y - [y - (2y + 8) + 3 - y^2] = y^2 - y - 5\]
34. \[-[-5t - (4 - t) + 7] - (t - 2) = 4t - 1\]
35. \[-[-(7 - 2s) - 8 + 3s] - (3s + 1) = -3s - 1\]
36. \[2a + \{5a - [3 - (a + 2)] + 6\} = -(2a + 5)\]
37. \[5n + \{n - [2 - (2 + 3n)] + n\} = -(2n - 6)\]
38. \[1.2x - \{10x + [2 - .6x - (3 + x + .2)] + 13\} = -1\]
39. \[1.4y - \{9y + [3y - .3 - (1.2y + y - 3)] + 1.7\} = -3\]

**MULTIPLICATION OF POLYNOMIALS**

### 6–3 The Product of Powers

You will recall that \(b^4\) (read \(b\) exponent four or \(b\) fourth) stands for \(b \cdot b \cdot b \cdot b\). Therefore,

\[
b^4 \cdot b^2 = \underbrace{(b \cdot b \cdot b \cdot b)}_{6 \text{ factors}} \cdot \underbrace{(b \cdot b)}_{4 \text{ factors}} = b^{4 + 2} = b^6
\]

In multiplying these two powers of the same base, you could have found the exponent by retaining the base and adding the exponents of the factors. In general, for positive integral exponents \(m\) and \(n\),

\[
b^m \cdot b^n = \underbrace{(b \cdot b \cdot \ldots \cdot b)}_{m \text{ factors}} \cdot \underbrace{(b \cdot b \cdot \ldots \cdot b)}_{n \text{ factors}} = b^{m + n}
\]

This result is usually stated as the rule of exponents for multiplication:

For all positive integers \(m\) and \(n\): \(b^m \cdot b^n = b^{m+n}\)

To multiply monomials, you may use this rule of exponents together with the commutative and associative properties to determine the numerical factor and the variable factors of the product:

\[
(2rz^3)(-3r^6z^5) = (2 \cdot -3)(r^1 \cdot r^6)(z^3 \cdot z^5) = -6 \cdot r^{1+6} \cdot z^{3+5} = -6r^7z^8
\]

Note that this rule of exponents applies only when the bases of the powers are the same. You cannot use it, for example, to simplify the product of \(x^8y^9\) because the bases of the powers \(x^8\) and \(y^9\) are different.
Find each of the following products.

### A

1. \((-3rs)(5rs^2)r^4\)
2. \((-6xy)(8x^2y)y^3\)
3. \((-3a)(-3a)(3a)\)
4. \((-4a)(3ab)(-5b)\)
5. \((-5x)(-6y)(-z)\)
6. \((6n)(-6n)(-6n)\)
7. \((6x)(-3xy)(-4y)\)
8. \((-la)(-b)(-2c)\)
9. \((igt)(igt)\)
10. \((8rs^2t)(-6r^2s)\)
11. \(tx(tx)\)
12. \(rs(rs)(-r)\)
13. \(ab(-b)(ab)\)
14. \(-rt(r^2t)(-rt)\)
15. \(-3x(4x^3y^2)(-y^4x^2)\)
16. \(-5a(2a^2b^5)(-ba^9)\)
17. \((.2^2r^3s^7)(.4^2rs^8)\)
18. \((.3^2x^5)(9^2z^3v^4)\)

### B

19. \((-e^2d^3f^3)(-dc^7f^3)\)
20. \((-w^2x^5y^4)(-x^3y^2w^2)\)
21. \((-a)^2(a^3b)(-b)^3\)
22. \((-k)^3(k^4h^2)(-h)^2\)
23. \((-2m)^3(dm)(.3d)^2\)
24. \((.4b)^2(cb)(-2c)^3\)

Simplify by finding each indicated product and then combining similar terms.

### C

25. \((-7a^2)(5b)(-2b) + (4ab)(-2ab)\)
26. \((5p^2)(-3q)(-qp^2) - (7qp^3)(7pq)\)
27. \((-3rst)(4r^2st) + (-6s^3)(2rt^2)(-r^2)\)
28. \((9v^2wz^3)(-2vw^2z) + (-3z^2)(-6v^3w^2)(wz^2)\)
29. \((-2^2c^5d^3)(5cd^7)(-5e) + (3^2d^6c^3e)(-d^2c^2)(cd^2)\)
30. \((6^2m^3n^2p)(-2m^4p^4n^5) - (-p^3m^2n^3)(8^2m^5n^4p^2)\)
31. \(x^3(2x + 1) + 3x^2(x^2 + x)\)
32. \(3r^2(r^2 - r) - r^3(2r - 1)\)
33. \(10^m(10^n + 10^7)\)
34. \(a^m(a^n + b^m) + b^m(a^n + b^n)\)

### 6–4 The Power of a Product

Consider \(2b^3\) and \((2b)^3\); they are not equal unless \(b = 0\).

\[
2b^3 = 2 \cdot b \cdot b \cdot b \\
(2b)^3 = 2b \cdot 2b \cdot 2b = 2^3 \cdot b^3 = 8b^3
\]

In general, for every positive integral exponent \(m\), you can see that:

\[
(ab)^n = (ab)(ab) \ldots (ab) \\
\vdots \quad (ab)^n = (a \cdot a \cdot \ldots a)(b \cdot b \cdot \ldots b) = a^n b^n
\]

\(m\) factors \(m\) factors
These results may be summarized as the rule of exponents for a power of a product:

**For every positive integer \( m \):** \( (ab)^m = a^m b^m \)

For example:

\[
(-3x)^4 = (-3)^4 x^4 = 81x^4, \text{ and } (4rs)^2 = 4^2 r^2 s^2 = 16r^2 s^2
\]

Of course, the product whose power you are to find may itself be a power:

\[
(b^2)^3 = b^2 \cdot b^2 \cdot b^2 = b^{2+2+2} = b^6 = b^{2 \cdot 3}
\]

In general,

\[
b^n \text{ is a factor } n \text{ times} \quad (b^n)^n = \underbrace{(b^m)(b^m) \ldots (b^m)}_{n \text{ terms}}
\]

\[
\therefore (b^n)^n = b^{m+m+\ldots+m} = b^{mn}
\]

This result gives you the rule of exponents for a power of a power:

**For all positive integers \( m \) and \( n \):** \( (b^m)^n = b^{mn} \)

You use both of these rules in the following illustration:

\[
(-7w^7v^3)^3 = (-7)^3 (w^7)^3 (v^3)^3 = -343w^{21}v^9
\]

**Oral Exercises**

Give each of the following indicated powers.

1. \((2xy)^3\)  
2. \((-3ab)^3\)  
3. \((-2rst)^2\)  
4. \((3klm)^2\)  
5. \((2^3)^3\)  
6. \((2^3)^2\)  
7. \((b^3)^3\)  
8. \((e^4)^4\)  
9. \((-2a^2)^3\)  
10. \((-3b^3)^2\)  
11. \((-5x^4)^2\)  
12. \((-4y^5)^2\)  
13. \((-a^2b)^3\)  
14. \((-cd^2)^4\)  
15. \((6m^7p)^1\)  
16. \((7k^4n^3)^1\)

What is the square of each of the following?

17. \(-.3xy^2\)  
18. \(.2a^2b\)  
19. \(7y^3w^2\)  
20. \(-5r^2s^3\)
Find each of the following indicated products.

1. \((\text{-}2m^2)^3\)
2. \((-3s^2)^3\)
3. \(-3x(4xy)^2\)
4. \(-6n(5mn)^2\)
5. \((-b)(-3b)^3\)
6. \((-r)(-6r)^3\)
7. \((8m)(2mn)^3\)
8. \((3y)(5xy)^3\)
9. \((5a)(-2a^2b)^2\)
10. \((-3s)(4xy)^3\)
11. \((-3r)(-6r)^3\)
12. \((3y)(5xy)^3\)
13. \((2b^2d)^4(-3db)^2\)
14. \((-3r)(-6r)^3\)
15. \((3st^2v)(-3s^2tv^3)^2\)

Simplify each of the following expressions.

16. \((3xy)^2 + (-5x)(-4xy)(-y)\)
17. \((-2c)(-3cd)^3 + (3c)^2(cd)^2(-6d)\)
18. \((-a)^2a + (2a^2)(-2a^2)^3 - (2a^2)^4\)
19. \((2b)^2(-25b^4) + (-4b)(-b)^2 + (-10b^3)^2\)
20. \((-zt)^3(2z^2t) + (t^2z)^2(3z^3) + (t^2z)(z^2t)^2\)
21. \((-mk)^5(3mk^2) + (m^2k)^3(2k^2)^2 + (-m^3k^2)(-k)^4k\)
22. \(2x^3y(x^2y + 3y^3) - (xy)^2(x^3 - xy^2)\)
23. \((-5c^2d)(1 - c^4d^5) + (-2cd)(-3c + c^5d^3)\)
24. \(3k^2m(m^4k - 1) - .2km[k + (km)^2]\)
25. \((ab)^3(a^2 - ab^2) + (-.3b^2a)[(-ba)^3 + a^4b]\)

6-5 Multiplying a Polynomial by a Monomial

The distributive property of numbers, together with the rules of exponents for multiplication, enables you to multiply any polynomial by a monomial:

\[5x(2x^2 + 3) = 5x \cdot 2x^2 + 5x \cdot 3 = 10x^3 + 15x.\]

This result is illustrated in the figure.
The largest rectangle is made up of the other two. Its area is equal to the sum of the areas of the other two rectangles.

You may multiply either horizontally or vertically, as shown here.

\[-7y^2(3y^3 - 2y^2 + 1) = -21y^5 + 14y^4 - 7y^2\]

To multiply a polynomial by a monomial, use the distributive property: multiply each term of the polynomial by the monomial, and then add the products.

**ORAL EXERCISES**

Read the product of each indicated multiplication.

1. \(7(a - 2)\)
2. \(15(2x - 1)\)
3. \(-5(x + y)\)
4. \(-3(2a + b)\)
5. \(8(2x - 3y)\)
6. \(9(5m - 2n)\)
7. \(a^2(a - .3)\)
8. \(x^3(x - .7)\)
9. \(-3x^4(-2 + x)\)
10. \(-5a^5(-3 + a)\)
11. \(-1(8 + 2b)\)
12. \(-1(3x + 5)\)
13. \(-3a(-2a - 4)\)
14. \(-5r(-7 - 3r)\)
15. \((2n - 4)(-1)\)
16. \((5r - s)(-1)\)
17. \(2x - 3y\)
18. \(4x - 7y\)
19. \(-3x^3 + 4x - 1\)
20. \(-y^2 + 2y + 5\)

**WRITTEN EXERCISES**

Find each of the following products.

1. \(-5(x^2 - 3x + 7)\)
2. \(-6(a^2 - 2a + 1)\)
3. \(a(a^2 - 2ab + b^2)\)
4. \(x(x^2 - 2xy + y^2)\)
5. \(b^2(-a + b + c)\)
6. \(r^2(-r - s + t)\)
7. \(-x^2(5x^2 - x + 2)\)
8. \(-y^2(7 - 3y - y^3)\)
9. \((2 - 3v - 4v^2)(-2v^4)\)
10. \((x^2 - 6x + 9)(-3x^3)\)
11. \(3x^2y(5 - 2xy^4 + 3x^2y^3 - y^5)\)
12. \(5rs^2(3 + 4r^5s - r^3s^4 - 7rs^7)\)
13. \(-2a^3b^2(a^4b - a^3b^2 + 2a^2b^4 - b^5)\)
14. \(-3c^2d^4(c^5d + 4c^4d^3 - c^2d^4 - 2d^5)\)
Solve each of the following equations.

15. \(3x + 5(x - 3) = 9\)
16. \(8n + 6(-n - 2) = 16\)
17. \(-6y - 0.5(10 - y) = 17\)
18. \(-7x + 0.8(7 - x) = 18\)
19. \(5x + 3 - 3(x - 2) = 1\)
20. \((8x + 7) - 5(x - 8) = 44\)
21. \(12(x + 1) + 3(4x - 2) = 18\)
22. \(7(8 - x) + 5(2 + 2x) = 96\)
23. \(2x - 2(x + 21) = 3x - 3(7 - x)\)
24. \(5n - (n - 8) = 14n - 7(2n + 8)\)
25. \(0.5(3y - 7) + 13 = 0.2(3y - 12) - (6y - 5)\)
26. \(0.8x - (-x + 5) = -(x - 3) - 0.4(2 + 2x)\)
27. \(5 - 3[2n - 2(5 - 3n)] = 4(2 - 3n)\)

**Problems**

Give each answer as a polynomial or as a directed number.

1. A man travels for 2 hours at \((45 + x)\) miles per hour, then at \((200 + x)\) miles per hour for 2 hours. Represent the total distance traveled.
2. An airplane flew at \(x\) miles per hour for 2 hours, then at \((x + 25)\) miles per hour for another 4 hours. The entire distance was 1150 miles. Find its rate during the four-hour period.
3. A plane traveled at 190 miles per hour for \(h\) hours. After refueling, it traveled at 210 miles per hour for \((h + 1)\) hours. The entire trip was 1210 miles. Find the number of hours in the second part of the trip.
4. Mr. Macy drove 303 miles one day. During the first \(x\) hours, he averaged 34 miles per hour, but during the next \((x + 3)\) hours, his average speed was 47 miles per hour. How long did the trip take?
5. Harry mowed a rectangular lawn \(l\) feet long and 50 feet wide. Bob mowed one \((l + 20)\) feet long and 60 feet wide. Bob mowed 1500 square feet more than Harry. Find the length of each lawn.
6. Mr. Bernard’s and Mr. Cayne’s houses stand on rectangular lots of equal depth \(d\). Mr. Bernard’s lot is 30 feet wider than Mr. Cayne’s. Find the depth of the lots if their areas differ by 3600 square feet.
7. Charles earned \(d\) dollars on each of two days and \((d - 1)\) dollars on each of the next three days. What were his daily earnings if he earned $19.50 during the five days?
8. Mr. Reynolds traveled for 2 hours on a jet airplane and for one-half hour more on a piston-driven aircraft. The average speed of the jet was 4.6 miles per minute more than that of the piston-driven craft. If Mr. Reynolds traveled 1467 miles on this trip, find each plane’s speed.
9. A factory produced \( x \) dresses daily for six days and \((110 + x)\) dresses daily for the next five days. During the eleven days it completed 6600 dresses. How many dresses were produced each day during this time?

10. A machine which caps 1500 bottles an hour operated 2 hours longer than one which caps 1200 bottles an hour. If a total of 16,500 bottles were capped, how long did each machine operate?

6-6 Multiplying Two Polynomials

To simplify the product \((2x + 3)(4x + 5)\), first treat \((4x + 5)\) as a number to be multiplied by the binomial \(2x + 3\). By the distributive property, you find

\[
(2x + 3)(4x + 5) = (2x)(4x + 5) + 3(4x + 5) \\
= (2x)(4x) + (2x)(5) + 3(4x) + 3(5) \\
= 8x^2 + 10x + 12x + 15 \\
= 8x^2 + 22x + 15
\]

The figure illustrates this product.

![Diagram](image)

To multiply one polynomial by another, use the distributive property: multiply each term of one polynomial by each term of the other, and then add the products.

Usually it is convenient to set up the multiplication of polynomials in vertical form, and to work from left to right, thus:

\[
\begin{align*}
4x + 5 \\
2x + 3
\end{align*}
\]

This is \(2x(4x + 5)\) \(\Rightarrow\) \(8x^2 + 10x\)

This is \(3(4x + 5)\) \(\Rightarrow\) \(12x + 15\)

This is \((2x + 3)(4x + 5)\) \(\Rightarrow\) \(8x^2 + 22x + 15\)

To check the accuracy of your multiplication, evaluate the factors and their product, using any numbers except 0 and 1.
WRITTEN EXERCISES

Give each of the following products in its simplest form.

**A**

1. \((x + 7)(x + 2)\)  
4. \((n - 9)(n + 4)\)  
7. \((2x + 3)(5x + 1)\)
2. \((a + 8)(a + 7)\)  
5. \((x - 7)(x - 6)\)  
8. \((7n + 4)(8n + 9)\)
3. \((y - 9)(y + 5)\)  
6. \((y - 3)(y - 8)\)  
9. \((7x - 3)(5x + 2)\)

10. \((12w + 5)(5w - 2)\)  
18. \((x + 9)(x + 9)\)
11. \((3y - 4)(4y - 6)\)  
19. \((m + 3n)(m + 3n)\)
12. \((6x - 5)(7x - 8)\)  
20. \((m - 4n)(m - 4n)\)
13. \((a + b)(a + b)\)  
21. \((r + 3s)(r - 3s)\)
14. \((x - y)(x - y)\)  
22. \((3a - b)(3a + b)\)
15. \((x - y)(x + y)\)  
23. \((2a - 3b)(a + 5b)\)
16. \((c + d)(c - d)\)  
24. \((2a - 5b)(a + 3b)\)
17. \((m - 5)(m - 5)\)

**B**

25. \((m - .3)(m^2 - 3m + .4)\)  
29. \((a^2 - 8)(a^2 + 1)\)
26. \((2n - .3)(n^2 - .6n - 1)\)  
30. \((y^2 + 40)(y^2 - 5)\)
27. \((a^2 + ab + b^2)(a - b)\)  
31. \((xy - .7)(xy - .8)\)
28. \((a^2 - ab + b^2)(a + b)\)  
32. \((ab - .1)(ab + .3)\)

**C**

33. \((3x^2 - x + 2)(x^2 + 2x + 1)\)
34. \((5y^2 - y + 3)(y^2 - 3y - 2)\)
35. \((2a^2 + a - .1)(a^2 + .3a + 4)\)
36. \((.4b^2 - b + 3)(b^2 + 2b + .5)\)
37. \((2a - 4b - 3c)(4a + 4b + 5c)\)
38. \((3m - m^2)(mp + p - p^2)\)
39. \((x - 1)(x^4 + x^3 + x^2 + x + 1)\)
40. \((y + 1)(y^4 - y^3 + y^2 - y + 1)\)

By using the properties of numbers, prove that each of the following statements is true for all values of the variables.

41. \((a + b)(c + d) = ac + bc + ad + bd\)
42. \((a - b)(c + d) = ac - bc + ad - bd\)
43. \((x + y)(x - y) = x^2 - y^2\)
44. \((x + y)(x + y) = x^2 + 2xy + y^2\)
6-7 Problems about Areas

The solution of problems concerned with areas requires the ability to multiply polynomials. In analyzing such problems, you will find sketches especially useful.

EXAMPLE. To support the weight of a building, an engineer determines that its foundation must have an area of 496 square feet bearing on the soil. The building is to be rectangular, and its foundation, 2 feet thick. If the inside length of the foundation is two times the inside width, find these dimensions.

Solution:

Let \( w \) = number of feet in width of inner rectangle.

Then \( 2w \) = number of feet in length of inner rectangle.

Total Area = Area of Concrete + Inner Area

\[
(2w + 4)(w + 4) = 496 + 2w \cdot w
\]

Solve the equation:

\[
(2w + 4)(w + 4) = 496 + 2w^2
\]

\[
2w^2 + 8w + 4w + 16 = 496 + 2w^2
\]

\[
12w + 16 = 496
\]

\[
12w = 480
\]

\[
w = 40
\]

\[
2w = 80
\]

(1) Is the length twice the width? \( 80 = 2(40) \) \( \checkmark \)

(2) Is the area of the top of the concrete 496 square feet?

Area of Concrete = Total Area - Inner Area

\[
496 \quad \div \quad 84 \cdot 44 \quad - \quad 80 \cdot 40
\]

\[
496 \quad \div \quad 3696 \quad - \quad 3200
\]

\[
496 \quad = \quad 496 \quad \checkmark
\]

Inside width of foundation: 40 feet

Inside length of foundation: 80 feet

Answer.
CHAPTER SIX

PROBLEMS

Make a sketch for each problem, and solve.

1. One end of a casting is a square. The other is a rectangle whose length is 3 inches greater than a side of the square end and whose width is 2 inches less than a side of the square end. The areas of the two ends are equal. Find a side of the square end.

2. Two hi-fi loud-speaker enclosures are a square and a rectangle, respectively. The length of the rectangle is 1 inch greater than a side of the square, and its width is 3 inches less than a side of the square. The area of the rectangle is 45 square inches less than the area of the square. How long is a side of the square?

3. The distance from home plate to first base in softball is 30 feet less than the corresponding distance in baseball. The square bounded by the bases of a softball diamond is 4500 square feet less in area than that of a baseball diamond. Find the distance from one base to the next in softball.

4. The distance from first to second base in indoor baseball is 33 feet less than the corresponding distance in softball. The square bounded by the bases of an indoor baseball diamond is 2871 square feet less in area than that bounded by the bases of a softball diamond. Find the distance from one base to the other in indoor baseball, using only the information given in this problem.

5. The Taylors plan to replace their croquet court by a badminton court. The badminton court will require 920 square feet less than the croquet court and will be 16 feet shorter and 10 feet narrower. If the croquet court is twice as long as it is wide, find the dimensions of the badminton court.

6. The standard basketball court for college women is twice as long as it is wide. The standard basketball court for college men is 4 feet longer, 5 feet wider, and 650 square feet greater in area than the women’s court. Find the dimensions of the men’s court.

7. A field is used for both lacrosse and field hockey. The length of the lacrosse field is 75 feet greater than its width. The length and width of the hockey field are 60 feet and 105 feet less than the length and width of the lacrosse field, and its area is 38,700 square feet less. Find the dimensions of the lacrosse field.

8. A rectangular field is planted in wheat. It is 500 feet longer than it is wide. In planting, a strip 10 feet wide is left unplanted on two adjacent sides. The area left unplanted is 44,900 square feet. What are the dimensions of the field?
9. The cross section of a rivet is in the shape shown. The base of the triangle is 3 inches longer, and its height is 2 inches shorter, than a side of the square, and its area is one-half that of the square. Find a side of the square.

10. Two crates are in the form of cubes. An edge of one crate is 1 foot longer than that of the other, and it requires 54 square feet more lumber to build. Find an edge of the smaller crate. (Disregard the thickness of the wood.)

11. A cross section (see diagram) of a circular water main is to have a wall 1\(\frac{1}{2}\) inches thick and an area of 478\(\frac{1}{2}\) square inches. Find its inner diameter. (Use \(\pi = \frac{22}{7}\).)

12. A circular concrete conduit has walls 2 inches thick, and the cross-sectional area of the concrete is 88 square inches. Find the inner diameter of the conduit.

13. The height of a box is 2 feet more than its length, and its width is 2 feet less than its length. The volume of the box is 80 cubic feet less than that of a cube having the same length. What are the dimensions of the box?

14. An open box is to be made from a piece of cardboard 8 inches long and 6 inches wide by cutting out a square from each corner and turning up the sides. Find the dimensions of the box, if the area of its base is 8 square inches less than the total area cut from the cardboard.

6-8 Powers of Polynomials

The area \(A\) of a square is found by using the equation \(A = s^2\), and the volume \(V\) of a cube, by using \(V = s^3\). The figure shows a cube in which each edge \(s\) is given as \((2x - 3)\). The base of this cube, then, has an area of \((2x - 3)^2\), read the square of \(2x - 3\). The volume is \((2x - 3)^3\), read the cube of \(2x - 3\).

In each of these expressions, the exponent shows how many times the polynomial is to be used as a factor. To find the product of these
factors, expressed as a sum of monomials, is to **expand the expression.**

To expand the expression \((2x - 3)^2\), find the product \((2x - 3)(2x - 3)\).

To expand \((2x - 3)^3\), find the product \((2x - 3)((2x - 3)(2x - 3))\).

\[
\begin{align*}
2x - 3 \\
2x - 3 \\
4x^2 - 6x \\
- 6x + 9 \\
4x^2 - 12x + 9 = (2x - 3)^2
\end{align*}
\]

\[
\begin{align*}
4x^2 - 12x + 9 \\
2x - 3 \\
8x^3 - 24x^2 + 18x \\
- 12x^2 + 36x - 27 \\
8x^3 - 36x^2 + 54x - 27 = (2x - 3)^3
\end{align*}
\]

**WRITTEN EXERCISES**

Expand each of the following expressions. Check your work by evaluating the factors and their products in each case.

**A**

1. \((a + b)^2\)  
2. \((a - b)^2\)  
3. \((a - b)^3\)  
4. \((a + b)^3\)  
5. \((3x + 2)^2\)  
6. \((5x + 2)^2\)  
7. \((7y + 3z)^2\)  
8. \((2m + 6n)^2\)  
9. \((r - 8s)^2\)  
10. \((c - 4d)^2\)  
11. \((3x - 1)^3\)  
12. \((2y - 1)^3\)

**B**

13. \((a + b + c)^2\)  
14. \((a - b + c)^2\)  
15. \((2x^2 + x - 3)^2\)  
16. \(.3y^2 - y + .2)^2\)  
17. \((x - y)^4\)  
18. \((x + y)^4\)  
19. \((2x^2 + x - 2)^3\)  
20. \((3y^2 - y + 2)^3\)  
21. \(.8x^2 - .9x)^3\)

**PROBLEMS**

Solve and check each of the following problems.

**A**

1. A side of the square opening of a ventilating duct is 3 inches longer than a side of another such opening. Their areas differ by 81 square inches. Find the length of a side of each opening.

2. Two square pieces of asbestos are used to insulate a corner stove. One piece is 10 inches shorter than the other, and its area is 1100 square inches less. What are the dimensions of the smaller piece?

3. The squares of two consecutive integers differ by 103. Find the integers.

4. The difference between the squares of two consecutive integers is 95. Find the integers.

5. A square picture is in a frame 1 inch wide. If the area of the frame is 24 square inches, find the width of the picture.
6. An inlaid checkerboard has a mahogany border 2 inches wide. The area of the border is 176 square inches. Find the length of the board.

7. The product of two consecutive integers exceeds the square of the smaller integer by 13. Find the integers.

8. The product of two consecutive integers is 17 less than the square of the larger integer. Find the integers.

9. An adjoining living room and bedroom have the same width, but the living room is 5 feet longer. If the bedroom is square and contains 75 square feet less area than the living room, what are the dimensions of the living room?

10. A rectangle is 5 feet longer than it is wide. A square with the same length has an area 105 square feet larger. Find the dimensions of the rectangle.

11. A circular skating rink is to be 28 feet larger in diameter than an adjoining circular restaurant. The area of the rink will be 3080 square feet larger than that of the restaurant. Find (a) the diameter of the rink and (b) the cost of linoleum for the restaurant floor at 35 cents per square foot. (Use $\pi = \frac{22}{7}$.)

12. A round iron rod 60 inches long has a diameter 1 inch less than that of another round rod of the same length, and its volume is $45\pi$ cubic inches less. Find (a) both diameters, (b) both volumes, and (c) the weights of both rods. (Iron weighs $\frac{5}{18}$ pounds per cubic inch.)

**DIVISION OF POLYNOMIALS**

**6-9 The Quotient of Powers**

Recall (p. 139) that dividing by a number is the same as multiplying by the reciprocal of the number. Therefore, another way to write the quotient $\frac{xy}{cd}$ is $xy \cdot \frac{1}{cd}$. You also know (p. 139) that $\frac{1}{cd}$ equals $\frac{1}{c} \cdot \frac{1}{d}$. Putting these facts together, you have

$$\frac{xy}{cd} = xy \left(\frac{1}{c} \cdot \frac{1}{d}\right)$$

Substitution principle

$$= \left(x \cdot \frac{1}{c}\right) \left(y \cdot \frac{1}{d}\right)$$

Commutative and associative properties

$$= \frac{x}{c} \cdot \frac{y}{d}$$

Meaning of division
This gives the property of quotients:

$$\frac{xy}{cd} = \frac{x}{c} \cdot \frac{y}{d}$$

for every replacement of $x$ and $y$ by numbers, and $c$ and $d$ by nonzero numbers.

In particular, if $c = 1$, you have: $\frac{xy}{d} = x \cdot \frac{y}{d}$;

and if $x = 1$, you have: $\frac{y}{cd} = \frac{1}{c} \cdot \frac{y}{d}$.

For example,

$$\frac{25 \cdot 28}{5 \cdot 14} = \frac{25 \cdot 28}{5 \cdot 14} = \frac{5 \cdot 2}{1} = 10$$

$$\frac{7 \cdot 15}{5} = \frac{7 \cdot 15}{5} = \frac{7 \cdot 3}{1} = 21$$

$$\frac{9}{2 \cdot 3} = \frac{1}{2} \cdot \frac{9}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

This property of quotients is helpful in simplifying quotients of powers; consider $\frac{a^7}{a^4}$:

$$\frac{a^7}{a^4} = \frac{a^{3+4}}{a^4} = \frac{a^3}{a^4} = a^{3-4} = \frac{a^3}{a^4}$$

Notice that you could have found the exponent in the quotient by retaining the base and subtracting the exponent in the denominator from the exponent in the numerator.

Similarly, whenever $m > n$, you may write

$$\frac{b^m}{b^n} = \frac{b^{m-n} \cdot b^n}{b^n} = b^{m-n} \cdot \frac{b^n}{b^n} = b^{m-n}$$

On the other hand,

$$\frac{b^2}{b^0} = \frac{b^2}{b^2} = \frac{1}{b^2} \cdot b^2 = \frac{1}{b^0} \cdot 1$$

$$\therefore \frac{b^2}{b^0} = \frac{1}{b^2} = \frac{1}{b^{0-2}}$$
Whenever \( m < n \),
\[
\frac{b^m}{b^n} = \frac{b^m}{b^{n-m}} \cdot \frac{1}{b^m} = \frac{b^{n-m}}{b^n} = \frac{1}{b^{n-m}}
\]

These are two important rules of exponents for division:

For positive integers \( m \) and \( n \) and nonzero \( b \),
\[
\text{if } m > n, \text{ then } \frac{b^m}{b^n} = b^{m-n}
\]
\[
\text{if } m < n, \text{ then } \frac{b^m}{b^n} = \frac{1}{b^{n-m}}
\]

When dividing monomials, you use these rules together with the property of quotients obtained above:
\[
\frac{15x^3y^4}{-5xy^3} = \frac{15 \cdot x^3 \cdot y^4}{-5 \cdot x \cdot y^3} = -3 \cdot x^{3-1} \cdot y^{4-3} = -3x^2y
\]
\[
\frac{-7a^{10}c^3}{-42a^8c^5} = \frac{-7 \cdot a^{10} \cdot c^3}{-42 \cdot a^8 \cdot c^5} = \frac{1}{6} \cdot a^{10-8} \cdot \frac{1}{c^{5-3}} = \frac{a^2}{6c^2}
\]

**ORAL EXERCISES**

Give each of the following quotients.

1. \( \frac{y^{10}}{y^4} \) 7. \( \frac{30x^2}{2x} \) 13. \( \frac{-45x^2}{5x} \) 19. \( \frac{36a^2b}{-12a^4b^2} \)
2. \( \frac{z^8}{z^2} \) 8. \( \frac{20r^2}{5r} \) 14. \( \frac{-170z^2}{5z} \) 20. \( \frac{20xy^2}{-10x^2y^3} \)
3. \( \frac{u^7}{u^{14}} \) 9. \( \frac{14x^3y}{-2x} \) 15. \( \frac{-5a^3b}{-15a^2b^4} \) 21. \( \frac{-5m^2n}{10m^2n^2} \)
4. \( \frac{v^8}{v^{16}} \) 10. \( \frac{21ab^2}{-3b} \) 16. \( \frac{-7r^3s}{-21rs^5} \) 22. \( \frac{7c^2d^3}{-28c^3d^3} \)
5. \( \frac{p^7}{-p^2} \) 11. \( \frac{-22m^2n}{-2m} \) 17. \( \frac{(-1.5xyz)^2}{-1.5xy} \) 23. \( \frac{-.3r^2s}{-3rs^2} \)
6. \( \frac{q^6}{-q^2} \) 12. \( \frac{-121bc^2}{-11c} \) 18. \( \frac{(-.6abc)^2}{-.6ac} \) 24. \( \frac{-11ab^2}{-.11a^2b} \)
Find each of the following quotients.

A
1. \( \frac{64a^2b}{8a} \)  
2. \( \frac{48b^2c}{6c^3} \)  
3. \( \frac{-40d^2e}{-5de^2} \)  
4. \( \frac{-96x^2y}{-8xy^2} \)  
5. \( \frac{45y^2z}{-5y^4} \)  
6. \( \frac{54r^4s}{-9r^2} \)  
7. \( \frac{-84s^2t}{7st^2} \)  
8. \( \frac{-132m^2n}{11mn^2} \)  
9. \( \frac{-8a^2b^7}{-56a^{12}b^3} \)  
10. \( \frac{-13g^2h^3}{-39g^3h} \)  
11. \( \frac{.3x^2y^7}{-5x^8y} \)  
12. \( \frac{.6rs^7}{-8r^2s^4} \)  
13. \( \frac{-5a^2b}{21(abc)^2} \)  
14. \( \frac{-4m^2n}{(4mnp)^3} \)  
15. \( \frac{-9abc}{-6c^2} \)  
16. \( \frac{-6xyz}{-9x^2} \)  
17. \( \frac{2.7p^5r^2}{(-3)^3pr^4} \)  
18. \( \frac{(-2)^4vw^9}{1.6v^2w^7} \)  
19. \( \frac{(-4)^3x^3z^5}{8x^{12}z^{10}} \)  
20. \( \frac{(-3)^4d^{11}h^4}{9^2d^8h^4} \)  

B
21. \( \frac{-3a^2b^y}{2ab} \)  
22. \( \frac{-2a^x y^b}{6x^b y^a} \)  
23. \( \frac{32(ab)^m}{8a^mb^n} \)  
24. \( \frac{51(a^m b^n)}{-13(ab^n)^m} \)

6-10 Zero as an Exponent (Optional)

If you use the rule of exponents to evaluate \( \frac{b^m}{b^m} \), you obtain two expressions:

\[
\frac{b^m}{b^m} = b^m \cdot \frac{1}{b^m} = b^{m-m} \cdot 1 = b^0 \quad \text{and} \quad \frac{b^m}{b^m} = b^m \cdot \frac{1}{b^m} = \frac{1}{b^{m-m}} = \frac{1}{b^0}.
\]

These two expressions seem to indicate that \( b^0 \) should represent a number that is its own reciprocal. The only number with this property is 1.

From these considerations, you make this definition of \( b^0 \) (read \( b \) exponent zero): \( b^0 \) is 1 for every nonzero \( b \). No meaning is assigned to the expression \( 0^0 \).

ORAL EXERCISES

Simplify each of the following. Assume that 0 is not a member of the replacement set of any of the variables.

1. \( (1000)^0 \)  
2. \( (-49)^0 \)  
3. \( (-\frac{3}{4})^0 \)  
4. \( (\frac{1}{1000})^0 \)  
5. \( a^5 \div a^0 \)  
6. \( e^6 \div e^0 \)  
7. \( f^0 \div f^0 \)  
8. \( g^0 \div g^4 \)
9. \((7b)^0\)
10. \((-3y)^0\)
11. \(b^0 \div b^m\)

Simplify each of the following.

1. \(\frac{3x^4}{x} - \frac{4x^2}{x^2} + 7 - 4x(3x^2)\)
2. \(-\frac{4y^5}{y^3} - 2y(-3y) + \frac{8y^3}{2y^3} + 4\)
3. \(-\frac{70de^7}{-7de} + (5e^3)^2 - (.10e^6)^0\)
4. \(-\frac{56r^8}{-8r^8} + (-7s^2)^2 + (.7s^2)^0\)
5. \(-\frac{81x^2y^6z^4}{(3x^2y^2z^2)^2} + (-3)^0\)

6. \((-2)^0 + \frac{(-2a^3b^2z^4)^2}{-16a^6b^4z^8}\)
7. \(\frac{x(x^2 + 1)}{x} - (x^2 + 1)^0\)
8. \(\frac{y^2(1 - y^2)}{y^2} + (y^2)^0\)
9. \(\frac{x^3(3x - 5)z}{x^3z} - (3x - 5)^0\)
10. \(\frac{y^5(4 - y)t}{y^5t} + (y - 4)^0\)

6–11 Dividing a Polynomial by a Monomial

One way to simplify the numerical expression \((93 + 48) \div 3\) is to use the distributive property:

\[
\frac{93 + 48}{3} = \frac{1}{3}(93 + 48) = \frac{1}{3}(93) + \frac{1}{3}(48) = 31 + 16 = 47.
\]

Similarly, you may simplify the algebraic expression \((ax + ay) \div a\):

\[
\frac{ax + ay}{a} = \frac{1}{a}(ax + ay)
= \frac{1}{a}(ax) + \frac{1}{a}(ay)
= \left(\frac{1}{a} \cdot a\right)x + \left(\frac{1}{a} \cdot a\right)y
= 1x + 1y = x + y
\]

The effect of this procedure is that of dividing each term of the polynomial \(ax + ay\) by the monomial \(a\).
EXAMPLE 1. \( \frac{3a^3 + 15a^2 - 9a}{3a} \)

Solution:

\[
\frac{3a^3 + 15a^2 - 9a}{3a} = \frac{3a^3}{3a} + \frac{15a^2}{3a} - \frac{9a}{3a}
\]

\[
= a^2 + 5a - 3
\]

EXAMPLE 2. \( \frac{ay^2 - y + a}{ay} \)

Solution:

\[
\frac{ay^2 - y + a}{ay} = \frac{ay^2}{ay} - \frac{y}{ay} + \frac{a}{ay}
\]

\[
= y - \frac{1}{a} + \frac{1}{y}
\]

To divide a polynomial by a monomial, use the distributive property: divide each term of the polynomial by the monomial.

**WRITTEN EXERCISES**

Find each of the following indicated quotients.

1. \( \frac{12x + 15y}{3} \)
2. \( \frac{5x + 10y}{5} \)
3. \( \frac{2x + 3x}{x} \)
4. \( \frac{6y + 16y}{y} \)
5. \( \frac{5a + b}{a} \)
6. \( \frac{8m - n}{n} \)
7. \( \frac{r^2 - 7r}{r} \)
8. \( \frac{2s^2 + 4s}{s} \)
9. \( \frac{4c^2 + 3c}{3} \)
10. \( \frac{3b^2 + 2b}{2} \)
11. \( \frac{8a^3 - 4a}{2a} \)
12. \( \frac{7e^3 + 14c}{7c} \)
13. \( \frac{12x^3 + 6x}{6x} \)
14. \( \frac{15y^2 - 5y}{5y} \)
15. \( \frac{24n^3 - 12n^2 + 15n}{3n} \)
16. \( \frac{50r^3 + 10r^2 - 35r}{5r} \)
17. \( \frac{8x^3 - 4x^2 - 2x}{-x} \)
18. \( \frac{2n^4 - 3n^3 - 4n^2}{-n} \)
19. \( \frac{-60m^3n^3 + 36m^2n^2 - 6mn}{-6mn} \)
20. \( \frac{12x^3y^3 - 6x^2y^2 + 18xy}{-6xy} \)
21. \( \frac{5x^4 - 15x^3 + 45x^2 - 10x}{5x} \)
22. \( \frac{35a^4 - 28a^3 - 56a^2 - 14a}{7a} \)
23. \( \frac{2.4a^2b^2 + .6ab^2 + 30a^2b}{.3ab} \)
24. \( \frac{3.2st^2 + .8s^2t + 40s^2t^2}{.4st} \)
WORKING WITH POLYNOMIALS

221

25. \[ \frac{32a^5 - 6a^4b + 24a^3b^2}{-8a^2} \]

26. \[ \frac{12h^4 - 24h^3k - 6h^2k^2}{-36h^2} \]

27. \[ \frac{30x^3y^3 - 45x^2y^2 + 15xy}{75xy^2} \]

28. \[ \frac{12r^2t^2 + 18r^2t^3 - 6r^3t^4}{21r^2t^4} \]

29. \[ \frac{21m^3n - 28m^2n^2 + 35mn^3}{-35mn^3} \]

30. \[ \frac{9r^2s + 18rs^2 - 27s^3}{-27rs^2} \]

31. If \[ \frac{a + b}{b} = 3.5 \], find \[ \frac{a}{b} \].

32. If \[ \frac{n^2 - x^2}{x^2} = -0.91 \], find the positive value of \( \frac{n}{x} \).

33. Solve the equation: \[ \frac{5a^2 + 2a}{a} = 3a - 8 \].

34. Solve the equation: \[ \frac{3x^2 - 7x}{x} = x - 13 \].

6–12 Dividing a Polynomial by a Polynomial

The adjoining example illustrates a division problem in arithmetic. Can you recognize these points? 1. The method uses repeated subtraction; first 20(12), then 2(12) is subtracted from 273. 2. The distributive property helps shorten the number of steps; without it you subtract 12 from 273 twenty-two times. 3. The check is a transformation of the division equation, 273 - (22)(12) = 9; the check is (22)(12) + 9 = 273. In general form, these statements are:

\[
\text{Dividend} - \text{Quotient} \times \text{Divisor} = \text{Remainder}
\]

\[
\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}
\]

Both equations are equivalent to a third:

\[
\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}
\]

which for the example is \( \frac{273}{12} = 22 \frac{9}{12} \). It is this last form which gives the complete quotient \( 22 \frac{9}{12} \).
When dividing polynomials, you follow a similar pattern, after the terms of divisor and dividend have been written in order of decreasing degree in a variable:

**EXAMPLE 1.** \( (x + 2)(x^2 + 5x + 6) \)

**Solution:**

\[
\begin{align*}
x^2 + x &= x \quad \text{Subtract } x(x + 2) \\quad x^2 + 2x \quad 3x + 6 \\
x + 3 \quad x^2 + 2x \quad 3x + 6 \quad 0 \quad \text{Subtract } 3(x + 2)
\end{align*}
\]

**Check:** \((x + 2)(x + 3) + 0 = x^2 + 5x + 6\)

In the following example the steps in the division are shown compactly. Example 2 also shows how to insert missing terms in a dividend, using 0 as a coefficient.

**EXAMPLE 2.** \( (m^2 + 2m + 1)(m^3 - 7) \)

**Solution:**

\[
\begin{align*}
m^2 + 2m + 1 \quad m - 2 \\
m^3 + 2m^2 + m \quad -2m^2 - m - 7 \\
\quad -2m^2 - 4m - 2 \\
\quad 3m - 5
\end{align*}
\]

**Check:** \((m^2 + 2m + 1)(m - 2) + (3m - 5) = m^3 - 7\)

When do you stop dividing? You stop when the remainder is zero or the degree of the remainder is less than that of the divisor.
Divide the polynomials as indicated. Express each answer as a complete quotient, and check.

1. \( \frac{x^2 + 5x + 6}{x + 2} \)
2. \( \frac{x^2 + 3x + 2}{x + 1} \)
3. \( \frac{x^2 + 7x + 12}{x + 3} \)
4. \( \frac{x^2 + 11x + 28}{x + 7} \)
5. \( \frac{y^2 - 13y + 42}{y - 6} \)
6. \( \frac{x^2 - 15x + 56}{x - 7} \)
7. \( \frac{r^2 - 5r - 7}{r + 1} \)
8. \( \frac{n^2 - 7n - 9}{n + 1} \)
9. \( \frac{4 - 8n + 3n^2}{3n - 2} \)
10. \( \frac{64 - 16z + z^2}{z - 8} \)
11. \( \frac{16x^2 - 49}{4x - 7} \)
12. \( \frac{25x^2 - 81}{5x + 9} \)
13. \( \frac{81 + a^2}{a + 9} \)
14. \( \frac{x^2 + 9}{3 + x} \)
15. \( \frac{y^2 - 6yz - 27z^2}{y - 9z} \)

16. \( \frac{w^2 - 11wx - 102x^2}{w - 17x} \)
17. \( \frac{3x^2 - 14xy + 25y^2}{x - 3y} \)
18. \( \frac{5x^2 - 13xy + 16y^2}{x - 2y} \)
19. \( \frac{6y^2 + 11y - 10}{2y + 5} \)
20. \( \frac{10n^2 + 7n - 12}{2n + 3} \)
21. \( \frac{8x^3 - y^3}{2x - y} \)
22. \( \frac{216b^3 + a^3}{a + 6b} \)
23. \( \frac{343y^3 + x^3}{x + 7y} \)
24. \( \frac{2x^2 + 11x - 18}{2x - 3} \)
25. \( \frac{12x^2 + 4x - 18}{2x + 3} \)

26. \( \frac{x^3 + 2x^2 + 11x^2 + 11x + 15}{x + 3} \)
27. \( \frac{x^3 - 5x^2 + 2x - 17x + 10}{x - 5} \)
28. \( \frac{6x^3 + x^2 - 18x - 33}{2x - 3} \)
29. \( \frac{2n^3 - 5n^2 + 21n - 14}{2n - 3} \)
30. \( \frac{x^4 - 2x^3 + 3x^2 - 4x^3 + 10x^2 - 12x + 9}{x^2 - 2x + 3} \)
31. \( \frac{y^4 + 2y - 1}{y^4 + 4y^3 + 2y^2 - 4y + 1} \)

32. One factor of \( x^3 + 1 \) is \( x + 1 \). Find the other factor.
33. One factor of \( y^3 - 1 \) is \( y - 1 \). Find the other factor.
34. Is \( 2y + 3 \) a factor of \( 6y^3 + 2y^2 + y - 9 \)? Justify your answer.
35. Is \( 3z - 2 \) a factor of \( 9z^3 - 4z^2 - 5z + 8 \)? Justify your answer.
36. If 718 is divided by a certain number, the quotient is 59, and the remainder is 10. Find the number.
37. If 395 is divided by a certain number, the quotient is 28, and the remainder is 3. Find the number.
Some 3600 years ago, the Pharaoh of Egypt had a prime minister named Joseph, of whom you surely have heard. Joseph was not the only subject of the Pharaoh whose name is known today. A certain Aahmesu, whose name means "born of the moon," is remembered also. His position in life was much humbler than Joseph's; probably he was a scribe. Nowadays he is known as Ahmes, the writer of the Ahmes Papyrus.

The Ahmes Papyrus is an ancient handbook of mathematics. In it there are eighty algebra problems, each with its solution. Many of these problems were of the "find-a-number" type. Here is one of Ahmes' number problems:

Heap, its two-thirds, its one-half, and its three-sevenths, added together, becomes thirty-three. What is the quantity?

Other problems dealt with everyday affairs, with bread and beer, with feeding livestock and storing grain. Some of these were practical; some were clearly just for fun. Here is one of the latter sort:

There are seven houses; in each are seven cats. Each cat kills seven mice. Each mouse would have eaten seven ears of spelt [wheat]. Each ear of spelt will produce seven hekats of grain. What is the total of these? [That is, how much grain was saved?]

A portion of the Ahmes Papyrus. This ancient book is now in the British Museum.
Chapter Summary

Inventory of Structure and Method

1. To add polynomials, combine their similar terms. To subtract a polynomial from another, add to the minuend the opposite of each term of the subtrahend.

You can check your work with polynomials by substituting a particular value (except 0 and 1) for each variable and evaluating each expression.

2. To multiply two powers with the same base, use the rule: \( b^m \cdot b^n = b^{m+n} \) for all positive integers \( m \) and \( n \).

To multiply monomials, multiply the numerical factors and the variable factors. When the base of a power is a product, the rule \((ab)^m = a^m b^m\) applies for every positive integer \( m \). When the base is itself a power, then for all positive integers \( m \) and \( n \), \((b^m)^n = b^{mn}\).

3. To multiply a polynomial by a monomial, apply the distributive property, multiplying each term of the polynomial by the monomial. To multiply a polynomial by a polynomial, use the distributive property to multiply one polynomial by each term of the other. Expand a power of a polynomial by using the polynomial as a factor as many times as shown by the exponent and by performing the indicated multiplications.

4. To divide two powers with the same base, use the rule: for positive integers \( m \) and \( n \) and nonzero \( b \), if \( m > n \), \( \frac{b^m}{b^n} = b^{m-n} \), and if \( m < n \), \( \frac{b^m}{b^n} = \frac{1}{b^{n-m}} \).

To divide monomials, divide the numerical factors and the variable factors. To divide a polynomial by a monomial, apply the distributive property, dividing each term of the polynomial by the monomial. To divide a polynomial by a polynomial, arrange the terms of divisor and dividend in order of decreasing degree in one variable, and then proceed as in arithmetic division. The process stops when the remainder is 0, or when its degree is less than that of the divisor. In general,

\[
\frac{\text{Dividend (D)}}{\text{divisor (d)}} = \text{Quotient (Q)} + \frac{\text{Remainder (R)}}{\text{divisor (d)}}
\]

5. (Optional) By definition, for any nonzero number \( b \), \( b^0 = 1 \).

Vocabulary and Spelling

- monomial (p. 197)
- polynomial (p. 197)
- binomial (p. 197)
- trinomial (p. 197)
- degree (monomial) (p. 197)
- degree (polynomial) (p. 197)
- expand (an expression) (p. 214)
Chapter Test

6-1 Find each of the following sums.

1. \((13x + 61) + (56x - 16)\)
2. \(3a^2 - 5a - 6\)

Check each addition by letting \(n = 2\) and \(r = 3\).

3. \((5n - 6) + (2n - 80) + (3n + 17) = 10n - 69\)
4. \((r^2 + r + 6) + (2r^2 - 2r - 5) = 3r^2 - r + 1\)

6-2 In each case, subtract the lower polynomial from the one above it. Check your work by letting \(x = 2\), \(y = 3\), and \(z = 4\).

5. \(2x - 3y - 6z\)
6. \(x^2 + y^2 - z^2\)

Solve each of the following equations.

7. \(30a + (6 - 2a) = 34\)
8. \(2n - [n - (2n + 8) - 1] = 0\)

6-3 Find each product.

9. \((3mn^3t^2)(-m^2t^3)\)
10. \((-r)^3(2rs^2)(3r^2s^4)\)

6-4 Simplify each expression.

11. \((-4m^2n^3)^3\)
12. \((-3x)^2(2xy^2z^5)^3\)

6-5 Find each product.

13. \(-4x(15x - 22y)\)
14. \(-t^2(6 + 2t - 3t^3)\)
15. \((c + 8)(c - 2)\)
16. \((8a - 5y)(2a - 4y)\)

6-7 A square garden is bordered by a path that is 2 feet wide. The area of the path is 184 square feet. How long is the garden?

6-8 A number is 5 more than another, and the difference of their squares is 135. Find the larger number.

6-9 Find the quotient. \(-48d^4h^3k^2 \div -3d^4hk^4\)

6-10 (Optional) Simplify.

20. a. \((-5)^0\) b. \(3c^0; (c \neq 0)\)

6-11 Find each quotient.

21. \(\frac{12a^3 - 3a}{3a}\)
22. \(\frac{16bc - 8b^2c^2 - c^3d}{-4bc^3d}\)
23. \(\frac{4n^2 - 29n + 45}{n - 5}\)
24. \((10x^3 - 21x^2 + 14x + 12) \div (5x - 3)\)
6-1 Adding Polynomials  

1. The terms $-3n$ and $\frac{5m}{2}$ are each ?.
2. The polynomial $3x + 2y + 5$ is a ?.
3. The degree of $8x + 4x^2y + 5x^3y^2$ is ?.
4. When a polynomial of second degree is added to a polynomial of third degree, the sum is a polynomial of the ? degree.

Find each sum, and arrange in order of decreasing degree in $n$.
5. $(3n^3 + 5 - 2n) + (n^2 - 6n - 8)$
6. $(2m^3n - 3m^2n^2) + (4m^2n - mn^3) + (-mn^3 - 7m^3n)$
7. It is best not to use ? or ? as a replacement for a variable in checking.

Check each addition, letting $a = 3$, $b = 2$.
8. $3a + b + 4$
   $2a - 3b - 2$
   $-6a - 2b - 5$
   $a - 3$
9. $2a^2b - 3ab^2 + b^3$
   $-a^2b + ab^2 - 2b^3$
   $-a^2b + 2ab^2 + b^3$
   $0$

6-2 Subtracting Polynomials  

10. To subtract a polynomial, add the ? of each of its terms.

Do each subtraction, and check, using suitable values for the variables.

In a vertical arrangement, subtract the lower polynomial from the one above it.
11. $(3t^2 - 2t + 1) - (t^2 - 2t + 2)$
12. $5 - 2c + c^2$
   $3 + 2c - c^2$
13. $4h^2$ \quad $1$
   $5h^2$ \quad $2h$
14. Subtract $2a + b + 7$ from $a - 2b + 3$.

Solve, and check.
15. $7k - (2k + 3) = 3k - 9$
16. $3 - [2n - (5 + n) + 3] = 2 + (6n + 3)$

6-3 The Product of Powers  

17. The product of $2^9$ and $2^9$ has the base ? with the exponent ?.
18. \( c^3 \cdot c = ? \)

19. \(-3a^2b^5(a^3bc^2)(-a^6b^3c^5) = ? \)

**6-4 The Power of a Product**

20. \((2ab)^3 = (?)^3 \cdot (?)^3 = ? \)

21. In the expression \((b^n)^m\) the base \(b^n\) is to be taken as a factor \(m\) times.

22. \((3p^3r^4)^2(-p^2r^3s)^5 = ? \)

**6-5 Multiplying a Polynomial by a Monomial**

23. \(2a(5a + 3b) = (_) (5a) + 2a(?) \)

24. \(-3yz(3yz^2 - 2y^2z) = ? \)

Solve, and check.

25. \(-2x + 4(x - 2) = -7 \)

26. \(8a - 5(a - 3) = 18a - 6(3a + 1) \)

27. A rectangle has a base represented by \((2n - 5)\) feet and an altitude, by \(n\) feet. Express its area as a binomial.

**6-6 Multiplying Two Polynomials**

28. \((2a - 3)(4a + 5)\) indicates that \((4a + 5)\) is to be multiplied by \(2a\) and then by \(3\). The final product is \(?\).

In Exercises 29-31, find the products, and check by evaluation, using \(a = 2, x = 3\).

29. \((5x + 2)(3x + 7)\)

30. \(7a - 3x\)

31. \(a + 9\)

\[ \frac{3a - x}{2a - 4} \]

**6-7 Problems about Areas**

Items 32-35 refer to this problem: The length of a rectangle is 3 times the width. If the length were made 4 inches smaller and the width, 2 inches longer, the area would be unchanged.

32. If you let \(s\) represent the width of the original rectangle, then the length is \(3s\) and the area is \(3s^2\).

33. Write another expression for the area of the new rectangle.

34. Write an equation by which you can solve this problem.

35. The dimensions of the original rectangle are \(3?\) by \(?\) inches.
36. A rectangular enclosure is 30 feet longer than it is wide. It consists of a concrete walk, 5 feet in uniform width, around a pool. If the area of the walk is 1200 square feet, find the dimensions of the pool.

6–8 Powers of Polynomials  
37. \((a + 1)^2 = ?\)  
38. \((x^2 - 5x - 1)^3 = ?\)

6–9 The Quotient of Powers  
40. \(-6m^3n^4 \div 15m^6n = ?\)

6–10 Zero as an Exponent (Optional)  
42. \((-\frac{2}{3})^0 = ?\)  
43. \((a - b)^0 = ?; (a \neq b)\)

6–11 Dividing a Polynomial by a Monomial  
44. \(\frac{5a - 3}{-3} = ?\)

6–12 Dividing a Polynomial by a Polynomial  
46. Before dividing one polynomial by another, arrange the terms of each polynomial in order of \(?\) or \(?\) degree in a given variable.

Find each quotient. Check either by multiplication or by evaluation.

47. \(\frac{6a^2 - 25a + 14}{3a - 2}\)

48. \(\frac{8c^3 - 12c^2 + 6c - 1}{2c - 1}\)

49. \(\frac{2a^3 - 13ab^2 - a^2b - 6b^3}{a + 2b}\)

50. \(\frac{9r^3 + 6r^2s - 11rs^2 + s^3}{3r - 2s}\)
Cumulative Review: Chapters 1-6

Complete each of the following statements.

1. \{12, 3, 9\} is a _J_ of \{3, 6, 9, 12\}.
2. _?_ is the graph of the solution set of the inequality \(2 < x \leq 5\).
3. The value of \((2a + b)(2a - b)\), when \(a = 3\) and \(b = 4\), is _?_.
4. \{2, 4, 6, 8, 10 . . . \} is _?_ under the operations of addition and _?_.
5. \(5r + r = 6r\) by the _?_ property.
6. \(2(x - 1) = 3\) and \(2y = 5\) are _?_ equations, since they have the _?_ solution set.
7. \(53 + (47 + 25) = (53 + 47) + 25\) by the _?_ property.
8. If the replacement set of \(x\) is \{-2, -1, 0, 1, 2\}, the solution set of \(3x \neq -3\) is _?_.

Express the following numbers in algebraic symbols.

9. Twice the cube of a number \(n\).
10. The sum of a number \(n\) and its square.
11. The difference of a number \(n\) and its reciprocal.
12. The sum of the squares of the two numbers \(a\) and \(b\).
13. Simplify: \(5(m^3 - n^2) - 9m^3 + 5n^2\)

Solve each of the following open sentences. Check your answers.

14. \(.6y + 51 = 3\) \hspace{1cm} 15. \(3n - \frac{3}{3}(5n - 6) = 2\)
16. \(4(3x + 5) - 2x = 3(4x - 5) + 25\)
17. \(3t - [2t - (3t + 5) - 7] = 0\)

Solve each inequality, and graph its solution set. (Optional)

18. \(-6 \leq 2x - 3 < 5\) \hspace{1cm} 19. \(|2r - 3| \geq 5\) \hspace{1cm} 20. \(\frac{5}{4}m + 6 \geq \frac{m}{2} + 3\)

In each case select the correct answer.

21. If \(b = -3\), then \(b^3\) is \((-9)\) \hspace{1cm} \((27)\) \hspace{1cm} \((-27)\).
22. An exponent indicates how many times a certain number is to be used as a (factor) (addend) (quotient).
23. In \(-ab\) the coefficient of \(ab\) is \((0)\) \hspace{1cm} \((1)\) \hspace{1cm} \((-1)\).
24. \(2a - b\) subtracted from 0 equals \((b - 2a)\) \hspace{1cm} \((2a + b)\) \hspace{1cm} \((2a - b)\).
25. $0$ subtracted from $2a - b$ equals $(b - 2a)$ $(2a + b)$ $(2a - b)$.

26. The reciprocal of $-p$ is $(p) \left( \frac{1}{p} \right) \left( -\frac{1}{p} \right)$.

27. If $12a^2 - 15ab$ is divided by $-3a$, the result is $(5b - 4a)$ $(-4a - 5b)$ $(4a - 5b)$.

28. $(-2r^3s^3)^3$ equals $(-8r^5s^6)$ $(8r^6s^9)$ $(-8r^6s^9)$.

Write the simplest expression which completes each statement correctly.

29. The sum of $(a^3 + b^3)$ and $(a^3 - b^3)$ is $\ldots$.

30. The difference of $(a^3 + b^3)$ and $(a^3 - b^3)$ is $\ldots$.

31. The product of $(a^3 + b^3)$ and $(a^3 - b^3)$ is $\ldots$.

32. $n^2 - 2n(4 - n) - 3 = \ldots$.

33. The result of multiplying $(3ax^2 - 15a^2x + 2a^3)$ by $-2ax$ is $\ldots$.

34. $4m^2(m^3 - 3n^4) - 5n^2(m^4n - 2m^3n^2) = \ldots$.

35. $(-6x^2y^2 + 3xy^2 - 2x^2y)$ divided by $-6x^2y^2 = \ldots$.

36. $(x^2 - 5x + 2)$ multiplied by $(3x - 2) = \ldots$.

37. $(8x^2 - 22x + 15)$ divided by $(4x - 5) = \ldots$.

Answer by writing an algebraic expression in simplest form.

38. Expand $(2r - 3s)^3$.

39. Divide $(10 - x^2 + 6x^3 - 27x)$ by $(3x - 5)$.

40. Express the perimeter of the shaded part of the figure; below left.

41. Give the area of the shaded straight-line figure as the sum of three algebraic terms.

42. Find the perimeter of the shaded part of the figure at the right below. (Leave the answer in terms of $\pi$.)

43. Express the area of the shaded curved-line figure as the difference of two algebraic terms. (Leave the answer in terms of $\pi$.)
Translate the rule into algebraic symbols by completing the open sentence.

44. To approximate the number of tons $T$ of one size of coal in a bin $l \times w \times h$ feet, divide 3 times the volume of the bin by 94; $T = ?$.

45. Solve for $k$, then evaluate: $L = \frac{mt - k}{t}; L = 99, m = 100, t = 4$.

46. Solve for $C$, then evaluate: $F = \frac{9C}{5} + 32; F = 40°$.

47. Solve for $x$: $\frac{x}{m} = \frac{1}{m}$

48. If $l$ is the length, $w$ the width, and $d$ the depth of a rectangular bin, all expressed in feet, then the number of bushels $b$ of corn which the bin can hold is its capacity (volume) divided by 2.5. Translate this rule into an open sentence whose left member is $b$.

Solve each of the following problems.

49. Two jet planes left Chicago at the same time, one for the east and the other traveling 25 miles per hour faster for the west. After four hours, the planes were 4900 miles apart. Find their rates.

50. Divide 75 into two parts such that three times the smaller is 5 less than twice the larger.

51. The sum of three numbers is 360. The second number is three times the first, and the third is 20 more than the first. Find the numbers.

52. Shirley has only nickels and dimes in her bank. There are 23 coins in all. If the value of all of the dimes exceeds the value of all of the nickels by 95¢, how many coins of each kind are there?

53. The length of a rectangular flower bed is twice its width. A concrete walk 3 feet wide surrounds the flower bed. The area of the walk is 216 square feet. Find the dimensions of the flower bed.

54. A dealer has two blends of tea, the orange pekoe being worth $.75 a pound more than the black pekoe. He mixes 20 pounds of the better tea with 40 pounds of the cheaper tea and sells the new blend at $1.50 a pound. How much is a pound of each original blend worth?

Extra for Experts

Negative Exponents

To apply the rule $b^m \div b^n = b^{m-n}$ when $m < n$, that is, when $m - n$ represents a negative number, you must understand the meaning of a nega-
tive exponent. Observe the following example:

\[ \frac{b^5}{b^2} = b^{5-2} = b^3 \quad \quad \frac{b^2}{b^5} = b^{2-5} = b^{-3}. \]

Since \( \frac{b^5}{b^2} \) and \( \frac{b^2}{b^5} \) are reciprocals, then \( b^3 \) and \( b^{-3} \) also must be reciprocals;

\[ b^3 = \frac{1}{b^{-3}} \text{ and } b^{-3} = \frac{1}{b^3}. \]

In general,

\[ b^n = \frac{1}{b^{-n}} \text{ and } b^{-n} = \frac{1}{b^n}; \quad (b \neq 0) \]

The other rules of exponents may now be extended to include negative exponents. Study the following illustrations.

1. \( \frac{a^3}{n^2} = a^3 \cdot \frac{1}{n^2} = a^3n^{-2} \)
2. \( 3x^{-1}y^2 = 3 \cdot \frac{1}{x} \cdot y^2 = \frac{3y^2}{x} \)
3. \( 10^{-3} = \frac{1}{10^3} = .001 \)
4. \( \left( \frac{2}{3} \right)^{-2} = \frac{1}{\left( \frac{2}{3} \right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4} \)
5. \( -5m^{-3} \cdot 2m^{-2} = -10m^{-5} \)
6. \( (3 \times 10^5)(4 \times 10^{-2}) = 12 \times 10^3 = 12,000 \)
7. \( \frac{8 \times 10^7}{4 \times 10^9} = 2 \times 10^{-2} = .02 \)
8. \( (-2n^2)^{-3} = \frac{1}{(-2n^2)^3} = \frac{1}{-8n^6} \)

**Questions**

Write with only positive exponents.

1. \( a^{-1}b^2 \)
2. \( 2a^{-3} \)
3. \( \frac{4b^3}{k^{-2}} \)
4. \( \frac{3m^{-1}}{2n^{-4}} \)

Express each fraction as a product of powers.

5. \( \frac{a}{b^2} \)
6. \( \frac{2r^{-2}}{s^{-5}} \)
7. \( \frac{x^2}{y^3z^{-4}} \)
8. \( \frac{p^{-2}}{q^{-1}r^{-3}} \)

Evaluate.

9. \( \left( \frac{1}{4} \right)^{-3} \)
10. \( (-4)^{-3} \)
11. \( 10^{-5} \)
12. \( (-\frac{3}{4})^{-2} \)

Simplify.

13. \( (3c^{-4})^2 \)
14. \( \frac{6ab^{-2}}{-2a^{-3}b^{-6}} \)
15. \( \frac{4.8 \times 10^{-8}}{2.0 \times 10^{-10}} \)
A Handy Way to Multiply by Nine

As a small child, you probably counted on your fingers, but did you ever multiply on them? Finger multiplication works very well when the multiplier is 9.

Suppose you wish to multiply 8 by 9. Spread your hands on the table, fingers outstretched. Bend the eighth finger from the left. Now you have your product, seven fingers to the left of the bent finger, two fingers to its right, or 72. (See Figure 6-1.)

You can multiply by 9 any number from 1 to 10 in this way. Try the other one-digit numbers.

Don’t think that multiplying by 9 on your fingers is limited to such small numbers. Here is how to proceed when the number to be multiplied has two digits; for example, 46. Start with the 6 of the 46; that is, begin by bending the sixth finger. Now, count off the 4 of 46 by putting the first four fingers together, as shown in Figure 6-2. Your fingers are now in three groups — four together at the far left, one alone just to the left of the bent finger, and four to the right of the bent finger. The product is 414.

The handy way of multiplying by 9 may be used for many numbers between 11 and 90, but not for all of them. It is up to you to discover which numbers cannot be multiplied this way, and why!
Merchandisers and Mathematics

Merchandisers purchase stock for retail and wholesale stores. The young lady in the photo, an assistant buyer in a department store, is checking price tags against an inventory list. Keeping a large stock in order, estimating the potential profit it represents, and calculating the amount of capital in merchandise, require a good grasp of mathematics.

Illustrated on the work pad are two such problems. The merchandiser has selected three styles of blouses wholesale priced at $2, $4, and $7. Estimating that the demand will be greatest for the $4 blouses and least for the $7 blouses, she decides to buy $x$ number of $2 blouses, $\frac{x}{2}$ number of $7 blouses, and 

$$2\left(x + \frac{x}{2}\right)$$

number of $4 blouses. Since her store has allotted about $350 for the purchase, she sets up the equation shown in Part I, and finds that she will buy 20, 10, and 60 blouses, respectively.

The buyer must also decide on the retail selling prices for this merchandise. The store would like a total markup of about 50%, but the per cent of markup usually varies for differently priced items. The buyer decides on a 25% markup for the lower-priced blouses, and a 50% markup for the medium-priced blouses. If the store is to realize a total markup of 50%, the higher-priced blouses must sell at $11.50, as shown in Part II.
Decay Process $\Lambda^0 \rightarrow P + \pi^- + 0^0 \rightarrow \pi^+ + \pi^-$
Special Products and Factoring

“I wonder what makes it tick?” This question has presented a challenge to many boys and girls and has led many scientists to new discoveries. Scientists and mathematicians, young or old, also share a curiosity about putting things together.

With the linear accelerator (upper photo) physicists attempt to determine the patterns of disintegration and combination of the elements they study (lower photo). Products like synthetic rubber represent chemists' attempts to duplicate natural products.

Like scientists, mathematicians analyze and combine the elements with which they work. These elements, such as numbers and sets, can be identified (or analyzed) more easily than the atom particles of physics. Also, they are not as complex as the compounds which the chemist tries to synthesize. Yet, the careful determination of relationships and the tracing of their patterns in order to explain their properties are the same. In this chapter you will learn to analyze and combine algebraic elements into forms with which you can work easily.

THE DISTRIBUTIVE PROPERTY IN FACTORING

7-1 Factoring in Algebra

It is often necessary to know how a number can be written as a product of two or more other numbers. Because

\[ 90 = 9 \cdot 10 \quad \text{and} \quad 90 = 5 \cdot 18, \]

9 and 10, and 5 and 18 are called factors of 90. Because \( 90 = \frac{1}{2} \cdot 180 \) you may think that \( \frac{1}{2} \) and 180 also can be called factors of 90. However, if fractions generally were allowed as factors, any nonzero fraction would be a factor of every number. Therefore, you usually specify a particular set of numbers to be used as factors.

Finding numbers belonging to a given set of numbers and having their product equal to a given number is called \textbf{factoring the number over the given set}. Hereafter, integers will be factored over the set of integers, unless another set is specified.
An important subset of the integers that often is chosen as a set of possible factors is the set of prime numbers. A prime number is an integer greater than one, having no positive integral factor other than itself and one. The first prime numbers are

- 2, 3, 5, 7, 11, 13, 17, 19, 23, ... 

To factor 90 over the set of primes, write

\[ 90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5. \]

The prime factors of 90 are 2, 3, and 5, with 3 occurring twice.

To express an integer as a product of primes, you usually can proceed in several ways. For example,

\begin{align*}
392 &= 7 \cdot 56 \\
   &= 7 \cdot 7 \cdot 8 \\
   &= 7 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \\
   &= 7^2 \cdot 2^3 \\
392 &= 2 \cdot 196 \\
   &= 2 \cdot 2 \cdot 98 \\
   &= 2 \cdot 2 \cdot 2 \cdot 49 \\
   &= 2 \cdot 2 \cdot 2 \cdot 7 \cdot 7 \\
   &= 2^3 \cdot 7^2
\end{align*}

In the second method you look systematically for the smallest prime factor of the number still to be factored at each stage. That is, you first try 2, and try it again and again until it no longer can be used; then you try 3, then 5, then 7, and so on until all the factors are prime numbers.

Once you know the prime factors of an integer, it is easy to list all its positive factors. The factors of 392, for example, are 1 and all possible products of one or both of the primes 2 and 7, each with an exponent less than or equal to its exponent in 392. These factors are

\[ 1, 2, 4, 8, 7, 14, 28, 56, 49, 98, 196, 392 \]

By factoring integers into products of primes, you can determine the largest integral factor of both of two integers. To find the greatest common factor of 392 and 1260, notice that

\[ 392 = 2^3 \cdot 7^2 \quad \text{and} \quad 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7, \]

so the largest power of 2 common to 392 and 1260 is 2² and the largest common power of 7 is 7. The greatest common factor is, therefore, 2² \cdot 7 or 28.

In algebra you often need to express a polynomial as a product.
Transforming a given polynomial into a product of other polynomials is called **polynomial factoring**. For example, each term of the polynomial $5x + 5y$ contains 5 as a factor; therefore, by the distributive property

$$5x + 5y = 5(x + y).$$

Both 5 and $x + y$ are factors of $5x + 5y$.

When factoring polynomials whose numerical coefficients are integers, you look for factors that are either integers or polynomials with integral coefficients. Some of the factors of $6x^2y$ are 1, 2, 3, 6, $2x^2$, and $3xy$. You see that $2x^3$ is not a factor of $6x^2y$, since there is no polynomial by which you can multiply $2x^3$ to obtain $6x^2y$.

### ORAL EXERCISES

Tell why each statement is true or why it is false.

**SAMPLE 1.**

What you say: True, because 7 is a prime and $42 = 7 \cdot 6$.

**SAMPLE 2.**

What you say: False, because there is no integer $a$ such that $12 = 9a$.

1. 6 is a factor of 48 over the set of integers.
2. 14 is a factor of 28 over the set of integers.
3. The smallest prime factor of 246 is 3.
4. 13 is a prime factor of 52. 9. 3 is a prime factor of 123.
5. 17 is a prime factor of 68. 10. 3 is a prime factor of 1234.
6. 1 is not a factor of 6. 11. 9 is not a factor of 801.
7. 10 is a factor of 10. 12. 9 is not a factor of 802.
8. $2^3$ is a factor of 144. 13. 3 is a factor of 0.

Name the monomial with the largest numerical coefficient and the greatest degree in each variable that is a factor of both monomials in each pair.

**SAMPLE 3.**

What you say: $6x^2y$.

<table>
<thead>
<tr>
<th></th>
<th>12x^2y^3, 42x^4y</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>42, 56p^2</td>
</tr>
<tr>
<td>15</td>
<td>72, 30v^3</td>
</tr>
<tr>
<td>16</td>
<td>3x, 18xy</td>
</tr>
<tr>
<td>17</td>
<td>5b, 40bc</td>
</tr>
<tr>
<td>18</td>
<td>60m^3n, 48m^2n</td>
</tr>
<tr>
<td>19</td>
<td>70rs^4, 105rs^3</td>
</tr>
<tr>
<td>20</td>
<td>$-21v^3w^2$, $14v^2w^5$</td>
</tr>
<tr>
<td>21</td>
<td>$75y^2z^4$, $-80y^7z^2$</td>
</tr>
<tr>
<td>22</td>
<td>$23xy$, $32a$</td>
</tr>
</tbody>
</table>
WRITTEN EXERCISES

Factor each integer over the set of primes.

A

1. 210
2. 182
3. 2310
4. 1155
5. 1500
6. 2000
7. 13
8. 17

Find all the positive integral factors of each number.

9. 35
10. 91
11. 63
12. 52
13. 100
14. 441
15. 726
16. 1690

Find the greatest common factor of each pair of integers.

17. 144, 630
18. 231, 294
19. 180, 1368
20. 4200, 3850

Give the second factor for each monomial.

21. $5x^2y = 5x (?)$
22. $-24a^2b^3 = -8ab^2 (?)$
23. $20ab^2c^3 = 4bc (?)$
24. $-14cd^3e^2 = -2cde (?)$
25. $18mn^4p^3 = 6n^2p^2 (?)$
26. $-21r^5s^3t^2 = 7r^2s^2t^2 (?)$
27. $38p^4d^3q^5 = -2pdq (?)$
28. $-39u^6v^4w^2 = 3u^4v^3w^2 (?)$
29. $28k^2j^2m^2 = -7mjk^2 (?)$
30. $-25m^8n^7 = 5n^7m^5 (?)$

For each pair of monomials, find the highest power of the first monomial that is a factor of the second.

SAMPLE. $2a; 12a^3b^2$

Solution: $(2a)^2 = 4a^2$, and $12a^3b^2 = (4a^2)(3ab^2)$.

$(2a)^3 = 8a^3$, and 8 is not a factor of 12.

$\therefore (2a)^3$ is not a factor of $12a^3b^2$, but $(2a)^2$ is.

(2a)$^2$, Answer.

B

31. $x; 7x^5y^2$
32. $y; -3x^2y^7$
33. $b^2; 15b^6c^4$
34. $h^2; 34a^5h^7$
35. $5r; 250r^4t^3$
36. $3v; 27v^4w^2$
37. $3rp; 192r^4p^6$
38. $7rq^3; 98r^3q^7$
39. $2x^2y^2; 6x^6y^6$
40. $3u^3v^3; 9u^9v^9$
41. $5a^2b^3; 625a^8b^7$
42. $4r^3s^2; 256r^{12}s^6$
43. $2pq^4; 16p^4q^{16}$
44. $7m^3n; 343m^9n^3$
45. $2c^5d^3e; 8c^{20}d^{12}e^4$
7–2 Identifying Common Factors

As each term of \(4ab + 6a\) has \(2a\) as a factor, using the distributive property, you have

\[4ab + 6a = 2a(2b + 3).\]

You see, \(2a\) is a monomial, and \(4ab + 6a\) is a polynomial. Thus, \(2a\) is a monomial factor of the polynomial \(4ab + 6a\). When you factor a polynomial, first see whether each term has the same monomial as a factor. A monomial is a common monomial factor of a polynomial if it is a factor of every term of the polynomial.

Be sure to use the greatest common monomial factor. The greatest common monomial factor of a polynomial is the common monomial factor having the greatest numerical coefficient and the greatest degree.

Examine this chart, observing how each polynomial is factored.

<table>
<thead>
<tr>
<th>Given Polynomial</th>
<th>Factors</th>
<th>Factored Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5x^2 - 3x)</td>
<td>(x)</td>
<td>(x(5x - 3))</td>
</tr>
<tr>
<td>(5y^2 + 35y)</td>
<td>(5y)</td>
<td>(5y(y + 7))</td>
</tr>
<tr>
<td>(6z^4 + 36z^3 + 60z^2)</td>
<td>(6z^2)</td>
<td>(6z^2(z^2 + 6z + 10))</td>
</tr>
<tr>
<td>(12cm - 12cn^2)</td>
<td>(12c)</td>
<td>(12c(m - n^2))</td>
</tr>
</tbody>
</table>

The associative and commutative properties for addition, together with the distributive property, enable you to factor a polynomial by grouping.

\[ax + by + ay + bx = (ax + bx) + (ay + by)\]

\[= (a + b)x + (a + b)y\]

\[= (a + b)(x + y)\]

In the last step, \((a + b)\) is treated as a single term in applying the distributive property.

Here is another polynomial that can be factored readily when you group the terms appropriately:

\[2vw - 15st - 10vt + 3sw = (2vw - 10vt) + (3sw - 15st)\]

\[= 2v(w - 5t) + 3s(w - 5t)\]

\[= (2v + 3s)(w - 5t)\]
Of course, there can be more than one convenient way to group terms:

\[
2vw - 15st - 10vt + 3sw = (2vw + 3sw) + (-15st - 10vt) \\
= w(2v + 3s) + (-5t)(3s + 2v) \\
= w(2v + 3s) + (-5t)(2v + 3s) \\
= (w - 5t)(2v + 3s)
\]

Because multiplication is commutative, this result is the same as the preceding one.

**Oral Exercises**

Give the greatest common monomial factor of each polynomial.

**Sample.** \(8s^2 - 24s\)

What you say: \(8s\).

1. \(2a^2 + 12a\)  
2. \(9b^2 - 81b\)  
3. \(12c^2 - 6\)  
4. \(9d^2 + 27\)  
5. \(e^2 + 9\)  
6. \(2f^2 - 7\)  
7. \(7g^2 - 28g\)  
8. \(13h^2 + 26h\)  
9. \(i^2 - 1\)  
10. \(4j^2 - 1\)  
11. \(b^2 + 2bx + x^2\)  
12. \(x^2 + 10xy + 25y^2\)  
13. \(3x^2 - 12x + 18\)  
14. \(18n^2 - 27n + 9\)  
15. \(6x^2 - 24y^2\)  
16. \(8k^2 - 72\)  
17. \(mn^2t - m^2nt^2\)  
18. \(a^2bx^2 + ab^2x\)  
19. \(60m^3n + 48m^2n\)  
20. \(70rs^4 - 105s^3r\)  
21. \(-21v^3w^2 + 14v^2w^5\)  
22. \(-80y^7z^2 + 75y^2z^4\)  
23. \(2x^4 + 6x^3 - 10x^2\)  
24. \(9y^5 - 6y^4 + 3y^3\)

**Written Exercises**

Write in factored form.

**Sample.** \(25x^2 - 15x\)

Solution: \(25x^2 - 15x = 5x(5x - 3)\)

A  
1. \(3x^2 + 12y^2\)  
2. \(7a^2 + 14b^2\)  
3. \(18x^2 - 12x\)  
4. \(27r^2 - 90r\)  
5. \(x^2 + 7x\)  
6. \(n^2 + 13n\)  
7. \(3x^2 - 21x^3\)  
8. \(5b^2 - 70b^3\)  
9. \(6x^2 - 4x\)  
10. \(8a^2 + 12a\)  
11. \(b^3 + b^2 + b\)  
12. \(a^3 - a^2 - a\)  
13. \(a^2b + ab^2\)  
14. \(xy^2 - x^2y\)  
15. \(15a^2c - 3c\)  
16. \(12x^2y + 2y\)  
17. \(5r^2s - 10rs^2\)  
18. \(3a^2b - 9ab^2\)  
19. \(t + 2mt\)  
20. \(3nx - x\)  
21. \(-12x^2 - 6x\)
Write each expression in factored form.

**SAMPLE.** \(x^2(x + 2) + 7(x + 2)\)

**Solution:** \(x^2(x + 2) + 7(x + 2) = (x^2 + 7)(x + 2)\)

22. \(y(y - 1) + 2(y - 1)\)
23. \(a(a - 8) + 9(a - 8)\)
24. \((4c + 5d)x - (4c + 5d)y\)
25. \((x + 1)(2x + 3) - (x + 1)\)
26. \((x - y)^2 + (x + y)(x - y)\)
27. \(2m(m - n) - (m + n)(m - n)\)
28. \((b - 3c)x^2 + (b - 3c)y^2\)
29. \(k^2 + 2k + kt + 2t\)
30. \(x^2 - x + xy - y\)
31. \(dg + dm - fg - fm\)
32. \(rs - rt - ks + kt\)
33. \(3b^2 + 2b + 12b + 8\)

34. \(12x^4 - 8x^2 + 20x\)
35. \(18y^4 + 30y^3 - 42y\)
36. \(5r^2t - 10rt + 5rt^2\)
37. \(18a^3c^2 - 9a^2c + 6ac^2\)
38. \(3x^2 - 2x + 6x - 4\)
39. \(10y^2 - 3 + 15y - 2y\)
40. \(x^2 + x(a + b)\)
41. \(3a^2 + 6a(x - y)\)
42. \(m^2 + 14a + 2m + 7am\)
43. \(2p^2 + 2a + 4ap + p\)
44. \(x^3 - 21 - 3x^2 + 7x\)
45. \(r^3 - s^3 - sr^2 + rs^2\)

**PROBLEMS**

Write an algebraic expression in factored form for the shaded area \(A\) of each figure on page 244.

**SAMPLE.**

![Diagram of shaded area](image)

**Solution:**

\[
A = \text{area of square} - \text{area of circle}
\]

\[
A = (2r)^2 - \pi r^2
\]

\[
= 4r^2 - \pi r^2
\]

\[
= r^2(4 - \pi), \text{ Answer.}
\]
7-3 Multiplying the Sum and Difference of Two Numbers

Certain products occur so often that you should recognize them at sight. Study each of the three examples below:

\[
\begin{align*}
  x + 3 & \quad 2y - 5z & \quad a + b \\
  x - 3 & \quad 2y + 5z & \quad a - b \\
  x^2 + 3x & \quad 4y^2 - 10yz & \quad a^2 + ab \\
  -3x - 9 & \quad 10yz - 25z^2 & \quad -ab - b^2 \\
  x^2 & \quad 4y^2 & \quad -b^2 \\

\end{align*}
\]

The examples illustrate this rule: the product of the sum and difference of two numbers is the square of the first number minus the square of the second number,

\[(a + b)(a - b) = a^2 - b^2.\]

\[\text{Figure 7-1}\]

**Oral Exercises**

Square each monomial.

**Sample.** \(-4n^3\)

What you say: \(16n^6\)

1. \(5x\)  \hspace{1cm} 3. \(2x^3\)  \hspace{1cm} 5. \(-3n^2\)  \hspace{1cm} 7. \(8rt\)  \hspace{1cm} 9. \(-2sm^2\)
2. \(3a\)  \hspace{1cm} 4. \(7b^2\)  \hspace{1cm} 6. \(-6x^3\)  \hspace{1cm} 8. \(12xy\)  \hspace{1cm} 10. \(-3tv^2\)
Find each product.

11. \((x + y)(x - y)\)  
12. \((c - d)(c + d)\)  
13. \((a - 1)(a + 1)\)  
14. \((b + 6)(b - 6)\)  
15. \((3a + b)(3a - b)\)  
16. \((1 + 5b)(1 - 5b)\)  
17. \((2m - n)(n + 2m)\)  
18. \((r + 3s)(3s - r)\)  
19. \((x^2 - 9)(x^2 + 9)\)  
20. \((y^3 - 4)(y^3 + 4)\)  
21. \((st - 8)(st + 8)\)  
22. \((mn - 7)(mn + 7)\)  
23. \((x - \frac{1}{4})(x + \frac{1}{4})\)  
24. \((z + \frac{1}{2})(z - \frac{1}{2})\)  
25. \(29 \times 31\) or \((30 - 1)(30 + 1)\)  
26. \(18 \times 22\) or \((20 - 2)(20 + 2)\)  
27. \(45 \times 55\)  
28. \(76 \times 84\)

**WRITTEN EXERCISES**

Multiply (a) by using the sum and difference of two numbers and (b) by using the distributive property.

**SAMPLE.**

\((83)(77)\)

**Solution:**

a. \((83)(77) = (80 + 3)(80 - 3) = 6400 - 9 = 6391\)

b. \((83)(77) = 80(77) + 3(77) = 6160 + 231 = 6391\)

Ordinary multiplication, which uses the distributive property, also can be used for this part.

**A**

1. \((9)(11)\)  
2. \((51)(49)\)  
3. \((22)(18)\)  
4. \((58)(62)\)  
5. \((33)(27)\)  
6. \((67)(73)\)  
7. \((36)(44)\)  
8. \((84)(76)\)  
9. \((55)(45)\)  
10. \((95)(85)\)

**B**

13. \((1010)(990)\)  
14. \((520)(480)\)  
15. \((5\frac{1}{2})(4\frac{1}{2})\)  
16. \((1130)(1070)\)  
17. \((640)(560)\)  
18. \((8\frac{1}{3})(9\frac{2}{3})\)  
19. \((1150)(1250)\)  
20. \((360)(240)\)  
21. \((4\frac{1}{4})(3\frac{3}{4})\)

**7-4 Factoring the Difference of Two Squares**

By the symmetric property of equality, the relation \((a + b)(a - b) = a^2 - b^2\) is reversible;

\[a^2 - b^2 = (a + b)(a - b)\]
This shows you how to factor an algebraic expression consisting of the difference of two squares.

\[ m^2 - 16 = (m + 4)(m - 4) \]
\[ 36x^2 - y^2 = (6x + y)(6x - y) \]
\[ 9m^2 - 49t^4 = (3m + 7t^2)(3m - 7t^2) \]
\[ -r^2s^2 + v^2w^2 = v^2w^2 - r^2s^2 = (vw + rs)(vw - rs) \]

If, as in \(49x^2y^4\), the degree in each variable is even and the numerical coefficient is the square of an integer, then the monomial is a square; \(49x^2y^4 = (7xy^2)^2\). Sometimes it is difficult to tell at sight whether a numerical coefficient is a square. However, Table 3 in the Appendix enables you to tell whether or not each integer from 1 to 10,000 is the square of an integer.

**ORAL EXERCISES**

Tell whether or not each of the following is the difference of two squares. If it is, give its factors.

1. \(r^2 - 81\)  
2. \(c^2 - d^2\)  
3. \(121 - a^2\)  
4. \(s^2 - 36\)  
5. \(x^2 - y^2\)  
6. \(z^2 - 1\)  
7. \(x^2 - 100\)  
8. \(16 - y^2\)  
9. \(56x^2 - y^2\)  
10. \(64x^2 - 10y^2\)  
11. \(-m^2 + n^2\)  
12. \(-1 + c^2\)  
13. \(-81x^2 - y^2\)  
14. \(25r^2 - 50s^2\)  
15. \(r^2 - 9s^2\)  
16. \(a^2b^2 - c^2d^2\)  
17. \(-r^2s^2 + y^2\)  
18. \(-c^2 + c^2d^2\)  
19. \(x^3y^2 - z^2\)  
20. \(-ab^2 - cd^2\)  
21. \(x^2 - 1\)  
22. \(-x^2 + 1\)  
23. \(x^4y^2 - x^4z^2\)  
24. \(a^2b^2 - a^4c^2\)

**WRITTEN EXERCISES**

Factor, and check by multiplication. You may refer to Appendix Table 3.

1. \(x^2 - 16\)  
2. \(r^2 - 9\)  
3. \(x^2 - 4y^2\)  
4. \(a^2 - 4b^2\)  
5. \(R^2 - r^2\)  
6. \(a^2 - b^2y^6\)  
7. \(16a^2 - b^4\)  
8. \(256x^4 - y^2\)  
9. \(4x^4 - z^2\)  
10. \(m^2 - 16n^4\)  
11. \(196b^2 - 121x^2\)  
12. \(289x^2 - 676y^2\)  
13. \(-9 + 4r^2t^2\)  
14. \(-144 + m^2n^2\)  
15. \(1 - 9n^2\)  
16. \(25m^2 - 1\)  
17. \(.09a^2 - 4\)  
18. \(.04b^2 - 49\)  
19. \(c^2 - .64\)  
20. \(.81 - d^2\)  
21. \(x^2 - \frac{1}{9}\)
22. $\frac{1}{16} - y^2$

Factor, and check by multiplication.

**SAMPLE.** $2a - 8a^3$

**Solution:**

$$2a - 8a^3 = 2a(1 - 4a^2) = 2a(1 + 2a)(1 - 2a)$$

25. $-x^4 + 25x^6$
26. $16s^2 - s^4$
27. $a^2 - a^4$
28. $c^3 - c^5$
29. $x^3y - xy^5$

30. $a^6b - a^2b^3$
31. $2p^3q^4 - 72pq^2$
32. $147x^2y - 3x^4y^3$
33. $2a^4 - 32$
34. $n - n^5$

35. $-a^{2n} + 1$
36. $-n^{2x} + r^2$
37. $-m^{5a} + m^a$
38. $-b^{2x} + b^6x$
39. $0.18r^4 - 0.08s^6r^2$

40. $(a + 2b)^2 - x^4$
41. $(3a - 1)^2 - y^6$
42. $-1 + (x - 1)^2$
43. $-9 + (x + 3)^2$

44. Show that the difference between the squares, a. of two consecutive integers equals the sum of the integers; b. of two consecutive odd integers equals twice the sum of the integers.

**QUADRATIC TRINOMIALS**

### 7–5 Squaring a Binomial: Plateau Section*

Each side of the square in Figure 7–2 is $(a + b)$ units in length. You can consider the square as being made up of four areas, as shown. The total area can be expressed as a square of a binomial $(a + b)^2$, or as a trinomial square $(a^2 + 2ab + b^2)$.

![Figure 7-2](image)

*In a Plateau Section, skills you have learned are applied in new ways; no new ideas are introduced.*
The binomial \((a + b)\) is squared at the right by the usual method of multiplication. Notice how each term in the product is obtained.

\[
\begin{align*}
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a - b)^2 &= a^2 - 2ab + b^2
\end{align*}
\]

1. Square the first term in the binomial.
2. Double the product of the two terms.
3. Square the second term in the binomial.

Now examine the square of a binomial difference. The binomial \((a - b)\) is squared by multiplication at the right. Again, notice how each term is obtained.

\[
\begin{align*}
(a - b)^2 &= a^2 - 2ab + b^2 \\
(a + b)^2 &= a^2 + 2ab + b^2
\end{align*}
\]

1. Square the first term in the binomial.
2. Double the product of the two terms.
3. Square the second term in the binomial.

Whenever you square a binomial, the product is a trinomial square, whose terms show this pattern.

\[
(a + b)^2 = a^2 + 2ab + b^2 \\
(a - b)^2 = a^2 - 2ab + b^2
\]

Knowing these relationships, you can write the square of a binomial without performing long multiplication.

\[
\begin{align*}
(x + 1)^2 &= x^2 + 2x + 1 \\
(2n + 3)^2 &= 4n^2 + 12n + 9 \\
y^2 - 2y + 1 &= (4m^2 - n^3)^2 = 16m^4 - 8m^2n^3 + n^6 \\
(-ab^2 + c^3)^2 &= (c^6 - 2ab^2c^3 + a^2b^4)
\end{align*}
\]

**ORAL EXERCISES**

Read each of the following as a trinomial square.

1. \((m + h)^2\)  5. \((x - y)^2\)  9. \((2a + 1)^2\)  13. \((5x + b)^2\)
2. \((c + d)^2\)  6. \((m - n)^2\)  10. \((3b + 1)^2\)  14. \((8a + b)^2\)
3. \((x + 3)^2\)  7. \((a - 4)^2\)  11. \((3x + 5)^2\)  15. \((3a - 2)^2\)
4. \((x + 5)^2\)  8. \((y - 8)^2\)  12. \((3n + 7)^2\)  16. \((2b - 3)^2\)
17. \((4c - d)^2\)  
18. \((2r - s)^2\)  
19. \((x - 6y)^2\)  
20. \((w - 10z)^2\)  
21. \((8r - 1)^2\)  
22. \((10a - 1)^2\)  
23. \((x + 9y)^2\)  
24. \((x + 11y)^2\)  
25. \((r^2 - 5)^2\)  
26. \((4 + s^2)^2\)  
27. \((xy + 2)^2\)  
28. \((3 - rs)^2\)

**PROBLEMS**

**A**  
1. Show that \((a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\) by considering the figure at the left.

![Ex. 1](image)

2. Show that \((a - b)^2 = a^2 - 2ab + b^2\) by considering the adjoining figure.

![Ex. 2](image)

**B**  
3. Show that \((a + b)^2 - (a - b)^2 = 4ab\) by considering the figure at the left.

![Ex. 3](image)

4. Show that \((a + b)^2 + (a - b)^2 = 2(a^2 + b^2)\) by considering the adjoining figure.

![Ex. 4](image)
5. Use the figure just above to show that

\[(a + b)^2 - b^2 = a^2 + 2ab.\]

6. Use the figure at right above to show that

\[a^2 - (a - b)^2 = 2ab - b^2.\]

7. Use the adjoining figure to show that

\[1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{8(9)}{2}.\]

By the same method, show that for any positive integer \(n\)

\[1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.

8. Use the adjoining figure to show that

\[1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 8^2.\]

By the same method, show that for any positive integer \(n\)

\[1 + 3 + 5 + \cdots + (2n - 1) = n^2.

7–6 Factoring a Trinomial Square

To factor a trinomial square, reverse the equations you use in squaring a binomial.

\[a^2 + 2ab + b^2 = (a + b)^2\]
\[a^2 - 2ab + b^2 = (a - b)^2\]
Before you use one of these equations as a rule for factoring, be sure that the expression to be factored is a trinomial square. Examine each term of the trinomial to see how it may have been obtained. Consider \( x^2 + 14x + 49 \). Is the first term a square? Yes, \( x^2 \) is the square of \( x \). Is the third term a square? Yes, 49 is \( 7^2 \). Is the middle term double the product of \( x \) and \( 7 \)? Yes, \( 14x = 2(x)(7) \). Therefore, the trinomial is a square. Since all its terms are positive, it is the square of a sum;

\[
x^2 + 14x + 49 = (x + 7)^2.
\]

**EXAMPLE.** Factor \( 196m^2 - 476mn + 289n^2 \).

**Solution:**

\[
196m^2 - 476mn + 289n^2 = 14m \cdot 14m - 2 \cdot 14m \cdot 17n + 17n \cdot 17n
\]

Thus, \( 196m^2 - 476mn + 289n^2 = (14m - 17n)^2 \).

**ORAL EXERCISES**

Is each trinomial the square of a binomial? Justify your answer.

1. \( x^2 + 2xy + y \)  
2. \( a^2 + 2ab - b^2 \)  
3. \( r^2 + 4r + 4 \)  
4. \( w^2 + 2w + 4 \)  
5. \( z^2 + 4z + 2 \)  
6. \( x^2 + 2x + 1 \)  
7. \( 4x^2 + 8x + 4 \)  
8. \( 9r^2 - 6x + 1 \)  
9. \( 25x^2 - 10x + 1 \)  
10. \( 36x^2 + 12x + 1 \)  
11. \( 81t^2 + 18t + u^2 \)  
12. \( a^2b^2 + 6ab + 9 \)  
13. \( x^2 - 2xy + y^2 \)  
14. \( x^2 + 2xy - y^2 \)  
15. \( r^2 - 2rs + s \)  
16. \( a^2 - 2a - 1 \)  
17. \( z^2 + 2z - 1 \)  
18. \( 9r^2 - 18r + 1 \)  
19. \( 25x^2 - 20x + 1 \)  
20. \( 36w^2 - 24w + 1 \)  
21. \( a^2 - 20a + 100b^2 \)  
22. \( 4a^2 - 24a + 9 \)  
23. \( x^2y^2 - 2xyz + z^2 \)  
24. \( t^2u^2 + 2tuv + v^2 \)

**WRITTEN EXERCISES**

Factor, and check.

**A**

1. \( x^2 + 2bx + b^2 \)  
2. \( g^2 - 2gh + h^2 \)  
3. \( a^2 - 12a + 36 \)  
4. \( y^2 + 16y + 64 \)  
5. \( b^2 + 14b + 49 \)  
6. \( n^2 + 18n + 81 \)  
7. \( 4a^2 - 4ab + b^2 \)  
8. \( 25a^2 - 10ab + b^2 \)  
9. \( 9x^2 + 6x + 1 \)  
10. \( 16x^2 + 8x + 1 \)
11. \(1 + 2n + n^2\)  
12. \(1 + 4b + 4b^2\)  
13. \(49x^2 - 28x + 4\)  
14. \(25x^2 - 30x + 9\)  
15. \(25a^2 + 60ab + 36b^2\)  
16. \(144n^2 + 120nx + 25x^2\)  
17. \(144x^2 - 24x + 1\)  
18. \(16x^2 - 24xy + 9y^2\)  
19. \(36a^2 + 60ab + 25b^2\)  
20. \(16r^2 + 40rt + 25t^2\)  
21. \(r^2 + 25 - 10r\)  
22. \(4mn + n^2 + 4m^2\)  
23. \(121a^2b^2 - 22ab + 1\)  
24. \(243m^2n^2 + 54mn + 3\)  
25. \(3k + 42k^2 + 147k^3\)  
26. \(8n + 8n^2 + 2n^3\)  
27. \(z^2 - 4a^2z + 4a^4\)  
28. \(x^4 + 2x^2y + y^2\)  
29. \(y^6 - 8y^3 + 16\)  
30. \(n^4 - 2n^2 + 1\)  
31. \(x^4 - 18x^2 + 81\)  
32. \(a^8 - 6a^4 + 9\)  
33. \(x^2 - 4xy + 4y^2 - 9\)  
34. \(a^2 + 6ab + 9b^2 - 1\)  
35. \(m^2 - x^2 + 2x - 1\)  
36. \(n^2 - y^2 - 6y - 9\)  
37. \(x^2 + kx + 9\)  
38. \(x^2 - 2kx + 25\)  
39. \(n^2 - 16n + k\)

### 7-7 Multiplying Binomials at Sight

To learn to write the product of two binomials of the form \((ax + b)(cx + d)\) at sight, study these examples which apply the distributive property.

**EXAMPLE 1.** \((3x + 2)(5x - 7)\)

**Solution:**

\[
\begin{array}{c}
5x - 7 \\
\downarrow \\
3x + 2 \\
\hline \\
15x^2 - 21x
\end{array}
\]

\[
\begin{array}{c}
10x - 14 \\
\downarrow \\
15x^2 - 11x - 14
\end{array}
\]

**EXAMPLE 2.** \((ax + b)(cx + d)\)

**Solution:**

\[
\begin{array}{c}
\frac{\text{first terms}}{ax + b} \\
\times \\
\frac{\text{last terms}}{cx + d}
\end{array}
\]

\[
\begin{array}{c}
acx^2 + adx \\
\downarrow \\
bcx + bd
\end{array}
\]

To write the terms in the trinomial product of two binomials at sight:

1. Multiply the first terms of the binomials.
2. Multiply the first term of each binomial by the last term of the other, and add these products.
3. Multiply the last terms of the binomials.
Each term of a trinomial like $15x^2 - 11x - 14$ has a special name. The first, $15x^2$, is the **quadratic term**; the second, $-11x$, is the **linear term**; and the third, $-14$, is the **constant term**. A **quadratic term** is a term of degree two. A **linear term** is a term of degree one. A **constant term** is a numerical term with no variable factor. The trinomial is itself called a **quadratic polynomial** because the term of the highest degree in it is a quadratic term.

**ORAL EXERCISES**

Give (a) the quadratic, (b) the linear, and (c) the constant terms, and (d) read the product as a quadratic trinomial.

1. $(a + 1)(a + 5)$
2. $(b + 4)(b + 1)$
3. $(v + 2)(v + 5)$
4. $(y + 4)(y + 7)$
5. $(c + 4)(c + 7)$
6. $(d + 8)(d + 5)$
7. $(s - 7)(s - 9)$
8. $(t - 7)(t - 8)$
9. $(u - 13)(u - 3)$
10. $(w - 14)(w - 2)$
11. $(a - 6)(a - 12)$
12. $(b - 7)(b - 12)$
13. $(x + 1)(x - 5)$
14. $(x + 4)(x - 5)$
15. $(x + 1)(x - 3)$
16. $(x - 3)(x + 2)$
17. $(y - 2a)(y + 5a)$
18. $(y - 3b)(y + 7b)$

**WRITTEN EXERCISES**

Write each product.

**A**

1. $(4a + 3)(3a + 4)$
2. $(9z + 2)(3z + 1)$
3. $(5b - 9)(2b - 4)$
4. $(2y - 3)(7y - 5)$
5. $(3c - 2)(4c + 3)$
6. $(6x - 9)(7x + 5)$
7. $(2w + 11v)(3w - 5v)$
8. $(4h + 7k)(7h - 12k)$
9. $(2x + 5y)(8x - 3y)$
10. $(6x - 7y)(8x + 5y)$
11. $(x + 7)(-x + 3)$
12. $(-x + 5)(x + 6)$
13. $(-s - 3)(-s - 3)$
14. $(-t - 8)(-t - 2)$
15. $(-2y + 3)(y - 8)$
16. $(y + 5)(-3y - 4)$
17. $(-2n - 4)(-3n - 5)$
18. $(-5m - 2)(-2m - 3)$
19. $(-1.2m + 2.3)(0.4m - 0.7)$
20. $(-3.1h - 3.2)(2.1h - 4.4)$

Solve, and check.

**C**

21. $(x + 5)(x - 5) = (x + 7)(x - 6)$
22. $(x - 4)(x + 4) = (x - 6)(x + 8)$
23. \( (2r + 1)(8r - 3) = (4r - 1)^2 \)
24. \( (9t - 1)(4t + 2) = (6t + 1)^2 \)
25. \( x^2 - 1 = (x - 1)^2 + 2x \)
26. \( (x - 2)(x - 1) = (x + 1)^2 - (5x - 1) \)

**7-8 Factoring the Product of Binomial Sums or Differences**

To factor the trinomial \( x^2 + 5x + 6 \), you recall that the product \((x + r)(x + s)\) is a similar trinomial. Compare the two.

\[
(x + r)(x + s) = x^2 + (r + s)x + rs
\]

\[
\downarrow \downarrow \downarrow
\]

\[
x^2 + 5x + 6
\]

You see that \( r + s = 5 \) and \( rs = 6 \), they would be exactly alike. With these clues, you can find two integers such that

\[
rs = 6 \quad \text{and} \quad r + s = 5.
\]

Observe that the product of the desired integers is positive, indicating that \( r \) and \( s \) are both positive or both negative. Observe next that their sum is positive, showing \( r \) and \( s \) both to be positive. There are two ways to express 6 as the product of two positive integers:

\[
6 = 1 \cdot 6 \quad \text{and} \quad 6 = 2 \cdot 3
\]

\[
\uparrow \uparrow \uparrow \uparrow
\]

\[
r \quad s
\]

\[
\downarrow \downarrow \downarrow \downarrow
\]

\[
r + s = 1 + 6 = 7 \quad \quad r + s = 2 + 3 = 5.
\]

The second set of factors satisfies both clues, so you conclude that

\[
x^2 + 5x + 6 = (x + 2)(x + 3).
\]

Of course, if you see the right pair of factors at the outset, there is no need to write the remaining possibilities. You check your conclusion by multiplying the factors, because the factored form is just another way of designating the original polynomial.

**Factor:** \( x^2 - 11x + 24 \)

**Solution:** \[
x^2 - 11x + 24 = (x - 3)(x - 8), \text{ Answer.}
\]

**Check:** \( (x - 3)(x - 8) = x^2 - 11x + 24 \) \( \checkmark \)
Each of the quadratic polynomials you have factored has a positive constant term. Because \(x^2 + 5x + 6\) has a positive linear term, the polynomial is expressed as a product of two binomial sums, \((x + 2)(x + 3)\). The second polynomial, \(x^2 - 11x + 24\), having a negative linear term, factors into the product of two differences, \((x - 3)(x - 8)\).

Not every quadratic trinomial can be written as a product of binomials having integral coefficients. To factor \(x^2 + 10x + 8\) you would have to find positive integers \(r\) and \(s\) such that

\[rs = 8 \quad \text{and} \quad r + s = 10.\]

The two ways of writing 8 as a product of positive integers are:

\[
\begin{align*}
8 &= 1 \cdot 8 \quad \text{and} \quad 8 = 2 \cdot 4 \\
r + s &= 9 \\
r + s &= 6.
\end{align*}
\]

In each case, \(r + s \neq 10\); therefore, \(x^2 + 10x + 8\) cannot be factored over the set of polynomials with integral coefficients. Such a polynomial is said to be prime, or irreducible over this set of polynomials. A polynomial which cannot be factored into polynomials of lower degree belonging to a designated set is said to be prime over that set of polynomials.

**ORAL EXERCISES**

For each trinomial, tell which two factors of the constant term have a sum equal to the coefficient of the linear term.

1. \(x^2 + 6x + 8\)  
2. \(x^2 + 5x + 6\)  
3. \(s^2 + 8s + 15\)  
4. \(t^2 + 12t + 35\)  
5. \(w^2 - 7w + 10\)  
6. \(w^2 - 13w + 22\)  
7. \(r^2 - 18r + 77\)  
8. \(z^2 - 10z + 21\)  
9. \(v^2 + 16v + 55\)  
10. \(m^2 + 19m + 34\)  
11. \(y^2 - 15y + 26\)  
12. \(x^2 - 7x + 12\)  
13. \(x^2 + 5x + 4\)  
14. \(x^2 - 9x + 8\)  
15. \(y^2 - 9y + 18\)  
16. \(r^2 + 10r + 9\)  
17. \(x^2 + 13x + 12\)  
18. \(w^2 + 8w + 7\)

**WRITTEN EXERCISES**

Factor each trinomial, and check by multiplication.

A  
1. \(n^2 + 14n + 33\)  
2. \(z^2 + 12z + 27\)  
3. \(x^2 + 11x + 18\)  
4. \(x^2 + 10x + 16\)  
5. \(h^2 - 13h + 36\)  
6. \(y^2 - 15y + 56\)
7. \(x^2 - 19x + 90\)  
8. \(a^2 - 47a + 90\)  
9. \(m^2 + 21m + 90\)  
10. \(r^2 + 33r + 90\)  
11. \(33 - 34b + b^2\)  
12. \(14 - 15k + k^2\)  
13. \(42 + 17c + c^2\)  
14. \(52 + 17s + s^2\)  
15. \(x^2 + 20x + 51\)  
16. \(y^2 + 52y + 51\)  
17. \(x^2 + 14xy + 24y^2\)  
18. \(y^2 + 26yz + 48z^2\)  
19. \(m^2 - 22mn + 72n^2\)  
20. \(s^2 - 21st + 20t^2\)  
21. \(b^2 - 23bc + 76c^2\)  
22. \(a^2 - 22ab + 57b^2\)  
23. \(z^2 - 29zb + 120b^2\)  
24. \(z^2 - 23zd + 120d^2\)

Determine all integral values of \(b\) for which the trinomial can be factored over the set of binomials with integral coefficients.

**SAMPLE.** \(x^2 + bx + 12\)

**Solution:** 12 can be factored into a product of two integers as follows:

\[1 \cdot 12, \ 2 \cdot 6, \ 3 \cdot 4, \ (-1)(-12), \ (-2)(-6), \ (-3)(-4).\]

The corresponding values of \(b\) are

\[13, \ 8, \ 7, \ -13, \ -8, \ -7, \ Answer.\]

**25.** \(y^2 + by + 20\)  
**26.** \(z^2 + bz + 63\)  
**27.** \(x^2 + bx + 1\)  
**28.** \(y^2 + by + 4\)  
**29.** \(m^2 + 2bm + 36\)  
**30.** \(n^2 + 2bn + 44\)

Find all positive integers \(c\) for which each trinomial can be factored over the set of binomials with integral coefficients.

**31.** \(x^2 + 5x + c\)  
**32.** \(x^2 + 7x + c\)  
**33.** \(y^2 - 6y + c\)

Show that each polynomial is prime over the set of polynomials with integral coefficients.

**34.** \(x^2 + 3x + 7\)  
**35.** \(x^2 + 15x + 9\)  
**36.** \(y^2 - 4y + 1\)  
**37.** \(y^2 - 3y + 3\)  
**38.** \(x^2 + 4\)  
**39.** \(x^2 + 9\)

**7–9** **Factoring the Product of a Binomial Sum and a Binomial Difference**

To factor \(x^2 + 2x - 15\), proceed as before to look for \(r\) and \(s\) such that

\[x^2 + 2x - 15 = (x + r)(x + s) = x^2 + (r + s)x + rs.\]

Your clues are:

\[rs = -15 \quad \text{and} \quad r + s = 2.\]

Here, the product \(rs\) is negative, indicating that one integer, say \(r\), must be positive, while \(s\) must be negative. But, \(r + s\) is 2, which means
that \( r, \) the positive member of the pair, must have the greater absolute value. On the basis of these conclusions, consider:

\[
\begin{align*}
15(-1) \quad &\quad \text{and} \quad 5(-3) \\
\uparrow \quad \uparrow \quad &\quad \uparrow \quad \uparrow \\
r \quad s \quad &\quad r \quad s
\end{align*}
\]

\[r + s = 14 \quad \quad r + s = 2\]

\[\therefore x^2 + 2x - 15 = (x + 5)(x - 3).\]

In factoring \( x^2 - 2x - 15, \) you would search for two integers of opposite sign, but with a negative sum. Therefore, the negative integer would have to have the larger absolute value. This trinomial is factored:

\[x^2 - 2x - 15 = (x - 5)(x + 3).\]

**WRITTEN EXERCISES**

Find the factors of each trinomial.

**A**

1. \( x^2 + 3x - 10 \)
2. \( y^2 + 5y - 14 \)
3. \( y^2 - 4y - 21 \)
4. \( x^2 - 2x - 15 \)
5. \( h^2 - 2h - 63 \)
6. \( n^2 + n - 56 \)
7. \( k^2 + k - 110 \)
8. \( w^2 - 4w - 96 \)
9. \( r^2 - 2r - 99 \)
10. \( x^2 - 13x - 30 \)
11. \( u^2 - 6u - 55 \)
12. \( b^2 + b - 132 \)
13. \( x^2 - 8x - 33 \)
14. \( n^2 - 8n - 48 \)
15. \( x^2 - x - 90 \)
16. \( b^2 - 5b - 24 \)
17. \( z^2 - 3zt - 4r^2 \)
18. \( t^2 - 3tu - 10u^2 \)
19. \( -y^2 - yz + 2z^2 \)
20. \( -x^2 - xy + 6y^2 \)
21. \( a^2 + 3a - 40 \)
22. \( c^2 + 9c - 36 \)
23. \( d^2 - 6d - 16 \)
24. \( m^2 - m - 30 \)

Determine all integral values of \( b \) for which the trinomial can be factored.

**B**

25. \( x^2 + bx - 12 \)
26. \( x^2 + bx - 14 \)
27. \( y^2 + by - 4 \)
28. \( y^2 + by - 16 \)
29. \( t^2 + 2bt - 28 \)
30. \( z^2 + 2bz - 20 \)

Find the two negative integers \( c \) of least absolute value for which each trinomial can be factored.

31. \( x^2 + 5x + c \)
32. \( x^2 + 7x + c \)
33. \( y^2 - 6y + c \)
Show that each of the following polynomials is prime over the set of polynomials with integral coefficients.

34. \( x^2 + 8x - 7 \)  
35. \( x^2 + 5x - 4 \)  
36. \( y^2 - 4y - 3 \)  
37. \( y^2 - 6y - 5 \)  
38. \( y^2 - 3 \)  
39. \( y^2 - 6 \)

7-10 General Method of Factoring Quadratic Trinomials

To factor a quadratic trinomial product whose quadratic term has a coefficient other than 1, you may use inspection and trial, as in this example:

Factor \( 6x^2 - 25x + 14 \).

First clue: The constant term is positive, and the linear term is negative.

\[ \therefore \text{ Both binomial factors are differences.} \]

Second clue: The product of the linear terms of the binomials is \( 6x^2 \), and the product of the constant terms of the binomials is 14.

\[ \therefore \text{ The possibilities to consider are as follows:} \]

\[
\begin{array}{ccc}
\text{Possible Factors} & \text{Corresponding Linear Terms} \\
(x - 1)(6x - 14) & -14x - 6x = -20x \\
(x - 14)(6x - 1) & -x - 84x = -85x \\
(x - 2)(6x - 7) & -7x - 12x = -19x \\
(x - 7)(6x - 2) & -2x - 42x = -44x \\
(2x - 1)(3x - 14) & -28x - 3x = -31x \\
(2x - 14)(3x - 1) & -2x - 42x = -44x \\
(2x - 2)(3x - 7) & -14x - 6x = -20x \\
(2x - 7)(3x - 2) & -4x - 21x = -25x \\
\end{array}
\]

Third clue: The linear term of the trinomial is \(-25x\). Only the last possibility satisfies all three clues.

\[ \therefore 6x^2 - 25x + 14 = (2x - 7)(3x - 2), \text{ Answer.} \]

One other clue can help reduce the number of possible factors. If a trinomial has no common factor, none of its binomial factors can have a
common factor. Thus, the above factor combinations containing \(6x - 14, 6x - 2, 2x - 14\), or \(2x - 2\) can be discarded at once, since each contains a common factor not in the given trinomial.

**EXAMPLE.** Factor: \(8x^2 + 2x - 3\)

**Solution:**

\[
8x^2 + 2x - 3 = ( - \,)( \, + \, )
\]

\[
= ( - 1)( \, + 3)
\]

\[
= (2x - 1)(4x + 3), \text{ Answer.}
\]

**Check:**

\[
(2x - 1)(4x + 3) = 8x^2 + 2x - 3 \checkmark
\]

Find the factors of each trinomial.

**A**

1. \(2y^2 + 7y + 3\)
2. \(3x^2 + 7x + 2\)
3. \(3n^2 - 4n + 1\)
4. \(3x^2 - 5x + 2\)
5. \(5x^2 - 2x - 7\)
6. \(2x^2 - 9x - 5\)
7. \(3a^2 + 2a - 1\)
8. \(5a^2 + 4a - 1\)
9. \(8x^2 - 14x + 3\)
10. \(3x^2 + 20x - 7\)
11. \(35y^2 - 22y + 3\)
12. \(13y^2 - 7y - 6\)
13. \(6x^2 + 25x + 21\)
14. \(14x^2 + 33x + 10\)
15. \(14x^2 - 15x - 11\)
16. \(12x^2 + 28xy - 5y^2\)
17. \(24x^2 - 14xy - 3y^2\)
18. \(6n^2 - 47ns - 63s^2\)
19. \(8m^2 + 14m - 15\)
20. \(10y^2 - 11y - 6\)

Express each dimension as a binomial with integral coefficients.

**C**

21. The area of a rectangle is \(6x^2 - x - 12\). What are the possible dimensions of the rectangle?
22. The area of a rectangle is \(4n^2 + 3n - 10\). What are the possible dimensions of the rectangle?
23. What are the possible height and base of a triangle whose area is \(15a^2 + 38a - 21\)?
24. What are the possible height and base of a triangle whose area is \(14b^2 - 25b - 25\)?
EXTENSION OF FACTORING

7–11 Combining Several Types of Factoring

Sometimes a common factor conceals:

<table>
<thead>
<tr>
<th>The difference of two squares</th>
<th>A trinomial square</th>
<th>A trinomial product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4y^3 - 36y$</td>
<td>$-x^2 + 4x - 4$</td>
<td>$ax^2 - 5axy - 50ay^2$</td>
</tr>
<tr>
<td>$= 4y(y^2 - 9)$</td>
<td>$= -1(x^2 - 4x + 4)$</td>
<td>$= a(x^2 - 5xy - 50y^2)$</td>
</tr>
<tr>
<td>$= 4y(y + 3)(y - 3)$</td>
<td>$= -1(x - 2)^2$</td>
<td>$= a(x + 5y)(x - 10y)$</td>
</tr>
</tbody>
</table>

Here is a procedure to follow in factoring:

1. Is there a common factor? If so, factor the polynomial by finding the greatest common factor. Then consider each factor.

2. If a factor is a binomial, is it the difference of two squares? You can factor such a binomial.

3. If a factor is a trinomial, is it a square? You can factor a trinomial square.

4. If a factor is a trinomial which is not a square, assume that it is the product of two binomials, and search for them. Of course, you cannot factor a prime trinomial. But never decide that an expression is prime until you have tried all the ways you know of factoring.

5. If a factor is neither a binomial nor a trinomial, can you show a common polynomial factor by grouping? If you can, try to factor each of the resulting factors as in Steps 2, 3, and 4.

6. After you factor a polynomial, write all the factors, including any monomial factor. The monomial factor may be a product, but all other factors should be prime; the factoring should be complete.

7. Always multiply the factors to see whether the product is the original expression; the factoring should be correct.

In general, to factor a polynomial product, first find any common factor; then find the binomial factors of the remaining product, and write all of the factors as an indicated product.
In Written Exercises 1–27, state the common factor, if any.

**ORAL EXERCISES**

**WRITTEN EXERCISES**

Factor each expression.

**A**

1. \(3x^2 - 27\)
2. \(14y^2 - 56\)
3. \(2w^2 + 36w + 162\)
4. \(8z^2 + 112z + 392\)
5. \(6x^2 - 11x + 5\)
6. \(5r^2 - 13r - 6\)
7. \(49n^2 - 14n + 1\)
8. \(25x^2 - 20x + 4\)
9. \(-5ax^2 - 5ay^2\)
10. \(-r^3 - r^3t\)
11. \(9x^2 - 24x + 16\)
12. \(4x^2 - 28x + 49\)
13. \(169a^2 - 49b^4\)
14. \(49y^4 - 144z^2\)
15. \(n^2 - 18n - 40\)
16. \(n^2 - 10n - 24\)
17. \(x^2 + 5x - 84\)
18. \(x^2 + 2x - 48\)
19. \(21 - 4m - m^2\)
20. \(161 - 160u - u^2\)
21. \(5s + 104 - s^2\)
22. \(3s + 180 - r^2\)
23. \(5 - 25y - 30y^2\)
24. \(3 - 21y - 24y^2\)
25. \(28n^2 + 87n + 54\)
26. \(6w^2 - 96w - 342\)
27. \(4t^2 - 28t - 480\)

**B**

28. \(az^2 - 12awz + 36aw^2\)
29. \(bx^2 - 14bxy + 49by^2\)
30. \(6y^3 + 3y^2 - 3y\)
31. \(6v^3 + 26v^2 + 8v\)
32. \(-x^2 + 2bx - b^2\)
33. \(-k^2 + 2kh - h^2\)
34. \(2n - n^2 - 1\)
35. \(4b - 4b^2 - 1\)
36. \(-14 - 9z - z^2\)
37. \(-16 - 10k - k^2\)
38. \(7a^2(a + 1) - 5a(a + 1) - 2(a + 1)\)
39. \(4b^2(b + 2) - 3b(b + 2) - (b + 2)\)
40. \(18x^2(x + 1) + 24x(x + 1) + 8(x + 1)\)
41. \(147y^2(y - 1) - 42y(y - 1) + 3(y - 1)\)
42. \(x^2 - 1 + 5x(x^2 - 1) + 4x^2(x^2 - 1)\)
43. \(y^2 - 4 + y(y^2 - 4) - 2y^2(y^2 - 4)\)
44. \(2w^4 - 162\)
45. \(3m^4 - 1875\)
46. \(-20y^2 + 43xy - 14x^2\)
47. \(-8m^2 + 50mn - 33n^2\)

**C**

48. \(x^4 - 7x^2 - 60\)
49. \(n^4 + 13n^2 - 30\)
50. \(y^4 - 5y^2 + 4\)
51. \(m^4 - 13m^2 + 36\)
52. \(5ax^2 - by^2 + bx^2 - 5ay^2\)
53. \(m^2r^2 - 9s^2 - 9r^2 + m^2s^2\)
54. \(y^3 - y^2 + y - 1\)
55. \(x^3 + x^2 - x - 1\)
In Exercises 56–59, the given binomial is a factor of the trinomial over the set of polynomials with integral coefficients. Determine \( c \) in each case.

56. \( 2x - 3; 10x^2 - 3x + c \)
57. \( 3y + 2; 21y^2 - y + c \)
58. \( 5w + 1; cw^2 + 3w - 1 \)
59. \( n - 3; cn^2 - 5n - 21 \)

### 7–12 Working with Factors Whose Product Is Zero

If you know that the product of two numbers is zero, what can you say about the numbers?

Let \( ab = 0 \) and \( a \neq 0 \).

Since \( a \neq 0 \), the reciprocal of \( a, \frac{1}{a} \), exists and is not zero. Using the multiplication property of equality, multiply each of the terms \( ab \) and \( 0 \) by \( \frac{1}{a} \):

\[
\frac{1}{a} (ab) = \frac{1}{a} \cdot 0.
\]

On the left, use the associative property, the product of reciprocals, and the multiplicative property of 1.

\[
\left( \frac{1}{a} \cdot a \right) b = 0
\]

\[
1 \cdot b = 0
\]

\[
b = 0
\]

On the right, use the multiplicative property of 0.

Similarly, if \( ab = 0 \) and \( b \neq 0 \), you can show that \( a \) must equal 0.
And, of course, if either \( a = 0 \) or \( b = 0 \), \( ab = 0 \).

A product is zero if, and only if, at least one of the factors is zero.

### WRITTEN EXERCISES

Solve each equation.

**SAMPLE.** \((x - 1)(x + 4) = 0\)

**Solution:** If \((x - 1)(x + 4) = 0\), either \((x - 1)\) or \((x + 4)\) must be zero. Thus, the solution sets of \(x - 1 = 0\) and \(x + 4 = 0\) together form the solution set of \((x - 1)(x + 4) = 0\).

\[\therefore\] The solution set is \(\{1, -4\}\), Answer.
1. $13(a - 2) = 0$
2. $15(a - 8) = 0$
3. $-11(5 + c) = 0$
4. $37 \left( 16 - \frac{2}{y} \right) = 0$
5. $29 \left( \frac{1}{b} + \frac{2}{7} \right) = 0$
6. $-31 \left( \frac{1}{b} + \frac{3}{7} \right) = 0$
7. $24(12 + c) = 0$
8. $0(z + 9) = 0$
9. $0(z - 19) = 0$
10. $107 \left( 18 - \frac{3}{y} \right) = 0$
11. $0 \left( \frac{2}{u} - \frac{4}{3} \right) = 0$
12. $0 \left( \frac{9}{2} - \frac{3}{v} \right) = 0$
13. $(x - 3)(x - 5) = 0$
14. $(x + 2)(x + 7) = 0$
15. $0 = (y + 10)(y - 7)$
16. $0 = (y - 8)(y + 3)$
17. $0 = v(2v + 1)$
18. $0 = v(3v + 1)$
19. $(5r + 1)(2r - 6) = 0$
20. $(6r + 3)(7r + 14) = 0$

7-13 Solving Polynomial Equations by Factoring

A polynomial equation is an equation whose left and right members are polynomials. A polynomial equation is in standard form when one of its members is zero and the other is a polynomial in which all similar terms have been combined. Standard form for $x^2 - 20x = 300$ is $x^2 - 20x - 300 = 0$. This equation is of degree two and is called a quadratic equation. The degree of a polynomial equation is the greatest of the degrees of the terms of the equation when written in standard form. An equation of degree one, like $3x + 5 = 2x - 1$, is called a linear equation, while an equation like $x^3 = 4x^2 + 5x$, whose degree is three, is a cubic equation. If you transform a polynomial equation into standard form, and if you can factor the left member, then you obtain its roots by finding the numbers for which at least one of those factors is zero.

EXAMPLE 1. Solve $x^2 - 20x = 300$.

Solution:

1. Put the equation into standard form. $x^2 - 20x - 300 = 0$
2. Factor the left member. $(x - 30)(x + 10) = 0$
3. Set each factor equal to zero. $x - 30 = 0 | x + 10 = 0$
4. Solve the resulting linear equations. $x = 30 | x = -10$
5. Check each apparent root in the original equation. $x^2 - 20x = 300$. 
(30)^2 - 20(30) \equiv 300 \\
900 - 600 \equiv 300 \\
100 + 200 \equiv 300 \\
\therefore The solution set is \{30, -10\}, Answer.

Several situations may arise when you try to solve a polynomial equation by factoring. First, the polynomial may have a common numerical factor. Since such a factor would be a nonzero number, you should eliminate it by applying the division property of equality. Second, two or more factors may be identical. Such factors will yield a double or multiple root, which should be written only once in the roster of the solution set. Of course, to use this method, you must be able to factor the polynomial in the equation.

**EXAMPLE 2.** Solve \(3x^2 - 12x + 12 = 0\).

**Solution:**

1. Divide each member by 3.
2. Factor the left member.
3. Set each factor equal to zero.
4. Solve the resulting linear equations.
5. Check in the original equation.

\[
x - 2 = 0 \quad x = 2
\]

\[
3(2)^2 - 12(2) + 12 \equiv 0
\]

\[
0 = 0 \checkmark
\]

\therefore The solution set is \{2\}, Answer.

In other problems you may have a common monomial factor which contains a variable. Such a factor may be zero and give you a root. Therefore, you should not eliminate it by division. The following cubic equation illustrates this situation.

**EXAMPLE 3.** Solve \(x^3 = 4x^2 + 5x\).

**Solution:**

1. Transform the equation into standard form.
2. Factor the left member.

\[
x^3 - 4x^2 - 5x = 0 \\
x(x^2 - 4x - 5) = 0 \\
x(x - 5)(x + 1) = 0
\]

(\text{cont. on p. 266})
3. Set each factor equal to zero.

4. Solve the resulting equations.

5. Check each value in the original equation.

\[
\begin{align*}
n^3 & = 4n^2 + 5n \\
0 & = 0 \\
0 & = 0
\end{align*}
\]

\[
\begin{align*}
(0)^3 & \neq 4(0)^2 + 5(0) \\
125 & \neq 100 + 25 \\
125 & = 125
\end{align*}
\]

\[
\begin{align*}
(-1)^3 & \neq 4(-1)^2 + 5(-1) \\
125 & \neq 125
\end{align*}
\]

\[x^3 = 4x^2 + 5x\]

\[
\begin{align*}
x = 0 & \quad x - 5 = 0 & \quad x + 1 = 0 \\
0 & \quad 5 & \quad -1
\end{align*}
\]

\[
\begin{align*}
x = 0 & \quad x = 5 & \quad x = -1
\end{align*}
\]

\[x = 0 \quad x = 5 \quad x = -1\]

\[\therefore \text{The solution set is } \{0, 5, -1\}, \text{ Answer.}\]

\[\text{WRITTEN EXERCISES}\]

Find the solution set of each equation.

**A**

1. \(w^2 + w - 90 = 0\)  
2. \(w^2 + 5w - 14 = 0\)
3. \(t^2 - 2t = 15\)
4. \(s^2 - 3s = 18\)
5. \(x^2 + 8x = -15\)
6. \(y^2 + 11y = -18\)
7. \(y^2 + 3 = 4y\)
8. \(x^2 + 5 = 6x\)
9. \(x^2 = 108 + 3x\)
10. \(m^2 = 66 + 5m\)
11. \(25y^2 - 100 = 0\)
12. \(h^2 - 256 = 0\)
13. \(4z^2 = 25\)
14. \(9z^2 = 16\)
15. \(x^2 - 24x = 0\)
16. \(y^2 - 16y = 0\)
17. \(2r^2 + 9r + 10 = 0\)
18. \(3r^2 + 13r + 14 = 0\)
19. \(2n^2 - 11n = -5\)
20. \(3x^2 + 9x = 0\)
21. \(6n^2 - 15n = 0\)
22. \(2r^2 - 17r = -21\)
23. \(3k^2 + 10 = 17k\)
24. \(10r^2 + 3 = 11t\)
25. \(3x^2 + 6x = 144\)
26. \(4x^2 + 8x = 140\)
27. \(7t^2 - 35t = 168\)
28. \(8y^2 - 48y = 216\)
29. \(x^2 + 9 = 6x\)
30. \(16 + 8m + m^2 = 0\)
31. \(4m = m^2\)
32. \(6s^2 + s = 0\)
33. \(n^3 - 6n^2 - 40n = 0\)
34. \(t^3 + 8t^2 - 84t = 0\)
35. \(4r^3 + 4r^2 + r = 0\)
36. \(9s^3 - 12s^2 + 4s = 0\)
37. \(x^4 - 17x^2 + 16 = 0\)
38. \(y^4 - 10y^2 + 9 = 0\)
39. \((2x - 1)^2 + 3x(x - 3) = 3(x - 2)(x - 1) - 5\)
40. \((3y^2 - 2)^2 - (2 + 5y^2) = (5y^2 + 1)(y^2 - 2)\)
Find an equation of the lowest degree having the given solution set.

**SAMPLE.**  
$x \in \{0, -1, 2\}$

*Solution:* Since $x = 0$, $x = -1$, or $x = 2$,
then $x = 0$, $x + 1 = 0$, or $x - 2 = 0$,
$\therefore x(x + 1)(x - 2) = 0$.
$x(x^2 - x - 2) = 0$,
$x^3 - x^2 - 2x = 0$, Answer.

43. $y \in \{1, 3\}$  
44. $z \in \{-1, -2\}$  
45. $w \in \{0, 1, -2\}$  
46. $r \in \{0, 2, 4\}$  
47. $p \in \{-1, 1, 3\}$  
48. $y \in \{-3, -2, 2, 3\}$

**7–14 Using Factoring in Problem Solving**

With the ability to solve polynomial equations by factoring, you can solve problems which would have baffled you only a few weeks ago. You must exercise your judgment, rejecting answers which are not sensible in the light of the conditions of the problem.

**EXAMPLE 1.** Mr. Gardner wishes to start a 100-square foot vegetable patch. Since he has only 30 feet of short chicken-wire fencing, he fences three sides of a rectangle, letting his garage wall act as the fourth side of the enclosure. How wide is the garden?

*Solution:*

Let $x =$ number of feet in width;

Then $(30 - 2x) =$ length in feet,

and $x(30 - 2x) =$ area in square feet.

$x(30 - 2x) = 100$

$30x - 2x^2 = 100$

$2x^2 - 30x + 100 = 0$  (Since $2 \neq 0$, divide by $2$.)

$x^2 - 15x + 50 = 0$

$(x - 5)(x - 10) = 0$

$x - 5 = 0 \quad x - 10 = 0$

$x = 5 \quad x = 10$

---

**Diagram:**

- **Garage**
- $x$ is the width of the garden.
- Length: $(30 - 2x)$ feet.
- Area: $x(30 - 2x) = 100$ square feet.
Check: Is the area of the vegetable patch 100 square feet?

If \( x = 5 \), \( 30 - 2x = 20 \).
\[
5 \times 20 = 100 \checkmark
\]

If \( x = 10 \), \( 30 - 20 = 10 \).
\[
10 \times 10 = 100 \checkmark
\]

The width is 5 feet or 10 feet, Answer.

The next problem has only one solution, although the quadratic equation used in solving it has two roots. The problem is interesting also because it employs a most important rule: \( d = rt + 16t^2 \). This rule applies to any object falling freely to the ground.

**EXAMPLE 2**

An object is ejected directly downward from an airplane 9600 feet above the ground. It starts to fall at 160 feet per second. How many seconds elapse before the object hits the ground?

**Solution:**

1. Let \( t \) = number of seconds before the object hits the ground.
   \( d \) = number of feet of fall = 9600 feet.
   \( r \) = rate at which fall starts = 160 feet per second.

2. \[
d = rt + 16t^2
\]
   \[
9600 = 160t + 16t^2
\]

3. \[
16t^2 + 160t - 9600 = 0
\]
   \[
t^2 + 10t - 600 = 0
\]
   \[
(t - 20)(t + 30) = 0
\]
   \[
t - 20 = 0 \quad t + 30 = 0
\]
   \[
t = 20 \quad t = -30 \quad \text{(Rejected)}
\]

4. Does an object take 20 seconds to fall 9600 feet when it starts falling at the rate of 160 feet per second?

\[
d = rt + 16t^2
\]
\[
9600 = (160)(20) + (16)(20)^2
\]
\[
9600 = 3200 + 6400
\]
\[
9600 = 9600 \checkmark
\]

The number of seconds elapsed is 20, Answer.
The root \(-30\) is rejected because the object could not have hit the ground before it was thrown. Remember that you solve the problem by reasoning that if \(t\) satisfies the requirements stated in the problem, then \(t\) must satisfy the equation obtained in Step 3. On the other hand, just because a value of \(t\) satisfies the equation, you cannot conclude that the value will satisfy the problem. The solution set of the equation gives the possible solutions of the problem. By checking these possibilities in the words of the problem, you find the actual solutions of the problem.

### Problems

Solve each problem, rejecting roots that do not fulfill its conditions.

1. A rectangle is 5 feet longer than it is wide. The area of the rectangle is 66 square feet. Find its dimensions.
2. A rectangle is 2 yards longer than it is wide. The area of the rectangle is 99 square yards. Find its dimensions.
3. An airplane can release objects with a downward speed of 64 feet per second. If an object is released when the airplane is at an altitude of 7680 feet, within how much time will the object hit the ground?
4. An object is thrown from an airplane at an altitude of 11,200 feet. It starts falling at 48 feet per second. Find how soon the object reaches the ground.
5. The number of telephone connections \(c\) that can be made through a switchboard to which \(n\) telephones are connected is given by the equation \(c = \frac{1}{2}n(n - 1)\). If a switchboard operator can make 325 connections through her board, how many telephones are connected to it?
6. The number of straight lines \(n\) that can connect \(p\) points is given by the equation \(n = \frac{p(p - 1)}{2}\). How many points has a figure if only 15 lines can be drawn connecting them?
7. Find two consecutive odd integers the sum of whose squares is 202.
8. Find two consecutive even integers the sum of whose squares is 100.
9. A builder decorates 36 square meters of a courtyard with a triangular garden whose base is 1 meter longer than its height. Find the base and height of the triangular garden.
10. Find the dimensions of a triangle whose area is 42 square centimeters and whose base and height together measure 19 centimeters.
The equation \( h = rt - 16t^2 \) is needed to solve the next four problems. It gives the height \( h \), in feet, which an object will reach in \( t \) seconds when it is thrown upward with a starting speed of \( r \) feet per second.

11. A ball was thrown upward with a starting speed of 64 feet per second. In how many seconds did it reach a height of 64 feet?

12. A Fourth of July rocket was shot directly upward with a starting speed of 96 feet per second. In how many seconds did it reach a height of 144 feet?

13. A bullet left a gun at 1600 feet per second. In how many seconds did the bullet hit the balloon 4656 feet directly overhead?

14. A man shot at a balloon 2080 feet directly above him. The bullet left the gun with a muzzle speed of 2096 feet per second. How soon did the bullet reach the balloon?

15. The plowed area of a field is a rectangle 80 feet by 120 feet. The owner plans to plow an extra strip of uniform width on each of the four sides of the field, in order to double the plowed area. How many feet should he add to each dimension of the field?

16. To make room for a barbecue pit, a man cuts the area of his garden in half by subtracting equal amounts from its length and width. If the garden originally is 30 feet by 40 feet, by how much should he reduce each dimension?

17. A ball is thrown directly upward with a starting speed of 48 feet per second. (a) When will the ball be 32 feet above the ground? Explain the two roots. (b) When will it return to the ground?

18. A bullet is fired directly upward with a muzzle velocity of 3216 feet per second. (a) When will the bullet be 3200 feet above the ground? Explain the two roots. (b) In how many minutes will it hit the ground?

19. What is the error in the following argument?

\[
\begin{align*}
x^2 - 1 &= 12 \\
\therefore (x + 1)(x - 1) &= 12 = 6 \cdot 2 \\
\therefore x + 1 &= 6 & \quad x - 1 = 2 \\
x &= 5 & \quad x = 3
\end{align*}
\]

\[\therefore \text{The solution set is \( \{3, 5\} \).}\]

20. Jo is \( n \) years old. Her brother Al is \( n^2 \) years old. In 8 years, Al will be twice as old as Jo is then. How old is Al now?

21. In a right triangle, the hypotenuse exceeds a leg by 2 inches, and the perimeter is 60 inches. If the area is 120 square inches, find the hypotenuse.

22. Show that the sum of the squares of any two consecutive numbers is an odd number.
County Agents and Mathematics

County agents are employed by federal, state, and county authorities to assist rural communities. Their chief tasks are to aid farmers with specific problems — such as crop failure, soil erosion, or marketing of produce — and to keep them informed of developments in agricultural research.

Agents often test soil to determine its pH, that is, whether the soil is acid, alkaline, or neutral. The degree of pH greatly affects the growth of many plants. Certain flowering shrubs — rhododendrons and azaleas, for example — grow best in somewhat acid soil, while cranberries require highly acid soils. Many vegetables, on the other hand, do best in neutral soils.

pH is measured in terms of a scale from 1 to 14; numbers from 1 to 7 indicate acidity in decreasing strength, 7 is neutral, and numbers from 7 to 14 indicate alkalinity in increasing strength. It is possible to alter the pH and thereby increase the productivity of a soil by adding lime, an alkaline substance, or sulphur, an acid-former.

The work pad shows a computation for modifying the composition of the topsoil on a particular farm. Soil samples taken over the 120-acre area showed a pH of 3. The county agent recommended that the farmer neutralize the soil by adding lime, and calculated that a total of 627 1/4 tons would be needed for the 120 acres.

The county agent in the photograph is examining a tract of land that had gone to waste because poor vegetation and low rainfall subjected it to destructive dust storms. The area has been seeded with a hardy grass which retains moisture and binds the soil, providing good grazing land. In calculating the complex factors of rainfall, wind erosion, and plant growth, the agent again used mathematics to solve a difficult problem.
Chapter Summary

Inventory of Structure and Method

1. To find the greatest common factor of a number of integers, factor each as a product of prime numbers. The factors of a polynomial with integral coefficients usually are limited to positive integers and polynomials with integral coefficients.

2. To factor a polynomial, use the distributive property to form a product of the greatest common factor, if any, and the polynomial sum of the other factors of each term. Next, consider the possibilities of factoring this polynomial sum.

3. Certain special products should be read and factored at sight: The sum of two numbers times their difference: \((a + b)(a - b) = a^2 - b^2\); the square of a binomial sum: \((a + b)^2 = a^2 + 2ab + b^2\); and the square of a binomial difference: \((a - b)^2 = a^2 - 2ab + b^2\).

4. To factor trinomial products such as \(ax^2 + bx + c, (a > 0)\):
   - If \(b\) and \(c\) are positive, both binomial factors are sums; if \(b\) is negative and \(c\) is positive, both binomial factors are differences; if \(c\) is negative, the binomial factors are a sum and a difference. By inspection and trial find factors of the quadratic and constant terms which produce binomials whose product contains a linear term with coefficient \(b\).

5. Factoring must be complete; each polynomial factor must be prime over the set of polynomials with appropriate coefficients. The correctness of factoring should be checked by multiplication.

6. To solve a polynomial equation by factoring: Transform the equation into standard form with the right member zero and the left member a polynomial in descending powers of the variable. Factor the left member. Set each factor equal to zero, applying the principle that if a product is zero at least one of its factors is zero. Solve the resulting linear equations. Check each possible root in the original equation. Write the solution set, listing multiple roots only once.

7. Problems leading to quadratic equations may have two answers. However, some problems have only one answer even though the equation has two roots. Therefore, all possible answers must be checked against the wording of the problem.

Vocabulary and Spelling

factoring a number (over a set of numbers) (p. 237)
prime number (p. 238)
greatest common factor (of two integers) (p. 238)  
factoring a polynomial (p. 239)  
common monomial factor (p. 241)  
greatest common monomial factor (of a polynomial) (p. 241)  
factoring by grouping (p. 241)  
quadratic polynomial (p. 254)  
quadratic term (p. 254)  
linear term (p. 254)  
constant term (p. 254)  
prime polynomial (over a set of polynomials) (p. 256)  
polynomial equation (p. 264)  
standard form (p. 264)  
degree of polynomial equation (p. 264)  
linear equation (p. 264)  
quadratic equation (p. 264)  
cubic equation (p. 264)  
multiple root (p. 265)

Chapter Test

7-1 1. Find the greatest common factor of 792 and 2520.  
2. $42xy^2z^3 = 6xyz(?)$

7-2 3. Write $80a^2 - 16ab$ in factored form.  
4. Write $(x + 1) - 2(x + 1)$ in factored form.  
5. Group the terms of $y^3 - 15 - 5y^2 + 3y$, and factor.

7-3 6. Find the square of $-9x^3$.  
7. Write the product of $(rs + t^2)$ and $(rs - t^2)$.

7-4 8. Factor $8m^2 - 50$.  
10. $(3a + b)^2 = ?$  
11. $(5k - 2m)^2 = ?$

7-5 12. Factor $25x^2 + 90xy + 81y^2$.  
13. Solve and check: $(3x - 5)(4x + 1) = (6x - 7)(2x - 1)$.  
Factor each trinomial.

7-6 14. $n^2 + 17n + 42$  
15. $r^2 - 23rs + 90s^2$  
16. $m^2 + 5m - 36$  
17. $k^2 - 7k - 18$  
18. $36n^2 + 95n + 56$  
19. $3a^2 - 23ab - 36b^2$  
19. $30r^2 + 10t - 100$

7-7 Find and check the solution set of each equation.

7-8 21. $(x - 2)(x + 5) = 0$  
22. $y(2y - 1) = 0$

7-9 23. $z^2 - z = 90$  
24. $49n^2 - 169 = 0$

7-10 25. A rectangle is 8 feet longer than it is wide. Its area is 105 square feet. Find its dimensions.
Chapter Review

7-1 Factoring in Algebra  Pages 237–240
1. \(360 = 2^2 \cdot 3^2 \cdot 5^2\)  2. \(-75ab^2c^3 = 15abc(\_\_\_\_)\).
3. The greatest common factor of 360 and 400 is \(_\_\_\_\_\_\_\_\_\_\_\_\_\_.
4. The highest power of \(2x^3y^3\) that is a factor of \(6x^9y^9\) is \(_\_\_\_\_\_\_\_\_\_\_\_\_.

7-2 Identifying Common Factors  Pages 241–244
5. Factor \(5x^4 - 10x^3 + 15x^2\).
6. Factor \(x^2 - 2x^3 + x\).
7. By the distributive property,
   \[n^2(n - 1) + 2(n - 1) = (n - 1)(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.\]
Factor each expression.
8. \(y(y - 3) + 5(y - 3)\)  10. \((2x - 1)(3x + 1) - (2x - 1)\)
9. \(r^2 + 3r + rs + 3s\)  11. \(2rt + 5st - 6ru - 15su\)

7-3 Multiplying the Sum and Difference of Two Numbers  Pages 245–246
Give the square of each number.
12. \(-16\)  13. \(8x\)  14. \(-7a^2b^3\)
Find each product.
15. \((5n - x)(5n + x)\)  16. \((3y + \frac{1}{3})(3y - \frac{1}{3})\)

7-4 Factoring the Difference of Two Squares  Pages 246–248
Factor each expression.
17. \(4x^2 - 25\)  18. \(8x^2 - 32y^2\)  19. \(2d^3 - d\)  20. \(x^4 - 81\)

7-5 Squaring a Binomial: Plateau Section  Pages 248–251
21. \((a + b)^2 = \_\_\_\_.\)  23. \((2x - 1)^2 = \_\_\_\_.\)
22. \((a - b)^2 = \_\_\_\_.\)  24. \((5h + 3k)^2 = \_\_\_\_.\)

7-6 Factoring a Trinomial Square  Pages 251–253
25. Which of these trinomials are squares?
   a. \(x^2 - 20x - 100\)  b. \(4m^2 - 24m + 9\)  c. \(y^2 + 2y + 1\)
Factor the trinomial squares.
26. \(16x^2 - 8x + 1\)  27. \(9z^2 + 30az + 25a^2\)

7-7 Multiplying Binomials at Sight  Pages 253–255
Items 28–30 refer to \((2x + 3)(4x - 5) = 8x^2 + 2x - 15\).
28. A term of degree two, such as \(8x^2\), is a \(_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_. term.
29. A term of degree one, such as $2x$, is a _?_ term.
30. A term with no variable factor, such as $-15$, is a _?_ term.

Give these products at sight.
31. $(8x + 3)(x + 7)$
32. $(5n - 4)(n - 12)$
33. Solve: $(5x - 8)(2x - 3) = (x + 2)(10x - 9)$

7-8 Factoring the Product of Binomial Sums or Differences

Pages 255–257

Items 34–36 refer to $x^2 + 8x + 12 = (x + r)(x + s)$.
34. To factor $x^2 + 8x + 12$, you must find two integers $r$ and $s$ whose product is _?_ and whose sum is _?_.
35. Since 12 is positive, $r$ and $s$ are both _?_ or both _?_.
36. Since $8x$ is positive, $r = _$ _and $s = _$ _, if $r > s$.

Factor each expression.
37. $n^2 + 15n + 44$
38. $x^2 - 23x + 42$

7-9 Factoring the Product of a Binomial Sum and a Binomial Difference

Pages 257–259

Items 39–41 refer to the relationship $x^2 + x - 42 = (x + r)(x + s)$.
39. To factor $x^2 + x - 42$, you must find two integers whose product is _?_ and whose sum is _?_.
40. Since $-42$ is negative, $r$ and $s$ are _?_ in sign.
41. Since $x$ is positive, if $r > s$, then $r = _$ _and $s = _$ _.

Factor each expression.
42. $x^2 - x - 42$
43. $x^2 - 2x - 35$
44. $x^2 + 2x - 35$

7-10 General Method of Factoring Quadratic Trinomials

Pages 259–261

Determine all the integral values of $b$ for which the trinomial can be factored over the set of binomials with integral coefficients.
45. $3x^2 + bx + 2$
46. $14y^2 - by + 3$
47. $10n^2 - bn - 6$

Write in factored form.
48. $20y^2 - 19y + 3$
49. $26z^2 + z - 6$

7-11 Combining Several Types of Factoring

Pages 261–263

50. If a factor is a binomial, determine whether it is the _?_ of two _?_.
51. If a factor is a trinomial, determine whether it is a _?_ or _?_.

7-12 Working with Factors Whose Product Is Zero

Pages 263–264

Solve each equation.
52. $3(x + 2) = 0$
53. $(y + 2)(2y - 5) = 0$
7-13 Solving Polynomial Equations by Factoring

54. Transform \( x^2 - 5x = 50 \) into standard form.

Solve by factoring.

55. \( x^2 + 9x = 70 \)  
56. \( x^2 = 19x - 84 \)

57. \( x^3 - 7x^2 - 30x = 0 \)

7-14 Using Factoring in Problem Solving

58. Twice the square of a certain integer is 3 less than 7 times the integer. Find the integer.

Extra for Experts

Scientific Notation

Scientists use very large and very small numbers.

The speed of light is 29,900,000,000 centimeters per second; the mass of a proton is .000,000,000,000,000,000,001,65 grams.

It is customary to express such numbers in a shorter way in scientific notation (or standard notation), as the product of a whole number or decimal between 1 and 10 and an integral power of 10. Thus, \( 29,900,000,000 = 2.99 \times 10^{10} \) and \( .000000000000000000000165 = 1.65 \times 10^{-24} \).

To determine a method of transforming a number from ordinary decimal form to scientific notation, study the change in the exponent of ten in the following:

\[
\begin{align*}
.00194 &= 1.94 \times .001 = 1.94 \times 10^{-3} \\
.0194 &= 1.94 \times .01 = 1.94 \times 10^{-2} \\
.194 &= 1.94 \times .1 = 1.94 \times 10^{-1} \\
1.94 &= 1.94 \times 1 = 1.94 \times 10^0 \\
19.4 &= 1.94 \times 10 = 1.94 \times 10^1 \\
194 &= 1.94 \times 100 = 1.94 \times 10^2 \\
1940 &= 1.94 \times 1000 = 1.94 \times 10^3
\end{align*}
\]

The effect of multiplying or dividing a number in the decimal system by 10 is to shift the position of the decimal point. Therefore, changing from one form to the other becomes a matter of counting the number of places you must shift the decimal point.

These examples illustrate a procedure for changing a number from one form to the other.
EXAMPLE 1. A starfish lays an average of 2,520,000 eggs annually. Express this number in scientific notation.

Solution: 

\[
2,520,000 = 2.52 \times 10^n.
\]

To find \( n \), place a caret (\(^{\wedge}\)) to correspond to the position of the decimal point in the answer: \( 2^{\wedge}520,000 \). Count the number of places from the caret to the decimal point. You count 6 to the right.

\[
\therefore 2,520,000 = 2.52 \times 10^6, \text{Answer}.
\]

EXAMPLE 2. The charge of an electron is .00000000048 electrostatic units. Express this number in scientific notation.

Solution: 

\[
.00000000048 = 4.8 \times 10^n
\]

Place the caret sign as above: .0000000004\(^{\wedge}\)8. Count the number of places from the caret sign to the decimal point. You count 10 to the left.

\[
\therefore .00000000048 = 4.8 \times 10^{-10}, \text{Answer}.
\]

EXAMPLE 3. Express in decimal form the average white blood cell count of a human male, \( 5.43 \times 10^6 \) per cubic millimeter.

Solution: Here \( n = 6 \), so count 6 places to the right of the caret to locate the decimal point.

\[
5.43 \times 10^6 = 5,430,000, \text{Answer}.
\]

EXAMPLE 4. Express \( 8.0 \times 10^{-4} \) without exponents.

Solution: Since \( n = -4 \), count 4 places to the left of the caret to locate the decimal point.

\[
8.0 \times 10^{-4} = .0008, \text{Answer}.
\]

Scientific notation shows the degree of accuracy of a measured or computed number. Thus, the 5.43 in \( 5.43 \times 10^6 \) is an example of three-digit accuracy, but the 8.0 in \( 8.0 \times 10^{-4} \) shows two-digit accuracy.

Two numbers expressed in scientific notation can be multiplied or divided readily. Thus,

\[
(4.3 \times 10^{-12})(1.92 \times 10^8) = (4.3)(1.92) \times (10^{-12})(10^8) = 8.3 \times 10^{-4}
\]
Questions

Express each number in scientific notation.

1. The diameter of the earth’s orbit: $1.86 \times 10^8$ miles
2. The diameter of a large molecule: $0.0000017$ centimeter
3. The length of the largest virus (parrot fever): $0.00000017$ centimeter
4. The number of molecules in 22.4 liters of a gas (Avogadro’s number): $6.02 \times 10^{23}$
5. The diameter of the sun: $1.30 \times 10^{11}$ centimeters
6. The electronic charge: $0.00000000000000000016$ coulomb

Express each number without using exponents.

7. $9.3 \times 10^7$ miles
8. $1.86 \times 10^5$ miles per second
9. $9.6 \times 10^{22}$ centimeters
10. $9.5 \times 10^{-6}$ centimeter
11. $6.67 \times 10^{-8}$ cgs unit
12. $1.6 \times 10^{-19}$ coulomb

Express the result of each operation in standard notation.

13. $(6.9 \times 10^{-6}) \div (2.3 \times 10^{-2})$
14. $(5.83 \times 10^{-9}) \times (1.39 \times 10^{-8})$
15. $\frac{1.82 \times 10^5}{9.1 \times 10^{17}}$
16. $(9.72 \times 10^8) \times (4.8 \times 10^{-20})$
17. $8.3 \times 10^3 + 2.7 \times 10^4 + 3.1 \times 10^4 - 6.7 \times 10^3 - 8.1 \times 10^4$

Just for Fun

Perfect and Amicable Numbers

In ancient and medieval times, learned men considered 6 a “perfect” number, whereas 8 was “deficient” and 12, “excessive.”

This sort of classification of numbers depends on the sum of their aliquot parts. The aliquot parts of a number are all its factors except itself. Notice the results when you add the aliquot parts of each of the three numbers mentioned above:

$6 \rightarrow 1 + 2 + 3 = 6 \rightarrow$ The sum of the aliquot parts equals the number, so the number is perfect.
The sum of the aliquot parts is less than the number, so the number is deficient.

The sum of the aliquot parts is greater than the number, so the number is excessive.

Now identify each of the following four consecutive numbers as perfect, deficient, excessive, or prime: 27, 28, 29, 30.

The Greek mathematician Euclid developed a rule for finding perfect numbers: \[ N = 2^p - 1 \] in which \( p \) is a prime number. The rule doesn’t give all the Perfect numbers, but it works with these values of \( p \): 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127. Try it with some of the smaller values. Check the perfection of your result.

The ancients called 220 and 284 “friends” or “amicable” because the aliquot parts of 220 total 284, and the aliquot parts of 284 total 220.

As time went on, mathematicians tried to find other numbers whose aliquot parts equaled each other. In 1636, Fermat told a friend that he had found another pair: 17,296 and 18,416. Below are the factors of these numbers; find their aliquot parts, and check Fermat’s statement that they are amicable.

17,296 \( \rightarrow \) 2 \( \cdot \) 2 \( \cdot \) 2 \( \cdot \) 2 \( \cdot \) 23 \( \cdot \) 47

18,416 \( \rightarrow \) 2 \( \cdot \) 2 \( \cdot \) 2 \( \cdot \) 2 \( \cdot \) 1151

A little over a hundred years later, Euler published a list of 64 pairs of amicable numbers, including the pair that Fermat had found and the pair that had been known since antiquity. All the other 62 are larger than Fermat’s pair. But there is a pair of amicable numbers smaller than Fermat’s, although larger than 220 and 284 which Euler completely missed.

The missing pair was found in 1866 by the Italian mathematician Paganini, who was then 16 years old. The numbers are 1184 and 1210. If he could find them, surely you can show that they are amicable!
Working with Fractions

Tempus fugit. To a child, it may seem that next Christmas will never come. An adult may feel that one Christmas follows too closely upon another. Why does time seem to pass more quickly as you get older? Psychologists explain this phenomenon in terms of the relationship of one year to the length of your life. When you are six years old, one year represents one-sixth of your whole life. When you are sixty, it is only one-sixtieth. Unconsciously, you compare numbers in terms of quotients; although you may be aware only of comparing numbers in terms of differences.

You will find comparing numbers by division to be a useful tool in solving problems in science (lower illustration). When you learn to think of two numbers in terms of their ratio, you have new insight into the properties of numbers.

FRACTIONS AND RATIOS

8–1 Defining Algebraic Fractions

Any indicated quotient of two algebraic expressions like \( \frac{2}{1}, \frac{5}{9}, \frac{7}{2}, \frac{x}{3}, \frac{1}{y}, \) and \( \frac{r^3 + 3r + 2}{5r - 10} \) is called an algebraic fraction. Since division by zero is not permitted, a fraction is defined only when its denominator is not zero. Do you see why the indicated numbers must be excluded from the replacement set of \( x \) in the fractions below? To find such excluded numbers, set the denominator of each fraction equal to zero, and solve the resulting equation.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>( \frac{1}{x} )</th>
<th>( \frac{1}{x - 2} )</th>
<th>( \frac{5x + 4}{x + 2} )</th>
<th>( \frac{7}{x^2 - 4} )</th>
<th>( \frac{3x + 6}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excluded Numbers</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>2 and -2</td>
<td>no exclusions</td>
</tr>
</tbody>
</table>
### ORAL EXERCISES

Give the value of the variable for which the fraction is not defined.

**SAMPLE.** \( \frac{x}{3x - 6} \)

**What you say:** When \( x = 2 \), \( 3x - 6 = 0 \).

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{5x} )</td>
<td>4</td>
<td>( \frac{b^2 - 1}{b + 3} )</td>
<td>7</td>
<td>( \frac{a - 7}{a - 7} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2y} )</td>
<td>5</td>
<td>( \frac{x}{5x + 10} )</td>
<td>8</td>
<td>( \frac{c + 2}{c + 2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{3a + 4}{a - 6} )</td>
<td>6</td>
<td>( \frac{t}{6t - 18} )</td>
<td>9</td>
<td>( \frac{d + 4}{d^2} )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{z - 3}{z^2} )</td>
<td>11</td>
<td>( \frac{3t - 15}{15 - 3t} )</td>
<td>12</td>
<td>( \frac{7k - 35}{35 - 7k} )</td>
</tr>
</tbody>
</table>

### WRITTEN EXERCISES

Express as fractions. Find any values of the variables for which the fraction is not defined.

**A**

1. \( 5 \div y \)
2. \( -7 \div z \)
3. \( 0.17 \)
4. \( 0.9 \)
5. \( .2x \)
6. \( 3y \)
7. \( (b - 1) \div b \)
8. \( a \div (a - 1) \)
9. \( k \div (6k - 12) \)
10. \( f \div (3f + 21) \)
11. \( (12x^2 + 4) \div 3 \)
12. \( (15y^2 - 10) \div 5 \)
13. \( (g - 2) \div (14g + 7) \)
14. \( (h - 5) \div (33h - 11) \)
15. \( 1 \div x(x - 1) \)
16. \( 1 \div y(y + 4) \)

Give the set of excluded values of the variables.

**B**

17. \( \frac{4c + 16}{c^2 - 8c + 12} \)
18. \( \frac{9d - 18}{d^2 + 9d + 14} \)
19. \( \frac{1}{5p^2 + 14p - 3} \)
20. \( \frac{1}{7g^2 - 5g - 2} \)
21. \( \frac{a - 2}{a^2 - 4} \)
22. \( \frac{b - 3}{b^2 - 9} \)
23. \( \frac{3y - 5}{y^2 + 9} \)
24. \( \frac{2x + 1}{x^2 + 4} \)
25. \( \frac{x + 4}{x^2 - 16} \)

State the restrictions on the values of the variables.

**C**

26. \( \frac{1}{a(a - b)} \)
27. \( \frac{cd}{c^2 + 2cd + d^2} \)
28. \( \frac{rs}{r^2 - 2rs + s^2} \)
WORKING WITH FRACTIONS

8-2 Reducing Fractions

Why do the fractions \( \frac{3}{2} \) and \( \frac{15}{10} \) name the same number? Using the property of quotients, you have
\[
\frac{15}{10} = \frac{3 \cdot 5}{2 \cdot 5} = \frac{3}{2} \cdot \frac{5}{5} = \frac{3}{2} \cdot 1 = \frac{3}{2}.
\]
Similarly, any fraction of the form \( \frac{3c}{2c} \) equals \( \frac{3}{2} \) because \( \frac{c}{c} \) equals 1 if \( c \neq 0 \).

This example illustrates the multiplication property of fractions:

Dividing or multiplying the numerator and denominator of a fraction by the same nonzero number produces a fraction equal to the given one.

\[
\frac{ac}{bc} = \frac{a}{b}, \text{ provided } c \neq 0
\]

Thus, \( \frac{20}{64} = \frac{5 \cdot 4}{16 \cdot 4} = \frac{5}{16} \) and \( \frac{24x}{12x^2} = \frac{2 \cdot 12x}{x \cdot 12x} = \frac{2}{x}, \text{ if } x \neq 0 \).

A fraction is said to be in lowest terms when its numerator and denominator have no common factor other than 1 and \(-1\). Reducing a fraction to lowest terms is the process of dividing the numerator and denominator by their greatest common factor.

**EXAMPLE 1.** Reduce \( \frac{15b + 5}{6b^2 - b - 1} \) to lowest terms.

**Solution:**

Factor the numerator and denominator.

\[
\frac{15b + 5}{6b^2 - b - 1} = \frac{5(3b + 1)}{(2b - 1)(3b + 1)}
\]

Divide numerator and denominator by the greatest common factor, \( 3b + 1 \).

\[
\frac{5(3b + 1)}{(2b - 1)(3b + 1)} \div (3b + 1) = \frac{5}{2b - 1}, \text{ if } b \not\in \left\{ \frac{1}{2}, -\frac{1}{3} \right\},
\]

Answer.

Recall: If \( b = \frac{1}{2} \) or \( b = -\frac{1}{3} \), the original fraction is meaningless.
EXAMPLE 2. Simplify \( \frac{6 - t}{t^2 - 36} \).

Solution:

\[
\frac{6 - t}{t^2 - 36} = \frac{6 - t}{(t + 6)(t - 6)} = \frac{-1(t - 6)}{(t + 6)(t - 6)} = \frac{-1}{t + 6}, \text{ if } t \not\in \{6, -6\}, \text{ Answer.}
\]

To show the common factor, express the numerator as a product having \(-1\) as a factor.

The fraction \( \frac{-1}{t + 6} \) also can be written in the form \( -\frac{1}{t + 6} \), because

\[
\frac{-1}{t + 6} = (-1) \cdot \frac{1}{t + 6} = -\frac{1}{t + 6}.
\]

You can reduce a fraction only when the numerator and denominator have a common factor. Compare the fractions below:

\[
\begin{array}{c|c}
2 \cdot 3 & 3 \\
\hline
2 & 1 \\
\end{array}
\]

\[
\frac{ab}{a} = \frac{b}{1} = b, \text{ if } a \neq 0.
\]

\[
\begin{array}{c|c}
2 + 3 & 2 \\
\hline
2 & 1 \\
\end{array}
\]

\[
\frac{a + b}{a}, \text{ if } a \neq 0.
\]

\( a \) and \( b \) are factors of the numerator. This fraction can be reduced because \( a \) is a common factor of the numerator and denominator.

\( a \) and \( b \) are not factors of the numerator. This fraction can not be reduced, for no factor (other than 1 or \(-1\)) is common to both numerator and denominator.

Hereafter, it will be assumed that the replacement sets of the variables include no value for which the denominator is zero.

Write each fraction in lowest terms, noting all restrictions on the values of the variables.

A 1. \( \frac{30h^2k}{30hk} \) 2. \( \frac{5st^2}{30st} \) 3. \( \frac{-21m^2n^2}{28m^3n^3} \)
4. \( \frac{-35x^2z^2}{63x^3y^3} \)
5. \( \frac{3a + 3b}{4a + 4b} \)
6. \( \frac{5c - 5d}{5c + 5d} \)
7. \( \frac{4x + 4y}{x^2 - y^2} \)
8. \( \frac{r^2 - 25}{3r + 15} \)
9. \( \frac{2a + 3b}{2ab} \)
10. \( \frac{5r + 3b}{5rs} \)
11. \( \frac{x^2 - 16}{4 - x} \)
12. \( \frac{1 - a^2}{a - 1} \)
13. \( \frac{m^2 - 4}{m^2 - 4m + 4} \)
14. \( \frac{x^2 - 16}{x^2 - 8x + 16} \)
15. \( \frac{3a + 3}{a^2 + 2a + 1} \)
16. \( \frac{5x + 10}{x^2 + 4x + 4} \)
17. \( \frac{r^2s - s}{r - 1} \)
18. \( \frac{2d^2 - 2}{d + 1} \)
19. \( \frac{x^3 + 1}{x + 1} \)
20. \( \frac{x - 4}{x^2 - 4} \)
21. \( \frac{6a^2 - 54}{2a^2 + 8a + 6} \)
22. \( \frac{8a^2 + 40a + 32}{32 - 2a^2} \)
23. \( \frac{5c^2 - 45c + 90}{180 - 5c^2} \)
24. \( \frac{3d^2 - 27}{24 - 11d + d^2} \)

25. \( \frac{c^2 + 4cd + 4d^2}{e^2 - 4d^2} \)
26. \( \frac{9a^2 + 6ab + b^2}{9a^2 - b^2} \)
27. \( \frac{r^2 - 3r}{r^2 - 4r + 3} \)
28. \( \frac{t^2 - 7t}{t^2 - 8t + 7} \)
29. \( \frac{a^2 - 3a - 4}{a^2 + 2a + 1} \)
30. \( \frac{b^2 + 5b - 6}{b^2 - 2b + 1} \)

Explain why each reduction is incorrect.

34. \( \frac{x + 1}{2x} = \frac{1 + 1}{2} = 1 \)
35. \( \frac{r - 5}{r + 5} = \frac{1 - 1}{1 + 1} = \frac{0}{2} = 0 \)
36. \( ay + y \div y = \frac{a}{1} = a \)
37. \( \frac{6 + b}{2 + b} = \frac{6}{2} = 3 \)
38. \( \frac{3 + t^2}{3 + t} = \frac{1 + t}{1 + 1} = \frac{1 + t}{2} \)
39. \( \frac{10 + 2x}{x^2} = \frac{10 + 2}{x} = \frac{12}{x} \)

Write each fraction in lowest terms, noting all restrictions on the values of the variables.

40. \( \frac{6n^2 - 13n - 5}{6n^2 - 17n + 5} \)
41. \( \frac{8n^2 + 10n - 3}{8n^2 - 14n + 3} \)
42. \( \frac{6m + 3m^2 - 3m^3}{6m^3 + 18m^2 + 12m} \)
43. \( \frac{10t^3 - 15t^2 - 10t}{4t - 16t^3} \)
44. \( \frac{2x^3 + 20x^2 + 50x}{4x^3 + 100x} \)
45. \( \frac{5p^3 + 20p^2 + 20p}{25p^3 + 100p} \)
46. \( \frac{6z^4 + 54xz^3 + 120x^2z^2}{3z^2 + 3xz - 36x^2} \)
47. \( \frac{4m^3n - 8m^2n^2 - 32mn^3}{6m^4 - 6m^3n - 36m^2n^2} \)
8–3 Ratio

To compare the daily output of two oil wells, one producing 750 barrels and the other, 250 barrels daily, you can say that the first yields three times as many barrels as the second. This comparison is made by computing the quotient of the numbers: \( \frac{750}{250} = 3 \). You also can say that the daily yields are in the ratio of 3 to 1 (\( \frac{3}{1} \) or 3:1).

A ratio of one number to another is the quotient of the first number divided by the second. You can express the ratio 5 to 4 by:

1. An indicated quotient using the division sign \( \div \) \( \to \frac{5}{4} \)
2. An indicated quotient using the ratio sign \( \propto \) \( \to 5:4 \)
3. A fraction \( \to \frac{5}{4} \)
4. A fraction in decimal notation \( \to 1.25 \)

By the multiplication property of fractions, the ratio 5:4 compares not only the numbers 5 and 4, but also 10 and 8, 15 and 12, -25 and -20, and 5n and 4n, where \( n \) is not zero. However, if you compare a 2-foot line to a 4-inch line, you must change the 2 feet to 24 inches, and then find the ratio, \( \frac{3}{4} \) or 6:1.

To compare the measures of two quantities of the same kind, express the measures in the same unit; then compute their quotient.

**EXAMPLE:** In ferric oxide the ratio of iron to oxygen, by weight, is 7 to 3. How many pounds of each element are in 500 pounds of ferric oxide?

*Solution:*

1. Choose a variable to use in representing the desired weights.

   \[ \text{Let } 7n = \text{pounds of iron.} \]
   \[ \text{Then } 3n = \text{pounds of oxygen.} \]

2. Form an equation.

   \[ 7n + 3n = 500 \]

3. Solve the equation.

   \[ 10n = 500 \]
   \[ n = 50 \]
   \[ \therefore 7n = 350, \; 3n = 150 \]
**WORKING WITH FRACTIONS**

**Check:** Are the weights in the ratio of 7 to 3?

\[
\frac{350}{150} = \frac{7}{3}
\]

Do the weights of iron and oxygen total 500 pounds?

\[
350 + 150 = 500
\]

\[
500 = 500
\]

Answer.

---

**ORAL EXERCISES**

Give each ratio in its lowest terms.

1. 4:8
2. 5:15
3. \(\frac{ab}{ad}\)
4. \(\frac{adx}{bdx}\)
5. \(\frac{200}{15}\)
6. \(\frac{369}{6}\)
7. \(\frac{3a}{3b}\)
8. \(\frac{10x}{10y}\)
9. \(6x\) to \(7x\)
10. \(2y\) to \(5y\)
11. \(\frac{9}{9}\)
12. \(\frac{2.5}{5}\)
13. \(x^2\) to \(4x^2\)
14. \(2y^3\) to \(y^3\)
15. \(\frac{12g}{60g^2}\)
16. \(\frac{6g^3}{9g^2}\)
17. \((x^2 + 1):(x^2 + 1)^3\)
18. \((y^2 + 4)^2:(y^2 + 4)\)
19. 3 feet to 15 feet
20. 4 meters to 16 meters
21. 1 quart to 1 gallon
22. 1 lb. to 1 oz.
23. 3 m. to 5 cm.
24. 1200 g. to 2 kg.
25. 35 cents to 1 dollar
26. 12 months to 2 years
27. 1 lb. 6 oz. to 2 lb. 1 oz.
28. 1 kg. 20 g. to 3 kg. 10 g.

---

**WRITTEN EXERCISES**

Give each ratio in its lowest terms.

1. The area of an 8- by 12-inch rectangle to that of one 4 by 36 inches
2. The area of a 6- by 9-foot rectangle to that of one 9 by 12 feet
3. The area of a 1- by 2-foot rectangle to that of one 8 by 20 inches
4. The area of a 9-inch square to that of a 1.5-foot square
5. A baseball player's 25 hits to his 100 times at bat
6. The cost per pound for screws selling at $15 for 25 pounds
7. $7,250,000 in assets to $2,500,000 in liabilities
8. A profit of $360 to a cost of $1200
9. Men to women in a college with 3500 women in 10,500 students
10. 175 pounds of sodium to chlorine in a compound of the two weighing 290 pounds

In Exercises 11–14, use the rule that in a triangle, Area = \( \frac{\text{Base} \times \text{Altitude}}{2} \).

11. The area of a triangle with a 12-yard base and a 6-yard height to that of one with an 18-yard base and a 5-yard height
12. The area of a triangle with a 14-meter base and a 20-meter height to that of one with a 28-meter base and a 5-meter height
13. The area of a triangle 9 inches high with a 1-foot base to that of one 4 inches high with a 1\( \frac{1}{2} \)-foot base
14. The area of a triangle 2 feet high with a 4-yard base to that of one 4 feet high with a 3-yard base

Find the ratio \( x:y \) in each case.

15. \( 4x = 3y \)
16. \( 5y = 2x \)
17. \( x = y \)
18. \( x = 7y \)
19. \( 3x - 2y = 0 \)
20. \( 6y - 9x = 0 \)
21. \( \frac{3x + 2y}{2y} = \frac{4}{3} \)
22. \( \frac{2x - y}{2x} = -\frac{3}{4} \)
23. \( \frac{9x - 4y}{6y} = \frac{5}{3} \)
24. \( \frac{4x + 6y}{3y} = \frac{3}{3} \)
25. \( \frac{x^2 - 2y^2}{y^2} = \frac{2x - 3y}{y} \)
26. \( \frac{x^2 + 7y^2}{y^2} = \frac{3y - 4x}{y} \)

**PROBLEMS**

A

1. Find the larger of two numbers in the ratio of 2 to 5, whose difference is 56.
2. Find the smaller of two numbers in the ratio of 5 to 3, whose sum is -24.
3. How many of 35 delegates should each of two cities, whose populations are in the ratio 2:5, send to a convention?
4. The ratio of tin to silver in 78 pounds of an alloy is 12:1. How much silver has the alloy?
5. An artist plans a landscape in which the land’s height has a ratio of 8:5 to the sky’s height. How high in the 104-inch mural is his horizon?
6. Sandy makes bread in which the ratio of whole-wheat to white flour is \( \frac{2}{3} \). How much white flour is in the total 8 cups?

7. A dime contains copper and silver in the ratio 1:9. How much of each is in 45 pounds of dimes?

8. Ross and Morgan need $24,200 in capital. Ross invests $500 for every $600 of Morgan. How much does each invest?

9. Mr. A charges $1.05 for 21 pounds of nails, and Mr. B charges 88 cents for 16 pounds of nails. Which is the better buy?

10. One worker assembles 34 units in 90 minutes. A second assembles 40 units in 1 hour 44 minutes. Which man works faster?

11. Divide a 30-cm. line into two parts with a ratio of 2:3.

12. Divide a line 21 cm. long into two parts whose ratio is \( \frac{3}{4} \).

13. Only 2 of every 7 of a city’s 1,343,790 dwelling units were built after 1950. How many of its dwelling units were over 10 years old in 1960?

14. In a 41,140-mile highway system, 3 of every 17 miles are primary, and the rest, secondary roadway. How many miles are secondary roadway?

15. In sulfuric acid, the ratio of sulfur to hydrogen is 16:1, and of oxygen to sulfur is 2:1. How much of each is in 490 lb. of sulfuric acid?

16. If the ratio of silt to clay is 1:1 and of sand to clay is 2:9, how much of each does 500 pounds of soil contain?

17. Find the largest of the three parts of $1200, having a ratio of 3:5:7.

18. Jack, Walt, and Randy receive a total of $6.75 for delivering newspapers. If they deliver in the ratio 2:3:4, what is Randy’s share?

8-4 Per Cent and Percentage Problems

The ratio of one number to another is often expressed as a per cent. The words per cent (\%) stand for divided by 100 or hundredths. Hence, 4\% is another way of writing \( \frac{4}{100} \) or .04; and 125\% is another name for \( \frac{125}{100} \) or 1.25; and 1\% = \( \frac{1}{100} \). Most important, 100\% = \( \frac{100}{100} = 1 \).

To write a ratio as a per cent, write the ratio as a fraction with denominator 100; then write the numerator followed by a per cent sign.

**EXAMPLE 1.** Express each number as a per cent: \( \frac{5}{6} \), 2.3.

**Solution:**

\[
\frac{5}{6} = \left(\frac{5}{6} \cdot 100\right)\frac{1}{100} = \frac{83\frac{1}{3}}{100} = 83\frac{1}{3}\%
\]

\[
2.3 = (2.3)(100)\frac{1}{100} = \frac{230}{100} = 230\%
\]
A percentage is a number equal to a per cent of another number called the base. Since per cent is the ratio of the percentage to the base, it is often called the rate, to avoid confusion with percentage. The key to per cent and percentage problems is this basic relationship:

\[
\frac{\text{percentage}}{\text{base}} = \text{rate} \quad \text{or} \quad p = rb, \quad \text{if} \quad b \neq 0.
\]

**EXAMPLE 2.** How much is a 20% markup on an item whose cost is $35?

**Solution:**

Let \( p = \text{markup (percentage)} \)

\[
\frac{p}{b} = r\quad \text{or} \quad p = rb
\]

\[
\begin{align*}
\frac{p}{35} &= \frac{20}{100} \\
p &= (.20)(35)
\end{align*}
\]

Show that the markup (percentage) is $7.

**WRITTEN EXERCISES**

Determine the indicated percentage.

A

1. 7% of 250
2. 2% of 2.5
3. 32% of 12.5
4. 90% of 1000
5. 75% of 6
6. 100% of 72
7. 200% of 12
8. 150% of 38
9. 300% of 1
10. 2.25% of 16
11. \( \frac{1}{5} \)% of 635
12. .03% of 1000

Find each number.

13. 24 is 30% of the number.
14. 17 is 25% of the number.
15. 5% of the number is 2.1.
16. 80% of the number is 5.2.
17. 100% of the number is 195.
18. 150% of the number is 63.
19. \( \frac{1}{3} \)% of the number is .72.
20. \( \frac{1}{6} \)% of the number is 20.4.

Determine each rate.

21. What % of 52 is 39?
22. What % of 35 is 21?
23. What % of 3 is 12?
24. What % of 6 is 15?
25. 1 is what % of 200?  
26. 3 is what % of 240?  
27. 60 is what % of 5?  
28. 500 is what % of 25?

1. If 2% of its output is defective, how many of a machine’s 1500-bolt output are good? 
2. How much copper is in 25 pounds of an alloy containing 5% copper? 
3. How much hydrogen chloride is in one quart of acid containing 40% hydrogen chloride? 
4. The average weight of men in World War II was 8% more than in World War I. If the average in the first war was 140 pounds, what was it in the second? 
5. A family spends $116 a month for rent. Find the total monthly income, if 20% is spent for rent. 
6. An agent received $47.50 as his 5 per cent commission on a car sale. What was the car’s price? 
7. Find the cost of a suit if 80 per cent of it is $42.40. 
8. A camera’s sale price of $7.50 is 60% of the list price. What was the list price? 
9. Ben sold a car for $1120, which was 160 per cent of its cost to him. What had the car cost him? 
10. What is the cost of a coat, if 175 per cent of it is $105? 
11. The salesman’s commission on a $1250 car is $62.50. What is his rate of commission? 
12. On $8000 Mr. Todd pays a tax of $1960. What per cent is this? 
13. Mr. Jones paid $2500 for a car. In a year, its value was $2000. By what per cent had the car depreciated in value? 
14. Mr. Williams bought a house for $12,000 and sold it for $13,800. By what per cent had the house increased in value? 
15. A camera and four rolls of film cost $11.55, 30 per cent below the original price. What was the original price? 
16. Rice contains only 5 per cent fat. How much rice would you have to eat to consume a quarter of a pound of fat? 
17. If a dealer’s expenses are 15%, and his profit, 30% of the selling price, for what does he sell a painting which cost $330? 
18. An article is marked $13 and a 15% discount is given. What profit does a dealer make if the article cost him $8?
MULTIPLYING AND DIVIDING FRACTIONS

8-5 Multiplying Fractions

When you read the equality (see page 216)

\[
\frac{xy}{cd} = \frac{x}{c} \cdot \frac{y}{d}
\]

from right to left, you see the rule for multiplying fractions: if \(c \neq 0\), \(d \neq 0\),

\[
\frac{x}{c} \cdot \frac{y}{d} = \frac{xy}{cd}
\]

When fractions are multiplied, the product is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the given fractions.

\[
\frac{m + 1}{m - 2} \cdot \frac{m - 3}{m + 2} = \frac{(m + 1)(m - 3)}{(m - 2)(m + 2)} = \frac{m^2 - 2m - 3}{m^2 - 4}
\]

\[
\frac{5r}{s} \cdot \frac{2r}{s} = \frac{5r}{1} \cdot \frac{2r}{s} = \frac{10r^2}{s}
\]

A product of fractions, not in lowest terms, should be reduced.

\[
\frac{n}{d} \cdot \frac{d}{n} = \frac{nd}{dn} = \frac{nd}{nd} = 1; \quad \frac{a - b}{a + b} \cdot \frac{a + b}{7} = \frac{(a - b)(a + b)}{7} = \frac{a - b}{7}
\]

You can simplify the multiplication of fractions by first factoring where possible.

\[
\frac{3x - 3}{5x + 2} \cdot \frac{25x^2 - 4}{x^2 - 2x + 1} = \frac{3(x - 1)}{5x + 2} \cdot \frac{(5x - 2)(5x + 2)}{(x - 1)(x - 1)}
\]

\[
= \frac{3(5x - 2)(x - 1)(5x + 2)}{(x - 1)(x - 1)(5x + 2)}
\]

\[
= \frac{3(5x - 2)}{x - 1} \quad \text{or} \quad \frac{15x - 6}{x - 1}
\]
ORAL EXERCISES

Determine each product.

1. \( \frac{1}{5} \cdot \frac{1}{3} \)
2. \( \frac{1}{2} \cdot \frac{1}{7} \)
3. \( \frac{3}{5} \cdot \frac{11}{4} \)
4. \( \frac{9}{4} \cdot \frac{3}{2} \)
5. \( \left( \frac{-1}{2} \right) \left( \frac{13}{7} \right) \)
6. \( \left( \frac{1}{4} \right) \left( -\frac{17}{5} \right) \)
7. \( \frac{x}{2} \cdot \frac{y}{3} \)
8. \( \frac{b}{4} \cdot \frac{c}{2} \)
9. \( \frac{7}{3} \cdot \frac{5}{6} \)
10. \( 8 \cdot \frac{6}{5} \)
11. \( m \cdot \frac{m}{4} \)
12. \( t \cdot \frac{1}{8} \)
13. \( \left( \frac{-2}{a} \right) \left( -\frac{7}{b} \right) \)
14. \( \left( -\frac{5}{r} \right) \left( -\frac{3}{s} \right) \)
15. \( \frac{2x}{3} \cdot \frac{x}{y} \)
16. \( \frac{3c}{d} \cdot \frac{c}{2} \)
17. \( \left( -\frac{1}{2c} \right) \left( \frac{1}{5c} \right) \)
18. \( \left( \frac{1}{4a} \right) \left( -\frac{1}{4a} \right) \)
19. \( \frac{5x^2}{3y} \cdot \frac{2x}{3y} \)
20. \( \frac{4k^2}{3m} \cdot \frac{2k}{5m} \)
21. \( \frac{a + b}{b} \cdot (2a - b) \)
22. \( \frac{3x - y}{3} \cdot (3x + y) \)
23. \( \frac{2}{t + 2} \cdot \frac{3}{t + 2} \)
24. \( \frac{k + 4}{5} \cdot \frac{k + 4}{5} \)
25. \( \frac{1}{v^2 + 1} \cdot \frac{1}{v^2 - 1} \)
26. \( \frac{1}{m^2 - 1} \cdot \frac{1}{m^2 - 1} \)
27. \( \frac{x + y}{3} \cdot \frac{6}{y + x} \)
28. \( \frac{u + v}{4} \cdot \frac{2}{v + u} \)
29. \( \frac{s}{s - r} \cdot \frac{r - s}{s} \)
30. \( \frac{k - t}{t} \cdot \frac{t}{t - k} \)

WRITTEN EXERCISES

Determine each product in lowest terms.

1. \( \frac{3}{7} \cdot \frac{5}{4} \cdot \frac{1}{7} \)
2. \( \frac{1}{8} \cdot \frac{3}{2} \cdot \left( -\frac{7}{5} \right) \)
3. \( \frac{3}{8} \cdot \frac{4}{9} \)
4. \( \frac{5}{7} \cdot \frac{14}{25} \)
5. \( \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \)
6. \( \frac{2}{9} \cdot \frac{18}{35} \cdot \frac{7}{4} \)
7. \( \frac{5c^2}{3} \cdot \frac{c}{4} \)
8. \( \frac{7x}{2} \cdot \frac{x^2}{8} \)
9. \( \frac{a \cdot b \cdot c}{b \cdot c \cdot a^2} \)
10. \( \frac{r \cdot s \cdot t}{s \cdot t \cdot r^2} \)
11. \( \frac{6x}{11y} \cdot \frac{77y}{15x} \)
12. \( \frac{9b}{7c} \cdot \frac{28c}{81b} \)
CHAPTER EIGHT

13. \( \frac{7ab}{5cd} \cdot \frac{15c^2d^2}{28ab^2} \)

14. \( \frac{2r^2s^2}{9t^2u^2} \cdot \frac{45tu}{14rs} \)

15. \( \frac{t - v}{t + 2v} \cdot \frac{t + v}{t - 2v} \)

16. \( \frac{3r + w}{r - w} \cdot \frac{3r - w}{r + w} \)

17. \( \frac{a^2 - b^2}{4} \cdot \frac{12}{a + b} \)

18. \( \frac{2}{1 - x^2} \cdot \frac{1 - x}{2} \)

19. \( \frac{4 - y^2}{14} \cdot \frac{7}{2 - y} \)

20. \( \frac{m^2 - n^2}{6} \cdot \frac{3}{m + n} \)

Give each indicated product in its lowest terms.

22. \( \frac{r^2 - s^2}{m^2 - n^2} \cdot \frac{m + n}{r - s} \)

23. \( \frac{a^2 - 49}{b^2 - 25} \cdot \frac{b - 5}{a + 7} \)

24. \( \frac{y^2 - 2y + 1}{3x^2} \cdot \frac{x}{y - 1} \)

25. \( \frac{b^2 + 2b + 1}{ab^2} \cdot \frac{a}{b + 1} \)

26. \( \frac{k^2 + k}{k^2} \cdot \frac{3k - 3}{k^2 - 1} \)

27. \( \frac{6m + 6n}{m^2 - n^2} \cdot \frac{m^2 - mn}{2m} \)

28. \( \frac{9x^2 - 1}{6x + 2} \cdot \frac{2}{3x - 1} \)

29. \( \frac{6a - 3b}{4a^2 - b^2} \cdot \frac{2a + b}{3} \)

30. \( \frac{s - t}{2} \cdot \frac{4s + 4t}{s^2 + 2st + t^2} \)

31. \( \frac{x^2 + 2x + 1}{5x - 5} \cdot \frac{15}{x + 1} \)

32. \( \frac{64r^2 - 1}{64r^2 + 16r + 1} \cdot \frac{8r + 1}{16r - 2} \)

33. \( \frac{x^2 + 3x + 2}{x + 1} \cdot \frac{x + 3}{x^2 + 5x + 6} \)

34. \( \frac{a^2 + 5a + 4}{a + 4} \cdot \frac{a + 5}{a^2 + 6a + 5} \)

35. \( \frac{c^2 - 5c - 6}{c^2 + 3c} \cdot \frac{c + 3}{6 - c} \)

36. \( \frac{2 - x}{2 + x} \cdot \frac{x + 2}{x - 2} \)

37. \( \frac{4 - b^2}{5b - 10} \cdot \frac{b + 2}{b} \)

38. \( \frac{2x^2 - 50}{5x^2} \cdot \frac{4x + 16}{3x - 15} \)

39. \( \frac{a^2 + ab}{b - a} \cdot \frac{a^2 - ab}{b + a} \)

40. \( \frac{3n^2 - 27}{6 - n - n^2} \cdot \frac{4 - 2n}{3 - n} \cdot \frac{3n + n^2}{6} \)

41. \( \frac{4k^2 - 9}{6 + k - 2k^2} \cdot \frac{2 - k}{15k - 10k^2} \cdot \frac{k^2}{2k + 3} \)

42. \( \frac{2x^2 + 7x + 3}{x^2 - 9} \cdot \frac{x^2 - 3x}{2x^2 + 11x + 5} \)

43. \( \frac{3n^2 + 7n + 2}{1 - 9n^2} \cdot \frac{3n^2 - 7n + 2}{4 - n^2} \)

44. \( \frac{4 - 16x^2}{16x^2 - 4x - 6} \cdot \frac{(4x^2 - 11x + 6)}{4x^2 - 8x^3} \cdot \frac{x^2}{x^6} \)

45. \( \frac{90a^2 - 6ab - 12b^2}{15a - 6b} \cdot \frac{3a^4 - a^2b^2}{81a^4 - b^4} \cdot \frac{(9a^2 + b^2)}{(9a^2 + b^2)} \)
8-6 Dividing Fractions

A quotient can be expressed as the product of the dividend and the reciprocal of the divisor (page 139).

\[ 6 \div 3 = 6 \times \frac{1}{3}; \quad 12 \div \frac{1}{3} = 12 \times 3; \quad \frac{4}{5} \div \frac{2}{15} = \frac{4}{5} \times \frac{15}{2} \]

Since the reciprocal of \( \frac{c}{d} \) is \( \frac{d}{c} \), if \( c \neq 0, d \neq 0 \), then

\[ \frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc} \]

**EXAMPLE.**

\[ \frac{t^2 - 9}{t + 3} \div \frac{t^2 - 6t + 9}{3 - t} \]

**Solution:**

\[ \frac{t^2 - 9}{t + 3} \div \frac{t^2 - 6t + 9}{3 - t} = \frac{t^2 - 9}{t + 3} \cdot \frac{3 - t}{t^2 - 6t + 9} \]

\[ = \frac{(t + 3)(t - 3)}{(t + 3)} \cdot \frac{(3 - t)}{(t - 3)(t - 3)} \]

\[ = \frac{(t + 3)(t - 3)(-1)(t - 3)}{(t + 3)(t - 3)(t - 3)} = -1 \]

**WRITTEN EXERCISES**

Find each quotient.

1. \( \frac{3}{7} \div \frac{11}{14} \)
2. \( \frac{12}{11} \div \frac{6}{7} \)
3. \( \frac{a}{b} \div \frac{a}{b^2} \)
4. \( \frac{c^2}{d} \div \frac{c}{d} \)
5. \( \frac{3b}{4c^2} \div \frac{15b^2}{16c^3} \)
6. \( \frac{13r^2}{20a^2} \div \frac{39r^3}{5a} \)
7. \( \frac{3g^2}{10} \div \frac{9g^2}{25} \)
8. \( \frac{12x^3}{35} \div \frac{48x^3}{77} \)
9. \( \frac{a - b}{4} \div \frac{a - b}{2} \)
10. \( \frac{2x + 1}{2} \div \frac{2x + 1}{6} \)
11. \( \frac{9}{11x} \div 18 \)
12. \( \frac{4a^2}{7} \div 8a \)
13. \( \frac{m^2 - n^2}{mn} \div (m - n) \)
14. \( \frac{a^2 + 2ab + b^2}{ab} \div (a + b) \)
15. \( \frac{x}{x^2 - 4x + 4} \div \frac{1}{x - 2} \)
16. \( \frac{m}{m^2 + 6m + 9} \div \frac{1}{m + 3} \)

17. \( \frac{1}{x^2 - 16} \div \frac{x - 4}{x + 4} \)

18. \( \frac{2x + 4}{3x + 9} \div \frac{4x + 8}{5x + 15} \)

19. \( \frac{ax - 2a}{bx - 3b} \div \frac{cx - 2c}{dx - 3d} \)

20. \( \frac{m^2 - n^2}{9m^2n^2} \div \frac{3m - 3n}{mn} \)

21. \( \frac{5a^2 - 5b^2}{a^2b^2} \div \frac{a + b}{10ab} \)

22. \( \frac{x^2 - y^2}{ax - ay} \div \frac{x^2 + 2xy + y^2}{x^2 + xy} \)

23. \( \frac{x^2 + 4x + 3}{x^2 - 4x - 5} \div \frac{x + 3}{x - 5} \)

24. \( \frac{a + 2a^2}{9 - a^2} \div \frac{1 - 4a^2}{3a + a^2} \)

25. \( \frac{r^3 - rs^2}{r^2s} \div \frac{r^2 + rs}{3rs + 3s^2} \)

26. \( \frac{3x + 3y}{x^4 - y^4} \div \frac{2x^2 + 2y^2}{6} \)

27. \( \frac{x^2 - 3x + 2}{x^2 - 1} \div \frac{2 - x}{1 - x} \)

28. \( \frac{5c^2 - 5cd}{cd + d^2} \div \frac{d^3 - dc^2}{cd^2} \)

29. \( \frac{14}{7t^2 + 7u^2} \div \frac{2u + 2t}{u^4 - t^4} \)

8–7 Fractions Involving Multiplication and Division

In the absence of parentheses the rule for order of performing multiplications and divisions is applied to an expression containing fractions. You replace only the fraction immediately following a division sign by its reciprocal.

**EXAMPLE.**

\[ \frac{(y + 1)^2}{15} \div \frac{y + 1}{5} \cdot \frac{3}{y + 1} \]

**Solution:**

\[ \frac{(y + 1)^2}{15} \div \frac{y + 1}{5} \cdot \frac{3}{y + 1} = \frac{(y + 1)^2}{15} \cdot \frac{5}{y + 1} \cdot \frac{3}{y + 1} \]

\[ = \frac{(y + 1)(y + 1)(5)(3)}{(y + 1)(y + 1)(15)} \]

\[ = 1, \text{ Answer.} \]

**Check:** Let \( y = 3 \). The check is left to you.

**WRITTEN EXERCISES**

Multiply or divide, and check by substitution.

**A**

1. \( \frac{3}{x^3} \cdot \frac{x^2}{5} \div \frac{6}{xy} \)

2. \( \frac{a^4}{4} \cdot \frac{3}{a^3} \div \frac{6a}{b} \)
WORKING WITH FRACTIONS

3. \( \frac{a^2 b}{c} \cdot \frac{c^3 d}{a^3} \div b^2 d \)

4. \( \frac{6r^2 s^3}{t} \div 9r^2 s^4 \cdot \frac{3t^4}{s^2} \)

5. \( \frac{x^2 - 1}{x^2} \cdot \frac{x}{x - 1} \div \frac{x + 1}{5} \)

6. \( \frac{3a}{3a + 2} \cdot \frac{6a + 4}{4a} \div \frac{a + 2}{2} \)

7. \( \frac{9m + n}{9mn} \div \frac{3m - n}{3m} \cdot \frac{12}{3m + n} \)

8. \( \frac{2c - d}{2c} \div \frac{8c + d}{8cd} \cdot \frac{16c + 2d}{4} \)

9. \( \frac{a^2 - b^2}{a} \cdot \frac{a + b}{a - b} \div \frac{a^2 + 2ab + b^2}{ab} \)

10. \( \frac{r^2 - 2rs + s^2}{rs} \cdot \frac{r + s}{r - s} \div \frac{r^2 - s^2}{r^2} \)

11. \( \frac{m}{m + 1} \div \frac{2m^2}{2m + 6} \cdot \frac{m^2 - 2m - 3}{m^2 - 9} \)

12. \( \frac{3x - 1}{3x^2} \div \frac{3x + 1}{3x} \cdot \frac{9x^2 + 6x + 1}{9x^2 - 1} \)

13. \( \frac{a^3}{3a} \div \frac{a^2 b^2 - 9}{b} \cdot \frac{3ab - 9}{ab} \)

14. \( \frac{4t}{s^2 t^2 - 4} \div \frac{16}{st} \cdot \frac{4st - 8}{st^2} \)

15. \( \frac{x^2 + 10x + 25}{x^2 + 10x} \cdot \frac{10x}{x^2 + 15x + 50} \div \frac{x + 5}{x + 10} \)

16. \( \frac{a^2 - 12a + 32}{8a} \cdot \frac{a^2 - 8a}{a^2 - 8a + 16} \div \frac{a - 8}{a - 4} \)

17. \( \frac{3a^3}{3a + c} \div \frac{9a^2}{9a^2 - c^2} \cdot \frac{3ab}{3a^2 - ac} \)

18. \( \frac{x^3 y^3}{x^3 - xy^2} \div \frac{xyz}{x^2 - xy} \cdot \frac{xy + yz}{xy} \)

19. \( \frac{ab + ac - ad}{abc} \div \frac{bcd^2}{bd + cd - d^2} \div d^2 \)

20. \( \frac{\nu^2 w^2 x^3}{\nu x - \nu w - x^2} \cdot \frac{uv - uw - ux}{uvw} \div x^2 \)

ADDING AND SUBTRACTING FRACTIONS

8–8 Combining Fractions with Equal Denominators

Consider the sum \( \frac{a}{b} + \frac{c}{b} \). Since \( \frac{a}{b} = a \left( \frac{1}{b} \right) \) and \( \frac{c}{b} = c \left( \frac{1}{b} \right) \), by
the distributive principle, you know that
\[ \frac{a}{b} + \frac{c}{b} = a \left( \frac{1}{b} \right) + c \left( \frac{1}{b} \right) = (a + c) \left( \frac{1}{b} \right) = \frac{a + c}{b}. \]

This chain of equalities gives the **rule for adding fractions:**

The sum of fractions with equal denominators is a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator of the given fractions.

**EXAMPLES.**

\[
\frac{4}{9} + \frac{7}{9} - \frac{y}{9} = \frac{4 + 7 - y}{9} = \frac{11 - y}{9}
\]

\[
\frac{5}{p + q} + \frac{q}{p + q} - \frac{q - p}{p + q} = \frac{5 + q - (q - p)}{p + q} = \frac{5 + p}{p + q}
\]

Sometimes you get a sum which can be reduced.

\[
\frac{7a}{a^2 - b^2} - \frac{5a + 4b}{a^2 - b^2} + \frac{2b}{a^2 - b^2} = \frac{7a - (5a + 4b) + 2b}{a^2 - b^2} = \frac{2a - 2b}{a^2 - b^2} = \frac{2(a - b)}{(a + b)(a - b)} = \frac{2}{a + b}
\]

**WRITTEN EXERCISES**

Give each sum in lowest terms.

A

1. \( \frac{4}{15} - \frac{8}{15} - \frac{7}{15} \)
2. \( \frac{3}{50} - \frac{29}{50} + \frac{1}{50} \)
3. \( \frac{8}{3z} - \frac{1}{3z} + \frac{2}{3z} \)
4. \( \frac{10}{7a} + \frac{15}{7a} - \frac{4}{7a} \)
WORKING WITH FRACTIONS

5. \( \frac{12a}{5r} - \frac{2a + 5}{5r} \)

6. \( \frac{2b}{9t} - \frac{3 - 7b}{9t} \)

7. \( \frac{2a + 1}{5} + \frac{4 - 2a}{5} \)

8. \( \frac{3b + 5}{7} + \frac{2 - 3b}{7} \)

9. \( \frac{3r}{r + s} + \frac{3s}{r + s} \)

10. \( \frac{x}{x - 3} - \frac{3}{x - 3} \)

11. \( \frac{a^2}{a + b} - \frac{b^2}{a + b} \)

12. \( \frac{z^2}{z - 4} - \frac{16}{z - 4} \)

13. \( \frac{3x - 3y}{2xy} - \frac{2x + 5y}{2xy} + \frac{8y - x}{2xy} \)

14. \( \frac{5}{3ab} + \frac{1 - 2a}{3ab} - \frac{6 - 2a}{3ab} \)

15. \( \frac{2xy}{x + y} + \frac{x^2 + y^2}{x + y} \)

16. \( \frac{m^2 + n^2}{m - n} - \frac{2mn}{m - n} \)

17. \( \frac{3}{x^2 - 3x - 4} + \frac{x - 2}{x^2 - 3x - 4} \)

18. \( \frac{2x}{x^2 - x - 2} - \frac{x + 2}{x^2 - x - 2} \)

19. \( \frac{1 + a}{a^2 + 3a - 4} - \frac{2a + a^2}{a^2 + 3a - 4} \)

20. \( \frac{5 - b}{b^2 - 16} - \frac{b^2 - 3}{b^2 - 16} \)

8-9 Adding Fractions with Unequal Denominators

To add \( \frac{5}{12} \) and \( \frac{41}{90} \), express them as fractions with equal denominators. Any positive integer having both 12 and 90 as factors will do, but for convenience you usually want the lowest common denominator (L.C.D.). To find the L.C.D. systematically, write 12 and 90 as the products of primes:

\[ 12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 \quad \text{and} \quad 90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5 \]

\[ \therefore \text{the L.C.D.} = 2^2 \cdot 3^2 \cdot 5 = 180. \]

To convert these fractions to the denominator 180, you do this:

\[ \frac{5}{12} = \frac{5 \cdot 3 \cdot 5}{12 \cdot 3 \cdot 5} = \frac{75}{180} \quad \text{and} \quad \frac{41}{90} = \frac{41 \cdot 2}{90 \cdot 2} = \frac{82}{180}. \]

To complete the problem, you add:

\[ \frac{5}{12} + \frac{41}{90} = \frac{75}{180} + \frac{82}{180} = \frac{75 + 82}{180} = \frac{157}{180}. \]
EXAMPLE

\[
\frac{a + 1}{a^2 - 2a - 3} - \frac{1}{a^2 + a} - \frac{3}{a^2 - 3a}
\]

Solution:

1. Find the L.C.D.

Factor each denominator.

\[
a^2 - 2a - 3 = (a - 3)(a + 1)
\]

\[
a^2 + a = a(a + 1)
\]

\[
a^2 - 3a = a(a - 3)
\]

Take each factor the greatest number of times it appears in any denominator.

\[
a(a + 1)(a - 3)
\]

2. Write with factored denominators.

\[
\frac{a + 1}{(a - 3)(a + 1)} - \frac{1}{a(a + 1)} - \frac{3}{a(a - 3)}
\]

3. Replace each fraction by an equivalent fraction.

\[
\frac{(a + 1)a}{(a - 3)(a + 1)a} - \frac{1(a - 3)}{a(a + 1)(a - 3)} - \frac{3(a + 1)}{a(a - 3)(a + 1)}
\]

4. Combine the fractions, and simplify.

\[
\frac{a(a + 1) - 1(a - 3) - 3(a + 1)}{a(a + 1)(a - 3)} = \frac{a^2 + a - a + 3 - 3a - 3}{a(a + 1)(a - 3)}
\]

\[
= \frac{a^2 - 3a}{a(a + 1)(a - 3)}
\]

\[
= \frac{a - 3}{a(a + 1)(a - 3)}
\]

\[
= \frac{1}{a + 1}, \text{ Answer.}
\]

Check: Let \(a = 2\). The check is left to you.

---

ORAL EXERCISES

Give the L.C.D. of these denominators.

1. 6, 9
2. 10, 4
3. 2, a
4. 3, b
5. 15, 6
6. 14, 4
7. \(ab, bc\)
8. \(m^2n^2, mn\)
WORKING WITH FRACTIONS

9. \( x, \ xy, \ y \)
10. \( a, \ b, \ ab \)
11. \( n - m, \ m - n \)
12. \( c - b, \ b - c \)

Give the lowest common denominator.

13. \( \frac{a}{a + b} - \frac{b}{a^2 - b^2} \)
14. \( \frac{1}{r^2 - s^2} + \frac{1}{r - s} \)
15. \( \frac{2}{x^2 + 2xy + y^2} - \frac{3}{x + y} \)
16. \( \frac{5}{a - b} + \frac{1}{a^2 - 2ab + b^2} \)
17. \( \frac{4x}{x + 4} + \frac{2x}{x + 3} \)
18. \( \frac{3t}{t + 2} + \frac{5t}{t + 3} \)
19. \( \frac{x}{x^2 - y^2} + \frac{y}{x + y} + \frac{xy}{y - x} \)
20. \( \frac{x}{x^2 - 9} + \frac{3}{x + 3} + \frac{3x}{3 - x} \)

WRITTEN EXERCISES

Combine the fractions.

1. \( \frac{4b}{a} - \frac{1}{2} \)
2. \( \frac{1}{3} + \frac{2a}{x} \)
3. \( \frac{7}{6c} + \frac{1 - 3c}{3c} \)
4. \( \frac{1}{10a} - \frac{2a + 5}{2a} \)
5. \( \frac{2a + 1}{5} - \frac{a - 3}{2} \)
6. \( \frac{7n - 4}{4} - \frac{3n + 2}{3} \)
7. \( \frac{2h + 1}{4} + \frac{h - 3}{6} \)
8. \( \frac{3h - 5}{9} - \frac{2h + 1}{6} \)
9. \( \frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3} \)
10. \( \frac{3}{a^3} - \frac{2}{a^2} + \frac{1}{a} \)
11. \( \frac{1}{ab} + \frac{a - 2}{bc} \)
12. \( \frac{4}{rs} - \frac{2 - t}{rt} \)
13. \( \frac{x}{bc} + \frac{x + c}{ac} - \frac{x + b}{ab} \)
14. \( \frac{a}{yz} - \frac{a + z}{xz} + \frac{a + y}{xy} \)
15. \( \frac{3}{xy^2} + \frac{2 - x}{x^2y} \)
16. \( \frac{4}{r^2s} - \frac{2 - r}{rs^2} \)
17. \( \frac{1}{a + b} + \frac{a}{a^2 - b^2} \)
18. \( \frac{d}{c^2 - d^2} - \frac{1}{d - c} \)
19. \( \frac{5 - 4k}{8} - \frac{2 - 3k}{6} \)
20. \( \frac{4 + 3k}{8} + \frac{k - 6}{12} \)
21. \( \frac{1}{6} + \frac{3x - 1}{4} - \frac{x + 2}{2} \)
22. \( \frac{2}{9} - \frac{2x - 3}{6} + \frac{x - 1}{2} \)
23. \( \frac{2x - 5}{2 - x} + \frac{x}{2x - 4} \)
24. \( \frac{3a}{2a + 6} - \frac{a - 1}{a + 3} \)
25. \( \frac{4x - 7}{x^2 - 3x + 2} - \frac{3}{x - 1} \)
26. \( \frac{6m - 13}{m^2 - 5m + 6} - \frac{5}{m - 3} \)
27. \( \frac{x}{x^2 - 25} - \frac{1}{2x + 10} \)
28. \( \frac{27}{x^2 - 81} + \frac{3}{2x + 18} \)
29. \( \frac{2a + 3b}{3a^2b} - \frac{a + 2b}{4ab^2} - \frac{1}{6ab} \)

30. \( \frac{3}{2m^2} + \frac{2m}{8m^2n} - \frac{3m + n}{4mn^2} \)
31. \( \frac{2}{x - 1} - \frac{3}{1 + x} - \frac{x - 5}{1 - x^2} \)
32. \( \frac{3}{2 + n} + \frac{2}{n - 2} + \frac{5n - 2}{4 - n^2} \)
33. \( \frac{2}{3 + y} + \frac{5}{y^2 - 9} + \frac{2y - 1}{3 - y} \)
34. \( \frac{3}{z^2 - 25} - \frac{1}{5 + z} - \frac{z + 1}{5 - z} \)

8–10 Mixed Expressions

A mixed numeral like \( 2\frac{3}{8} \) denotes the sum of an integer and a fraction. When you transform it into an improper fraction, you write the integer as a fraction with denominator 1 and add two fractions with unequal denominators.

\[ 2\frac{3}{8} = \frac{2}{1} + \frac{3}{8} = \frac{16}{8} + \frac{3}{8} = \frac{19}{8} \]

The sum or difference of a polynomial and a fraction is called a mixed expression. A mixed expression can be written as a single fraction:

\[ x + \frac{3}{x} = \frac{x}{1} + \frac{3}{x} = \frac{x^2}{x} + \frac{3}{x} = \frac{x^2 + 3}{x} \]

\[ 2 - \frac{a - b}{a + b} = \frac{2}{1} - \frac{a - b}{a + b} = \frac{2(a + b)}{a + b} - \frac{a - b}{a + b} = \frac{a + 3b}{a + b} \]

If the numerator is a polynomial, you can change a fraction to a mixed expression by doing the division:
WORKING WITH FRACTIONS

\[
\frac{3y^2 - 2}{3y} = y - \frac{2}{3y}
\]

\[
\frac{2c^2 + 5c - 6}{c - 1} = 2c + 7 + \frac{1}{c - 1}
\]

**ORAL EXERCISES**

Read each expression as a single fraction.

1. \(2\frac{3}{5}\)  
2. \(5\frac{1}{6}\)  
3. \(-3\frac{3}{8}\)

4. \(-7\frac{6}{8}\)  
5. \(a + \frac{3}{b}\)  
6. \(4 + \frac{x}{y}\)

7. \(x - \frac{3}{2x}\)  
8. \(y + \frac{6}{7y}\)  
9. \(3 - \frac{s}{t + 2}\)

10. \(1 + \frac{n}{m - 3}\)  
11. \(8 + \frac{a}{b - c}\)  
12. \(4 - \frac{xy}{x - y}\)

Read each fraction as a mixed expression.

13. \(\frac{x^2 + 2}{x}\)  
14. \(\frac{a^2 + 5}{a}\)  
15. \(\frac{b^2 + 2b + 2}{b}\)  
16. \(\frac{z^2 + 3z + 7}{z}\)

17. \(\frac{ab + b + c}{b}\)  
18. \(\frac{2y^2 + 4y + 7}{y}\)

**WRITTEN EXERCISES**

Express each mixed expression as a single fraction, in simplest form.

1. \(y + \frac{1}{y + 1}\)  
2. \(b + \frac{a}{a + b}\)  
3. \(1 + \frac{x - y}{x + y}\)

4. \(3 + \frac{a + b}{a - b}\)

5. \(a + 1 + \frac{1}{a - 1}\)  
6. \(x - 2 + \frac{4}{x + 2}\)  
7. \(\frac{2}{x - 1} + 1\)

8. \(\frac{4}{x - 2} + 1\)  
9. \(\frac{x}{y} + 2 + \frac{y}{x}\)

10. \(\frac{a}{b} + 2 + \frac{b}{a}\)  
11. \(s - 3 + \frac{5s - 6}{s - 2}\)

12. \(x + 5 - \frac{x - 20}{x - 4}\)

Change each fraction to a mixed expression.

13. \(\frac{1324}{15}\)  
14. \(\frac{2591}{13}\)  
15. \(\frac{12x^3 + 6}{3x}\)
8–11 Complex Fractions (Optional)

A complex fraction is a fraction whose numerator or denominator contains one or more fractions. Complex fractions may be changed to simple ones by two methods.

**Method I:** Multiply the numerator and denominator by the L.C.D. of all the fractions within them.

**Method II:** Express the fraction as a quotient, using the sign $\div$, and divide.

**EXAMPLE 1.**

\[
\frac{5}{6} \div \frac{7}{4}
\]

**Solution:**

Method I

\[
\frac{5}{6} \times \frac{4}{7} = \frac{5 \times 4}{6 \times 7} = \frac{20}{42} = \frac{10}{21}
\]

Method II

\[
\frac{5}{6} \div \frac{7}{4} = \frac{5 \div 7}{6 \div 4} = \frac{5}{6} \times \frac{4}{7} = \frac{20}{42} = \frac{10}{21}
\]

**EXAMPLE 2.**

\[
\frac{y - 3}{2y^2} \div \frac{3y + 1}{4y}
\]

**Solution:**

Method I

\[
\frac{y - 3}{2y^2} \times \frac{4y}{3y + 1} = \frac{2(y - 3)}{y(3y + 1)}
\]

Method II

\[
\frac{y - 3}{2y^2} \div \frac{3y + 1}{4y} = \frac{y - 3}{2y^2} \times \frac{4y}{3y + 1} = \frac{4y(y - 3)}{2y^2(3y + 1)} = \frac{2 \cdot 2y(y - 3)}{y(3y + 1)} = \frac{2(y - 3)}{y(3y + 1)}
\]
Simplify each fraction.

1. \( \frac{7}{8} \)  
2. \( \frac{16}{21} \)  
3. \( \frac{17}{16} \)  
4. \( \frac{18}{45} \)  
5. \( \frac{9c}{4ac} \)  
6. \( \frac{7s^2t}{11s} \)  
7. \( \frac{k - m}{m} \)  
8. \( \frac{x^2}{x} \)  
9. \( \frac{x^2 - 16}{x - 4} \)  
10. \( \frac{a^2 - 25}{a + 5} \)  
11. \( \frac{m}{n} \)  
12. \( \frac{3/4 + 3/8}{7/10 - 2} \)  
13. \( \frac{m + 2}{m - 2} \)  
14. \( \frac{x^2 - b^2}{x + b} \)  
15. \( \frac{m}{n} \)  
16. \( \frac{p}{t - 1} \)  
17. \( \frac{a^2 + b^2}{ab} + 2 \)  
18. \( \frac{x^2 - 9y^2}{6xy} \)  
19. \( \frac{p + q}{p^2 + q^2} - \frac{1}{q} \)  
20. \( \frac{r - s}{r^2 + s^2} - \frac{1}{r} \)  
21. \( \frac{2a + 1}{a - 1} \)  
22. \( \frac{b - 3}{b + 3} \)  
23. \( \frac{1}{2 + \frac{1}{3}} \)  
24. \( \frac{1}{a + \frac{1}{a}} \)  
25. \( \frac{1}{b + \frac{1}{b}} \)  
26. \( \frac{2 + 6}{x} \)  
27. \( \frac{4 - \frac{8}{y}}{y - 2} = \frac{1}{y} \)

Determine each solution set.

28. \( \frac{9c}{4ac} \)  
29. \( \frac{7s^2t}{11s} \)  
30. \( \frac{k - m}{m} \)  
31. \( \frac{x^2}{x} \)  
32. \( \frac{x^2 - 16}{x - 4} \)  
33. \( \frac{a^2 - 25}{a + 5} \)  
34. \( \frac{m}{n} \)  
35. \( \frac{3/4 + 3/8}{7/10 - 2} \)  
36. \( \frac{m + 2}{m - 2} \)  
37. \( \frac{x^2 - b^2}{x + b} \)  
38. \( \frac{m}{n} \)  
39. \( \frac{p}{t - 1} \)  
40. \( \frac{a^2 + b^2}{ab} + 2 \)  
41. \( \frac{x^2 - 9y^2}{6xy} \)  
42. \( \frac{p + q}{p^2 + q^2} - \frac{1}{q} \)  
43. \( \frac{r - s}{r^2 + s^2} - \frac{1}{r} \)  
44. \( \frac{2a + 1}{a - 1} \)  
45. \( \frac{b - 3}{b + 3} \)  
46. \( \frac{1}{2 + \frac{1}{3}} \)  
47. \( \frac{1}{a + \frac{1}{a}} \)  
48. \( \frac{1}{b + \frac{1}{b}} \)  
49. \( \frac{2 + 6}{x} \)  
50. \( \frac{4 - \frac{8}{y}}{y - 2} = \frac{1}{y} \)
FRACTIONS IN OPEN SENTENCES AND PROBLEMS

8-12 Open Sentences with Fractional Coefficients

Methods previously used to solve open sentences may be used also when the numerical coefficients are fractions.

EXAMPLE. \( \frac{5x}{4} + \frac{x + 3}{10} = 3 \)

Solution 1: 
\[
\frac{5x}{4} + \frac{x + 3}{10} = 3
\]
\[
20 \left( \frac{5x}{4} + \frac{x + 3}{10} \right) = 20(3)
\]
\[
20 \left( \frac{5x}{4} \right) + 20 \left( \frac{x + 3}{10} \right) = 20(3)
\]
\[
25x + 2x + 6 = 60
\]
\[\rightarrow 27x + 6 = 60\]
\[27x = 54\]
\[x = 2\]

The check is left to you.

ORAL EXERCISES

State the L.C.D., and read each open sentence after multiplying all its terms by the L.C.D.

SAMPLE. \( \frac{4p}{21} + \frac{p}{6} = 2 \)  

What you say: The L.C.D. is 42; \( 8p + 7p = 84. \)

1. \( \frac{2x}{3} + \frac{x}{5} = 3 \)
2. \( \frac{h}{2} + \frac{5h}{7} = 10 \)
3. \( \frac{s}{3} - s > \frac{7}{5} \)
4. \( \frac{1}{2}m < \frac{5}{8}m - 6 \)
WORKING WITH FRACTIONS

5. \( \frac{1}{3}n + \frac{1}{2}n = n - 2 \)
6. \( \frac{1}{4}g + \frac{1}{3}g = g + 3 \)
7. \( \frac{t}{4} - t \leq \frac{11}{12} \)
8. \( \frac{x}{3} \geq \frac{7x}{9} - 9 \)
9. \( \frac{x^2}{2} + \frac{x}{6} - \frac{1}{3} = 0 \)
10. \( \frac{u^2}{5} - \frac{u}{2} + \frac{7}{10} = 0 \)
11. \( b^2 - \frac{2b}{39} + \frac{3}{13} = 0 \)
12. \( q^2 + \frac{5q}{22} - \frac{7}{2} = 0 \)

WRITTEN EXERCISES

Solve. If the sentence is an inequality, graph its solution set.

1. \( \frac{b}{5} - \frac{b}{10} = \frac{1}{10} \)
2. \( \frac{k}{4} - \frac{k}{8} = \frac{3}{8} \)
3. \( \frac{3y}{4} + \frac{1}{4}y > \frac{7}{2} \)
4. \( \frac{1}{3}x + \frac{1}{3}x < \frac{9}{2} \)
5. \( \frac{3}{8}g - \frac{3}{8}g = 1 \)
6. \( \frac{3}{7}g - \frac{1}{3}g = 4 \)
7. \( .02c \geq .01c - .1 \)
8. \( \frac{2c}{3} + 1 \leq \frac{c}{2} \)
9. \( 3z - \frac{7}{3}z = 1 \)
10. \( .5s - 1.4s = .9 \)
11. \( \frac{n}{5} - \frac{n}{3} < \frac{6}{5} \)
12. \( \frac{x}{7} - \frac{x}{4} > \frac{6}{7} \)
13. \( \frac{4}{3}h = \frac{1}{2}h + 45 \)
14. \( \frac{3}{4}n = 26 - \frac{1}{3}n \)
15. \( \frac{3m - 5}{2} - \frac{m}{3} = 8 \)
16. \( \frac{3m}{2} + \frac{8 - 4m}{7} = 3 \)
17. \( \frac{n + 3}{2} \geq \frac{n - 8}{5} + 1 \)
18. \( \frac{n - 3}{4} \leq \frac{2n - 5}{5} + 1 \)
19. \( \frac{3}{8}(2r + 5) - \frac{5}{8}(3r - 1) = 1 \)
20. \( \frac{2}{3}(s - 3) - 3 = \frac{1}{3}(2s + 5) \)
21. \( \frac{n + 3}{8} - \frac{n - 2}{6} = 1 \)
22. \( \frac{n + 5}{12} - \frac{n + 3}{8} = 1 \)
23. \( .04x + .06(20,000 - x) = 960 \)
24. \( .03k + .05(1000 - k) = 34 \)
25. \( \frac{2w + 5}{4} - \frac{10w + 13}{8} = 2w + 1 \)
26. \( \frac{3w + 5}{3} = 3w + 2 - \frac{3w - 4}{6} \)
27. \[.15(y - 5) - .02(4y - 3) + .9 = 0\]
28. \[.08(4y + 5) - .03(2y - 3) = .36\]
29. \[\frac{2t + 3}{9} - \frac{4t - 1}{6} - \frac{9 - 8t}{18} = 0\]
30. \[\frac{1 - 6t}{10} - \frac{2t + 3}{6} - \frac{13 + 6t}{4} = 0\]

8-13 Investment Problems

The simple interest \(i\) on \(p\) dollars at the interest rate \(r\) per cent per year for \(t\) years is given by the equation \(i = prt\). Thus, the interest on $100 invested at 6\% for 2 months is

\[i = prt = 100(\frac{6}{100})^{\frac{2}{12}} = $1.\]

Investment problems may concern money invested at different rates of interest.

**EXAMPLE**

An alumni association sets up a fund to grant annual scholarships totaling at least $1800. They invest 25\% of the fund in stocks yielding 3\%, the rest in bonds at 5\%. How large is the fund?

**Solution:**

1. Let \(x\) = the number of dollars in the fund
   
   \[25\% \text{ of } x \text{ or } .25x = \text{ the number of dollars invested at } 3\%\]
   
   \[75\% \text{ of } x \text{ or } .75x = \text{ the number of dollars invested at } 5\%\]
   
   \[.03)(.25x) = \text{ yearly income from stocks}\]
   
   \[.05)(.75x) = \text{ yearly income from bonds}\]

2. \[
\begin{array}{ccc}
\text{Stock income} & + & \text{Bond income} \\
(.03)(.25x) & + & (.05)(.75x) \\
\hline
\leq & \text{least amount for scholarships} \\
\hline
\end{array}
\]

\[\leq 1800\]

3. \[
\begin{array}{ccc}
75x & + & 375x \\
450x & \geq & 18,000,000 \\
x & \geq & 40,000 \\
\end{array}
\]

4. The check is left to you.
WORKING WITH FRACTIONS

1. How much interest is earned in 2 years on $500 invested at 4% per year?
2. A man invests $600 at 5% interest per year. How much interest does he receive after 3 years?
3. John invests a sum at 4% per year. After half a year, he receives $20 interest. Find the investment.
4. An investment at 6% simple annual interest earned $360 in 5 years. Find the investment.
5. A man borrowed $1200 for 2 years at 6 per cent per year. What amount did he pay back at the end of the time? \[ a = p(1 + rt) \]
6. A man invested $750 at simple interest of 3% per year. What amount of money did he have after 4 years?
7. Mr. Collins invests a sum at 4%, and an equal amount at 6%. His return totals $40 a year. How much money is invested at each rate?
8. Mr. Samson invested half a sum at 3%, half at 4%. His total annual income was $189. How much did he invest at each rate?
9. Mr. Turner invested a sum of money at 7 per cent, and twice that sum at 3 per cent. His yearly return was $390. How much did Mr. Turner invest at each rate?
10. Mr. Martin risked a little money in an investment at 9%. He invested seven times as much money at 3%. After one year he received $750. What did he invest at each rate?
11. Mr. Paxton invested $2000 more than his wife. The income from both investments at 5% was $250 a year. How much did each invest?
12. Tom and Sally Avery each have money in postal savings, at 2% per year. Each year they use their $12 interest to buy a joint birthday present for their father. Tom’s savings total $100 more than Sally’s. How much has each?
13. A man invests $10,000, part at 4 per cent per year, and the remainder at 6 per cent. From this he receives $500 annually. How much does he invest at each rate?
14. A man invested some money at 5 per cent and $800 less at 3\(\frac{1}{2}\) per cent. He received $210 a year from these investments. How much was each?
15. The Best Mortgage Company invests $35,000 at 8% per year. How much must it invest at 10% per year for an annual income of $5000 from both investments?
16. Mrs. Able invests $7000 at 4%. Mr. Able wishes to invest enough at 6% for their combined annual income to total $1000. How much should he invest?
17. One sum is invested at 4%, and another, $500 larger than the first is invested at 5%. The interest on the second amount exceeds that on the first by $33 a year. How much is invested at each rate?

18. Part of $6000 is invested at 3\(\frac{1}{2}\)%, and the rest, at 6%. The income at 3\(\frac{1}{2}\)% exceeds the income at 6% by $67.50 per year. Find the investment at each rate.

19. Thirty per cent of a fund is invested at 5% per year. The rest is invested at 4%. How much is invested at each rate, if the total income is $860?

20. A man leaves 60% of his estate to his wife and the rest to his son. The wife invests at 5.5% per year and the son, at 4.5% per year. Find the wife’s yearly income, if the son’s is $5400.

8–14 Per Cent Mixture Problems

Another type of problem involving per cent is the per cent mixture problem.

EXAMPLE

A solution of antifreeze is said to be 60% antifreeze, if 60% of the solution is pure antifreeze and 40% is water. How much water must be added to 10 quarts of a 60% antifreeze solution to obtain a 25% antifreeze mixture?

Solution:

Let \(y\) = number of quarts of water to be added.

Amount of original solution: 10 quarts
Amount of final solution: \((10 + y)\) quarts
Per cent pure antifreeze in original solution: 60%
Per cent pure antifreeze in final solution: 25%
Amount of pure antifreeze in original solution: 60% of 10 quarts
Amount of pure antifreeze in final solution: 25% of \((10 + y)\) quarts

\[
\text{Amount of pure antifreeze in final solution} = \text{Amount of pure antifreeze in original solution}
\]

\[
.25(10 + y) = .60(10)
\]

Show that 25% of the new solution equals 60% of the original.
PROBLEMS

1. How much water must be added to 2 quarts of a disinfectant containing 30% active ingredient to form a solution containing 20% active ingredient?

2. To reduce 16 ounces of a 25% solution of antiseptic to a 10% solution, how much distilled water should a nurse add?

3. How much water must be added to a barrel containing 48 pounds of a 10% brine (salt and water) to obtain a 6% brine?

4. How much water must be evaporated from 100 pounds of a 4% brine to get a 5% brine?

5. If a dairyman has 200 pounds of milk testing 3% butterfat, how many pounds of skimmed milk must he remove to have 3.6% butterfat?

6. In certain localities, “approved milk” must contain 3.2% butterfat. How much skimmed milk may a farmer add to 200 pounds of milk testing 4.8% butterfat to obtain a milk testing 3.2% butterfat?

7. How much alcohol must be added to a pint of tincture of arnica, containing 20% arnica and 80% alcohol to reduce it to a 10% arnica solution?

8. One quart of a 10% iodine solution (90% is alcohol) can be reduced to a 2% solution by adding how much alcohol?

9. A solution contains 40 grams of sugar in 200 grams of water. How much sugar must be added to make a 50% sugar solution?

10. A brine contains 45 pounds of water and 3 pounds of salt. How much salt must be added to obtain a 10% salt solution?

11. In 14-carat gold are 14 parts, by weight, of gold and 10 parts of other metal (usually copper). Coin gold is 90% gold and 10% copper. How many ounces of pure gold must be added to 24 ounces of 14-carat gold to make an alloy of coin gold?

12. How many ounces of copper must be alloyed with 185 ounces of pure silver to produce sterling silver, which contains 92.5% pure silver?

13. A chemist has a solution of 50% pure acid and another of 80% pure acid. How many ounces of each will make 300 ounces of a solution which is 72% pure acid?

14. The capacity of an automobile cooling system is 16 quarts. If it is full of a 15% antifreeze solution, how many quarts must be replaced by a 90% solution to give 16 quarts of a 65% solution?
8–15 Fractional Equations

An equation like \( \frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6} \) which has a variable in the denominator of one or more terms, is called a fractional equation.

**Example**

Solve \( \frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6} \)

**Solution:**

\[
\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}
\]

\[
(z^2 - 36) \cdot \frac{60}{z^2 - 36} + (z^2 - 36) \cdot 1 = (z^2 - 36) \cdot \frac{5}{(z - 6)}
\]

\[
60 + z^2 - 36 = 5z + 30
\]

\[
z^2 - 5z - 6 = 0
\]

\[
(z - 6)(z + 1) = 0
\]

\[
z = 6 \quad \mid \quad z = -1
\]

When you test 6 and \(-1\) in the original equation, notice what happens:

\[
\frac{60}{z^2 - 36} + 1 = \frac{5}{z - 6}
\]

Let \(z = 6\).

\[
\frac{60}{6^2 - 36} + 1 \equiv \frac{5}{6 - 6}
\]

\[
\frac{60}{0} + 1 \equiv \frac{5}{0}
\]

The fractions \(\frac{5}{0}\) and \(\frac{5}{0}\) are meaningless, so 6 is not a root of the given equation.

Let \(z = -1\).

\[
\frac{60}{(-1)^2 - 36} + 1 \equiv \frac{5}{-1 - 6}
\]

\[
\frac{60}{-35} + 1 \equiv \frac{5}{-7}
\]

\[
-\frac{12}{7} + 1 \equiv -\frac{5}{7}
\]

\[
-\frac{5}{7} = -\frac{5}{7}\checkmark
\]

\(\therefore\) The solution set is \(\{-1\}\).
Multiplying the given equation by $z^2 - 36$ leads to an equation that is *not equivalent* to the given one. This new equation has the extra root 6 because the multiplier $z^2 - 36$ represents zero when $z$ is replaced by 6.

Multiplying an equation by a variable expression which can represent zero may produce an equation having roots not satisfying the original equation. Therefore, test each root found by substituting it in the original equation. Only values producing true statements belong to the solution set.

**WRITTEN EXERCISES**

Solve each equation.

1. \[ \frac{12}{s + 2} = \frac{4}{s - 2} \]
2. \[ \frac{7}{t - 3} = \frac{2}{t + 2} \]
3. \[ \frac{5}{x} = \frac{x - 3}{2} \]
4. \[ \frac{6}{2p - 1} = \frac{z}{2} \]
5. \[ \frac{t}{t - 4} = \frac{t + 4}{6} \]
6. \[ \frac{5}{v + 6} = \frac{v - 6}{v} \]
7. \[ \frac{3}{4x} + \frac{1}{x} = \frac{7}{8} \]
8. \[ \frac{2}{3x} + \frac{1}{x} = \frac{5}{9} \]
9. \[ \frac{n + 5}{2n} - \frac{7}{3n} = \frac{5}{12} \]
10. \[ \frac{4}{3n} - \frac{n + 4}{6n} = 2 \]
11. \[ a - \frac{2}{a - 3} = \frac{a - 1}{3 - a} \]
12. \[ c - \frac{c}{1 - c} = \frac{2 - c}{c - 1} \]
13. \[ \frac{4}{3t - 2} + \frac{7}{3t} - \frac{1}{t} = 0 \]
14. \[ \frac{3}{4k} + \frac{4}{3k - 1} - \frac{2}{k} = 0 \]
15. \[ \frac{2x}{x + 2} - 2 = \frac{x - 8}{x - 2} \]
16. \[ \frac{4y}{y - 3} - 3 = \frac{3y - 1}{y + 3} \]
17. \[ 5c - \frac{c + 1}{c + 1} + 6 = 0 \]
18. \[ 4k - \frac{k - 1}{k - 1} = 3 \]
19. \[ \frac{p + 2}{3p - 6} - \frac{2}{3p + 6} + \frac{7}{9} = 0 \]
20. \[ \frac{3n - 2}{15} - \frac{16 - 3n}{n + 6} = \frac{n + 3}{5} \]
22. \[ \frac{2}{3m + 12} - \frac{1}{9m - 3} - \frac{m - 2}{3m^2 + 11m - 4} = 0 \]

23. \[ \frac{2t + 1}{2t - 3} - 1 = \frac{t - 4}{2t + 3} - \frac{7t}{9 - 4t^2} \]

24. \[ \frac{2t + 3}{t - 1} - 4 - \frac{2}{t + 3} = \frac{5 - 6t}{t^2 + 2t - 3} \]

25. \[ \frac{1}{3s - 2} + \frac{1}{3s + 4} = \frac{6}{9s^2 + 6s - 8} \]

26. \[ \frac{1}{2d + 3} + \frac{1}{2d + 1} = \frac{2}{4d^2 + 8d + 3} \]

8-16 Work Problems

An equation for work is \( w = rt \), where \( w \) is the amount of work done, \( r \) is the rate of doing work, and \( t \) is the time worked. If several persons work together, the amount of work done is assumed to be the sum of the individual amounts.

**EXAMPLE 1** A deck of punched cards can be read by one electronic reader in 20 minutes. A second can read the same deck in 12 minutes. In how much time could both readers together process the cards?

**Solution:**

Let \( x = \) number of minutes needed for the readers processing the cards together.

\[ \frac{1}{20} = \text{rate of first reader (one-twentieth of the job in one minute)} \]

\[ \frac{1}{12} = \text{rate of second reader (one-twelfth of the job in one minute)} \]

\[ 1 = \text{work done (one whole job)} \]

Work done by both = work done by first plus work done by second.

\[ 1 = \left( \frac{x}{20} \right) + \left( \frac{x}{12} \right) \]

\[ (60)1 = (60) \left( \frac{x}{20} + \frac{x}{12} \right) \]

\[ 60 = 3x + 5x \]

\[ 60 = 8x \]

\[ \frac{15}{2} = x \]
Check: In \( \frac{15}{2} \) minutes, the first reader processes \( \frac{1}{2} \times \frac{15}{20} \) of the deck.

In \( \frac{15}{2} \) minutes, the second reader processes \( \frac{1}{2} \times \frac{15}{12} \) of the deck.

Does \( \frac{1}{2} \times \frac{15}{20} + \frac{1}{2} \times \frac{15}{12} = 1 \) (the job)? \( \frac{3}{8} + \frac{5}{8} = 1 \checkmark \)

It takes \( 7\frac{1}{2} \) minutes when both readers are used, Answer.

Sometimes those doing a job may not work for the same time. In such cases, the values substituted for time in the equation may differ.

**EXAMPLE 2**

How long would it take to process the deck of punched cards of Example 1 if the first reader stopped after working 5 minutes?

**Solution:**

Let \( t \) = number of minutes required to process deck (time second reader operated).

\( 5 \) = number of minutes first reader operated

\[ 1 = \frac{1}{20}(5) + \frac{1}{12}t \]

Steps 3 and 4 are left to you.

**PROBLEMS**

1. One pipe can fill a tank in 5 hours. A second can fill it in 3 hours. How long will it take both pipes together to fill the tank?

2. A press can print one day’s newspapers in 4 hours. A high-speed press can do the job in half that time. How fast can both presses together do the job?

3. A payroll is prepared by two computers in 6 hours. The faster computer can do the job itself in 10 hours. In what time can the slower computer do the job?

4. An air conditioner lowers the temperature 10 degrees in 12 minutes. With a second air conditioner also working, this change takes 4 minutes. How long would the second device need to produce that change?
5. One lathe can ream holes in a shipment of metal parts in 3 hours; a second also can do it in 3 hours, but a third needs 4 hours. How long will the job take, if all three lathes are used?

6. Each of two incinerators can process a day’s refuse in 20 hours. Together with a third incinerator, they process the refuse in 6 hours. In what time can the third incinerator do the job?

7. One bulldozer clears land twice as fast as another. Together they clear a large tract in 1 1/2 hours. How long would the larger bulldozer take?

8. One pump fills a tank twice as fast as another. If together they fill the tank in 16 minutes, how long does the larger pump take?

9. A swimming pool has two inlet pipes. One fills the pool in 3 hours, the other, in 6 hours. The outlet pipe empties the pool in 4 hours. Once the outlet pipe was left open when the pool was being filled. In how many hours was the pool full?

10. The hot water faucet fills a tub in 40 minutes, and the cold water faucet, in 30 minutes. The tub can be drained in 20 minutes. If both faucets are open while the drain is open, how soon will the tub be full?

11. It takes Mr. Shea 8 hours to paint his barn. If he works 1 hour and then asks a painter to help him, they finish in 3 more hours. In what time can the painter do the whole job?

12. One machine labels 1200 cans in one hour, and a second labels 900 cans in an hour. If the faster machine starts 1 hour before the slower, how long will it take to label 8200 cans?

13. A job can be done by 8 men in 3 hours, or by 15 boys in 5 hours. How long would it take 3 men and 25 boys together?

14. A man contracts to build a road in 72 days, a job requiring 60 men. The man hires 50 men who work for a while until he realizes that he must hire 30 more to finish on time. How many days do these 30 men work?

15. Together, a man and his two sons assemble an electric train in 12 minutes. The job would take the man alone 10 minutes less than it would take either son. How soon could both sons, together, set up the train?

16. Together, three men paint a barn in 6 hours. Alone, the first man takes twice as long as the second, and the second takes 6 hours longer than the third. In how many hours can the slowest man paint the barn?

8-17 Motion Problems

You can solve certain motion problems by using fractional equations.
EXAMPLE

An airplane that is flying 600 miles per hour in calm air can cover 2520 miles with the wind in the same time that it can cover 2280 miles against the wind. Find the speed of the wind.

Solution:

Let \( s \) = the speed, or rate, of the wind in miles per hour.

Since the plane's speed in calm air is 600 miles per hour,

\[
\begin{align*}
600 + s &= \text{speed with the wind.} & \text{Distance with wind} &= 2520 \text{ miles.} \\
600 - s &= \text{speed against the wind.} & \text{Distance against wind} &= 2280 \text{ miles.}
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>( r )</th>
<th>( t )</th>
<th>( d )</th>
<th>( r )</th>
<th>( t )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>With wind</td>
<td>600 + s</td>
<td>( \frac{2520}{600 + s} )</td>
<td>2520</td>
<td>time with the wind</td>
<td>( \frac{2520}{600 + s} )</td>
<td>time against the wind</td>
</tr>
<tr>
<td>Against wind</td>
<td>600 - s</td>
<td>( \frac{2280}{600 - s} )</td>
<td>2280</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
(600 + s)(600 - s) \cdot \frac{2520}{600 + s} = (600 + s)(600 - s) \cdot \frac{2280}{600 - s}
\]

\[
2520(600 - s) = 2280(600 + s)
\]

\[
\therefore 63(600 - s) = 57(600 + s)
\]

\[
63(600) = 57(600) + 57s
\]

\[
6 \cdot 600 = 120s
\]

\[
30 = s
\]

Is the rate of the wind 30 miles per hour?

The speed of the plane with the wind: \( 600 + 30 = 630 \) miles per hour.

\( \therefore \) with the wind it travels 2520 miles in \( \frac{2520}{630} \), or 4 hours.

The speed of the plane against the wind: \( 600 - 30 = 570 \) miles per hour.

Against the wind it travels 2280 miles in \( \frac{2280}{570} \), or 4 hours. \( \checkmark \)

\( \therefore \) the speed of the wind is 30 miles per hour, Answer.
A

1. Jim Black rows 9 miles downstream in the same time that he rows 3 miles upstream. The current flows at 6 miles per hour. How fast does Jim row in still water?

2. A motorboat goes 25 miles downstream in the time it goes 15 miles upstream. The current flows at 5 miles per hour. What is the boat's rate in still water?

3. A man rows 4 miles an hour in still water. He takes as long rowing 4 miles upstream as 12 miles downstream. How fast is the current?

4. A plane can fly 180 miles per hour in calm air. It can travel 800 miles with the wind in the same time as it can travel 640 miles against the wind. Find the speed of the wind.

5. A river steamer travels 36 miles downstream in the same time that it travels 24 miles upstream. Its engines drive it in still water at 12 miles an hour more than the rate of the current. Find the rate of the current.

6. In still water Jim's outboard motor drives his boat 4 times as fast as the current in Pony River. He takes a 15-mile trip up the river and returns in 4 hours. Find the rate of the current.

7. Chris can paddle his canoe at 5 miles per hour on Loon Lake. In Fallow Run, he paddles 4 miles upstream and returns. The upstream trip takes 4 times as long as the downstream trip. How fast is Fallow Run?

8. Leo swims at 2 miles per hour in still water. After he swims down a river for a quarter of a mile, returning takes three times as long as swimming downstream. Find the rate of the current.

9. A sound made at one end of a 8250-foot railroad rail reaches an observer at the other end 7 seconds before he hears it in air. Find the speed of sound in the rail, if sound travels 15 times faster in the metal than in air.

10. The total resistance $R$ in an electrical circuit consisting of two resistances of $a$ ohms and $b$ ohms connected in parallel is given by the equation $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$. Find the larger of the two parallel resistances if it is 2 ohms more than the smaller, and if the total resistance is two-thirds of the smaller resistance.

11. A boat takes 1 hour longer to sail 36 miles up a river than to return. If the river flows at 3 m.p.h., find the speed of the boat in still water.

12. A man rows 1 mile up a river in order to board a motorboat which takes him 10 miles down the river. The man rows at 4 m.p.h., and the
motorboat makes 12 m.p.h. If the trip takes 1 hour and 40 minutes, find the speed of the current.

13. On a 6400-kilometer rocket test range, one rocket takes 8 minutes longer than a second, which travels 40 kilometers a minute faster. Find the speed of the second rocket.

14. A machine folds and closes 5500 cartons. A second machine does the same job in 10 minutes less, processing 5 cartons more each minute. How many cartons does the second machine handle in a minute?

Just for Fun

Fractured Fractions

With certain fractions, you can obtain interesting number patterns by breaking the numerator into two equal factors and at the same time breaking the denominator into an indicated sum. One of these fractions has the value 121, or \((11)^2\):

\[
121 = \frac{484}{4} = \frac{22 \cdot 22}{1 + 2 + 1}.
\]

Notice the numerator of the “fractured” fraction. Each factor is made up of two \textit{twos}. Two is the middle digit in 121, the value of the fraction. Now look at the denominator. It contains the same digits as the number 121, but they are separated by plus signs.

Another fractured fraction is shown below. Its value is 12,321, or \((111)^2\):

\[
12,321 = \frac{333 \cdot 333}{1 + 2 + 3 + 2 + 1}.
\]

Each factor in its numerator is made up of three \textit{threes}: 3 is the middle digit in the number 12,321. The denominator is the indicated sum of the digits in 12,321.

What is the value of this fraction?

\[
1 + 2 + 3 + 4 + 3 + 2 + 1.
\]

Can you write fractured fractions having the following values?

123,454,321
12,345,654,321
1,234,567,654,321
123,456,787,654,321
12,345,678,987,654,321
Home Economics and Mathematics

Home economics is a deceptively simple title for a field including subjects as varied as nutrition, fashion, interior design, and child psychology. Planning a budget, calculating the mortgage on a home, determining how much can be saved by buying a product in a "king-size" container as opposed to the "regular" size, calculating the amount of carpet needed to cover a particular floor area — these are some of the mathematical problems frequently encountered by homemakers as well as professional home economists.

The technician in the photograph is applying her training in home economics to the testing of new food products at the experimental kitchens of a leading food company. She is using a paddle mechanism to determine the gel strength, after different setting intervals, of fruit preserves made with a new jelling agent. The meter in the foreground is used for determining the relative acidity of fruit products. In these experiments as in other aspects of home economics, a knowledge of mathematics is a definite asset.

The work pad illustrates one application of mathematics to the field of dietetics. A hospital dietician must plan a meal for a patient whose doctor has prescribed a diet with a 7:4 protein-to-carbohydrate ratio. The dietician’s computations on a tentative menu show that she can arrive at the desired ratio by (1) adding about 44 grams of protein to the meal planned or (2) omitting the potato. Omitting the potato would satisfy the 7:4 ratio without adding an excess amount of food to the meal total and is the preferred alternative.
Chapter Summary

Inventory of Structure and Method

1. The multiplication property of fractions: For each $a$, each $b$, and each $c$, other than zero, $\frac{ac}{bc} = \frac{a}{b}$. This property permits you to reduce a fraction to lowest terms by dividing its numerator and denominator by the same common nonzero factor.

2. The property of quotients $\frac{xy}{ab} = \frac{x}{a} \cdot \frac{y}{b}$ gives the rule for multiplying fractions: The product is the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the fractions. The product should be expressed in lowest terms. By first factoring numerators and denominators, you may be able to reduce the product of fractions. To divide fractions, multiply the dividend by the reciprocal of the divisor; thus, $\frac{ab}{c} \div \frac{ac}{b} = \frac{ab}{c} \cdot \frac{b}{ac} = \frac{b^2}{c^2}$.

3. To find the lowest common denominator of several fractions, factor each denominator, and find the product of these different factors, each taken the greatest number of times it appears in any one denominator. To add and subtract fractions, use the multiplication property to replace each fraction by one equal to it and having as its denominator the lowest common denominator of the given fractions. The sum of fractions with equal denominators is the fraction whose numerator is the sum of the numerators and whose denominator is the common denominator of the fractions.

4. To change an algebraic fraction to a mixed expression, divide the numerator by the denominator. To change a mixed expression to a fraction, write the polynomial as a fraction whose denominator is 1, and add this fraction to the fractional part of the expression.

5. Equations whose numerical coefficients are fractions, and fractional equations having the variable in the denominator of a fraction often occur in the solution of investment problems, per cent mixture problems, work problems, and motion problems. You simplify equations by multiplying each member of the lowest common denominator of the terms of the equation. Whenever the L.C.D. may represent zero, the roots of the resulting equation may not satisfy the original equation.

6. A ratio is used to compare like measurements or to express a rate.

Vocabulary and Spelling

- algebraic fraction (p. 281)
- reducing a fraction (p. 283)
- multiplication property of fractions (p. 283)
- fraction in lowest terms (p. 283)
322
CHAPTER EIGHT
ratio (p. 286)
per cent (p. 289)
percentage (p. 290)
base (p. 290)
rate (p. 290)

lowest common denominator
(L.C.D.) (p. 299)
mixed expression (p. 302)
complex fraction (p. 304)
fractional equation (p. 312)

Chapter Test

8-1 1. For what values of the variable is \( \frac{v^2}{v^2 - 16} \) not defined?

8-2 Express in lowest terms.

2. \( \frac{21r - 35s}{7r + 14s} \)

3. \( \frac{z^2 - 9}{z^2 + 2z - 15} \)

4. \( \frac{2 - t}{t^2 - 8t + 12} \)

8-3 Express the ratio in lowest terms.

5. 54 volts to 81 volts

6. How long are two parts of a 16-meter line in the ratio 3 to 5?

8-4 7. At what discount was a $3.75 item bought for $3.30?

Express in lowest terms.

8-5 8. \( \frac{5p \cdot 2p}{6q \cdot qt^2} \)

9. \( \frac{a^2 - 25}{8ab} \cdot \frac{2b}{a + 5} \)

10. \( \frac{m^2 - 2mn + n^2}{m + n} \cdot \frac{m + n}{m^2 - mn} \)

8-6 11. \( \frac{1}{2y^2 - 50} + \frac{y + 5}{2y - 10} \)

12. \( \frac{6a^2b^2}{a^2 - b^2} + \frac{3a^2b^2}{a^2 - 2ab + b^2} \)

8-7 13. \( \frac{c + d}{c - 3d} \cdot \frac{c^2 - 2cd - 3d^2}{c - d} \cdot \frac{c^2 + 2cd - 3d^2}{c + 3d} \)

8-8 14. \( \frac{4y}{y + 3} + \frac{4}{y + 3} - \frac{y - 5}{y + 3} \)

8-9 15. \( \frac{m + p}{3mp} - \frac{2m + 3}{6m^2} \)

17. \( \frac{x + 1}{x - 1} - \frac{3x^2 - 1}{1 - x^2} \)

16. \( \frac{h + 2}{h + 3} + \frac{h + 3}{h + 2} \)

8-10 18. Write as a single fraction: \( w + v - \frac{wv + v^2}{2w + v} \)

19. Change to a mixed expression: \( \frac{4x^2 + 8x + 5}{4x} \)
WORKING WITH FRACTIONS

8-11 20. (Optional) Simplify.

\[
\begin{align*}
\text{a. } & \frac{2x - 2a}{x^2 + 2ax + a^2} \\
\text{b. } & \frac{x - a}{x + a} \\
\text{c. } & \frac{5 - \frac{n}{k}}{25 - \frac{n^2}{k^2}}
\end{align*}
\]

8-12 21. Graph the solution set of the inequality \( \frac{x - 1}{3} \leq \frac{x}{2} + 1 \).

8-13 22. Investments at 3% and 5% total $1000. What sum at 3% yields $6 more income than the other?

8-14 23. How many pounds of salt would change 75 pounds of a 4% salt solution to a 10% solution?

8-15 24. Solve:
\[ \frac{x^2 - 7}{x^2 - 16} - \frac{3 - x}{4 - x} = 0 \]

8-16 25. Henry can chop a bin of wood in 24 minutes. How fast can George do the job if the two boys together do it in 15 minutes?

8-17 26. Tom’s boat goes 48 miles with a current of 2 m.p.h. in the time it goes 32 miles upstream. How fast is the boat in still water?

Chapter Review

8-1 Defining Algebraic Fractions

1. A fraction may not have a denominator whose value is \( \frac{5}{2} \).

For which values of the variables are Exercises 2-5 not defined?

2. \( \frac{1}{y^2 - 9} \)

3. \( \frac{x + 3}{x - 2} \)

4. \( \frac{ut}{x} \)

5. \( \frac{a^2 - b^2 + c^2}{a^2 - 2ab + b^2} \)

8-2 Reducing Fractions

Reduce Exercises 6-8 to lowest terms.

6. \( \frac{4x^2y}{6xy^2} \)

7. \( \frac{a - b}{a^2 - b^2} \)

8. \( \frac{4x + 3y}{4xy} \)

9. The term \( 2 - a \) is the product of \(-1\) and \( \frac{1}{2} \).

10. \( \frac{2 - a}{a - 2} = \) ?

11. \( \frac{x^2 - 2x + 1}{1 - x} = \) ?

12. \( \frac{4 - y^2}{2y^2 - y - 6} = \frac{y}{2y + 3} \), if \( y \notin \{ \_ , \_ \} \)
CHAPTER EIGHT  

8-3 Ratio  

13. Similar quantities to be compared by ratios are expressed in the ___. ___.

Express each ratio in lowest terms.
14. 18 liters to 6 liters  
15. 50 centimeters to 3 meters  
16. The area of an eight-inch square to that of a rectangle 1 foot by 6 inches

8-4 Per Cent and Percentage Problems  

17. 17% = ___  
18. ___ = 25%  
19. ___ = 25%  

20. How much is a tax which equals 28% of an $8000 income?  
21. What per cent of a 500-mile trip did a boy drive if his father drove 259 miles?

8-5 Multiplying Fractions  

Find the products.
22. \( \frac{x \cdot u}{t \cdot v} \)  
23. \( \frac{2atx}{3bz} \cdot \frac{5a^2xz}{7bt^3} \)  
24. \( \frac{5}{m + n} \cdot \frac{5}{m - n} \)

8-6 Dividing Fractions  

25. \( \frac{r + t}{r - t} \cdot \frac{r - t}{t} \)  
26. \( \frac{3z - 2}{2z + 3} \cdot \frac{2z + 3}{3z + 2} \)  
27. \( \frac{a^2 + 2a + 1}{a^2 - 2a + 1} \cdot \frac{2a - 2}{5a + 5} \)

8-7 Fractions Involving Multiplication and Division  

31. Simplify: \( \frac{r + 3k}{3} \cdot \frac{r - 3k}{3} \div \frac{3r + k}{k} \)

8-8 Combining Fractions with Equal Denominators  

Combine in Exercises 32–34.
32. \( \frac{9s}{4t} + \frac{3s}{4t} \)  
33. \( \frac{6g}{2g + 1} + \frac{3}{2g + 1} \)  
34. \( \frac{b^2 + 1}{b^2 - 1} - \frac{2b}{b^2 - 1} \)

8-9 Adding Fractions with Unequal Denominators  

Combine the fractions.
35. \( \frac{4}{2a + b} - \frac{2}{a + 2b} \)  
36. \( \frac{s^2 + rs}{r^2 - s^2} + \frac{r}{r - s} \)
37. \[ \frac{x^2 - 1}{3x + 3} - \frac{x^2 - 1}{4x + 4} \]

38. \[ \frac{2n - 3}{n - 5} + \frac{2}{5 - n} \]

39. \[ \frac{a}{a + 2} + \frac{6}{a - 2} - \frac{8}{a^2 - 4} \]

40. \[ \frac{y + 1}{y + 2} - \frac{2}{y + 4} + \frac{4}{y^2 + 6y + 8} \]

8-10 Mixed Expressions

Change to a single fraction.

41. \[ a^2 - \frac{a^3 - 1}{a} \]

42. \[ x + y - \frac{2xy}{x + y} \]

Change to mixed expressions:

43. \[ \frac{8m^2 - 16m + 2}{8m} \]

44. \[ \frac{w^2 + 5w + 7}{w + 2} \]

8-11 Complex Fractions (Optional)

Simplify each fraction.

45. \[ \frac{n}{x} - \frac{n^2}{x^2} \]

46. \[ \frac{9 - \frac{a^2}{h^2}}{3 + \frac{a}{h}} \]

8-12 Open Sentences with Fractional Coefficients

Find the solution sets.

47. \[ \frac{z}{3} + \frac{z}{5} = 45 - \frac{z}{15} \]

48. \[ \frac{3}{2} n \geq \frac{4}{3} n + 9 \]

8-13 Investment Problems

49. Mr. Keith invested half his money at 3\% and half at 5\% for an annual income of $120. How much is each investment?

50. Mrs. Willis invested $2500 at 4\%. How much must she invest at 5\% to make her income from both investments $325?

51. A man invested some money at 5\% and twice as much at 6\%. If his annual income from both was $289, find each amount.

52. Mr. Clety invests part of $7000 at 3\% and the remainder at 7\% for a total annual yield of $350. Find each investment.

8-14 Per Cent Mixture Problems

Items 53–57 refer to this problem: How many pounds of water evaporate from 100 pounds of 3\% salt brine to make a 4\% salt solution?

53. The number of pounds of brine is originally _____.

pages 302–304

pages 304–305

pages 306–308

pages 308–310

pages 310–311
54. Originally, __?__ per cent of the brine is salt.

55. There are __?__ pounds of salt in the original solution, and __?__ pounds of salt in the final solution.

56. The per cent of salt in the final solution is __?__.

57. If x pounds of water evaporate, __?__ pounds of brine remain.

58. A science teacher diluted 100 milliliters of pure acid to make a 20% acid solution. How much water did he add?

8–15 Fractional Equations

Pages 312–314

Solve each equation.

59. \[ \frac{3}{m} + \frac{2m - 3}{6m} - \frac{m + 1}{2m} = 0. \]

60. \[ \frac{n + 5}{4 - n^2} + \frac{3}{n - 2} = \frac{1}{n + 2} \]

8–16 Work Problems

Pages 314–316

Items 61–63 refer to this situation: In 24 hours two machines do a job which the faster machine can do alone in 40 hours.

61. The faster machine can produce __?__ of the order in 1 hour, and its rate is __?__.

62. The second machine completes the order in x hours at the rate of __?__.

63. When the machines work together, the first machine produces __?__, and the second, __?__ of the order.

64. Joe can wash his car in 30 minutes. If Arno helps, they do the job in 18 minutes. How fast can Arno wash the car alone?

8–17 Motion Problems

Pages 316–319

For 65–66 refer: The current of Rock River flows at 4 miles per hour. John rows 1 mile upstream in the time he rows 5 miles downstream.

65. John's rate in still water is s. His rate downstream is __?__, and his rate upstream is __?__.

66. In terms of s, the time John travels downstream is __?__.

67. A pilot flies 600 miles with a tail wind of 25 m.p.h. Against the same wind, he flies only 450 miles in the same time. Find his rate in still air.

Cumulative Review: Chapters 1–8

In each case select the correct answer.

1. If \( R = \{0, 1, 4, 9, \ldots\} \) is the set of squares of integers, then \( R \) is closed under (addition) (multiplication) (neither).
2. In the set of directed numbers \(|y| < 3\) is equivalent to 
\((y < -3)\) \((y < 3)\) \((-3 < y < 3)\).

3. The solution set of \(\frac{(r - 2)(r + 5)}{r - 2} = 1\) is \((-4)\) \((-5)\) \((2, -4)\).

4. If \(a + 1 = b\), then \((a > b)\) \((a < 2b)\) \((a < b)\).

5. If \(c\) is a directed number, then \(-c\) is \((always)\) \((sometimes)\) \((never)\) a positive number.

6. \(-(2a - k) = (2a + k)\) \((-2a - k)\) \((-2a + k)\).

7. \(\frac{(10r)(5r^2)}{5r} = (2r)\) \((10r^2)\) \((2r^2)\).

8. The set of values of \(x\) for which \(\frac{5(x - 3)}{(x^2 + x - 12)}\) is undefined is 
\((-4)\) \((-3, -4)\) \((\emptyset)\).

9. If \(ax - a > 2a\) and \(x - 1 < 2\), then \((a = 1)\) \((a < 0)\) \((a > 0)\).

Draw a conclusion about the value of \(t\) from each statement.

10. \(2t = 0\) 11. \(5s(t - 3) \neq 0\) 12. \(6s^2t^3 < 0\) 13. \(\frac{t + 1}{t + 1} = 1\)

Which statements are true and which are false? Justify your answers.

14. For every number \(m\), \(\frac{34 - m}{17} = 2 - m\).

15. For each number \(x\), \(5(6x^2 + x) - 2(4x^2 - 3x) = 22x^2 - x\).

16. \(y + 2\) is a factor of \(y^4 + 7y^3 + 10y^2\).

17. If \(A\) and \(B\) are polynomials of degree 3, then \(A + B\) is a polynomial of degree 3.

18. The solution set of \(2k^2 + k = 1\) is a subset of the set of integers.

Complete each statement.

19. In factored form the L.C.D. of \(\frac{2}{2 - y}\) \(\frac{8}{y^2 - 4}\) \(\frac{4}{(2 - y)^2}\) is ___.

20. If \(z \notin \{-3, 3\}\), \(\frac{27(z - 3)^2}{z^2 - 9} = ___.

21. The ratio of acid to the total volume of a solution containing 12 cc. of acid in 16 cc. of water is ___.

22. If \(x - 2\) is a factor of \(x^3 - 8\), another factor is ___.

23. The ratio of \(\frac{7xy}{11ab}\) to \(\frac{11a^2}{7x}\) is ___.

24. The numbers which equal their reciprocals are ____ and ___.
Find the solution set of each open sentence.

25. \( \frac{2(n - 3)}{7} - \frac{2 - n}{3} = \frac{3n - 2}{21} \)

26. \( \frac{3x - 4}{x - 4} - \frac{x + 4}{x + 1} = 2 \)

27. \( 1 + \frac{2}{t - 3} = \frac{3}{(t - 3)^2} \)

28. \( \frac{1}{y} - \frac{1}{y + 1} = \frac{1}{y(y + 1)} \)

Express each polynomial as a constant multiplied by a product of irreducible polynomials with integral coefficients.

29. \( 7k^4 - 63k^2 \)

30. \( 60 + 7h - h^2 \)

31. \( 12cr - 8cs - 15dr + 10ds \)

32. \( 4z^3t - 20z^2t^2 + 25zt^3 \)

33. \( 49(n - k)^2 - 25 \)

34. \( x^2 - \frac{6}{8}x - \frac{1}{6} \)

Express each exercise as a fraction in lowest terms.

35. \( \frac{1}{y + 2} - \left( \frac{3y}{y^2 - 4} + \frac{2y}{2 - y} \right) \)

36. \( \frac{1}{(x - 3)^2} + \frac{12}{(x + 3)^2} - \frac{8}{x^2 - 9} \)

37. \( \frac{4a^2 - 9b^2}{a^2 - 2ab + b^2} + \frac{2a^2 + 5ab + 3b^2}{a^2 - b^2} \)

38. \( \left( \frac{2g}{k} \right)^2 \cdot \frac{48k^2s}{(5tg^2)^2} = \left( \frac{6gs}{5t^2} \right)^3 \)

39. \( \frac{1}{t - v} - \frac{1}{t + v} \)

40. A dealer bought an air conditioner for $150. By selling it at a discount of 25% on the list price, he made a profit of 20% on the cost. Find the list price.

41. Two circles whose areas differ by \( 36\pi \) square cm. have radii differing in length by 3 cm. Find the length of the radius of the larger circle.

42. What quantities of silver 70% pure and 75% pure mixed together make 20 pounds of silver which is 73% pure?

43. The average of a number and its reciprocal is \( \frac{17}{5} \). Determine the smallest such number.

44. A plane traveled 1200 miles in \( 3\frac{1}{2} \) hours. Its average speed for the last 600 miles was 100 m.p.h. less than its average for the first 600. Find the average speed for the last 600 miles.

45. One pipe fills a tank in 40 minutes, and another fills it in 60 minutes. A third pipe empties the tank in 30 minutes. In what time will the tank fill if all three pipes are open?

Extra for Experts

Divisibility of Integers

Knowing whether one integer is an integral multiple of (is divisible by) another helps in factoring, reducing fractions, and checking computations.
An analysis of the general form of the decimal numeral of an integer will enable you to develop criteria of divisibility.

\[ N = a_n10^n + a_{n-1}10^{n-1} + \ldots + a_310^3 + a_210^2 + a_110 + a_0 \]

Here, the \( a's \) represent the digits in the numeral, and the powers of ten correspond to place values. If you were discussing 8,037,291, then \( a_6 = 8, a_5 = 0, a_4 = 3, \ldots a_1 = 9, a_0 = 1. \)

Many of these criteria are based on the following property of integers:

\[ \text{If } N = p + q, \text{ and } q \text{ is a multiple of } r, \text{ then } N \text{ is divisible by } r \text{ if, and only if, } p \text{ is a multiple of } r. \]

You may be acquainted with some of these criteria of divisibility.

<table>
<thead>
<tr>
<th>Divisor</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Is the last digit 0, 2, 4, 6, or 8?</td>
</tr>
<tr>
<td>3</td>
<td>Is the sum of the digits divisible by 3?</td>
</tr>
<tr>
<td>4</td>
<td>Is the integer named by the last two digits divisible by 4?</td>
</tr>
<tr>
<td>5</td>
<td>Is the last digit 0 or 5?</td>
</tr>
<tr>
<td>7</td>
<td>From the right, group the digits by threes, and mark these groups alternately positive and negative; then, total the signed groups. Is this sum divisible by 7?</td>
</tr>
<tr>
<td>9</td>
<td>Is the sum of the digits divisible by 9?</td>
</tr>
<tr>
<td>10</td>
<td>Is the last digit 0?</td>
</tr>
<tr>
<td>11</td>
<td>Mark the digits alternately positive and negative from the right; then, total the signed digits. Is this sum divisible by 11?</td>
</tr>
<tr>
<td>13</td>
<td>Compute the sum as in the test for 7. Is this sum divisible by 13?</td>
</tr>
</tbody>
</table>

**EXAMPLE 1.** Is 72,135 \( \div 9 \) an integer?

**Solution:** \[ 7 + 2 + 1 + 3 + 5 = 18; 18 \div 9 = 2. \] \[ \therefore 72,135 \div 9 \text{ is an integer, Answer.} \]

**EXAMPLE 2.** Is 386,749 \( \div 11 \) an integer?

**Solution:** \[ 9 - 4 + 7 - 6 + 8 - 3 = 11; 11 \div 11 = 1. \] \[ \therefore 386,749 \div 11 \text{ is an integer, Answer.} \]

**EXAMPLE 3.** Is 296,348,026 \( \div 13 \) an integer?

**Solution:** \[ 026 - 348 + 296 = -26; -26 \div 13 = -2. \] \[ \therefore 296,348,026 \div 13 \text{ is an integer, Answer.} \]
EXAMPLE 4. Prove the divisibility test for 3 and for 9.

**Solution:** Use the commutative property, and reverse the order of the terms in the general form of the integer $N$.

\[ N = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 \ldots + 10^{n-1}a_{n-1} + 10^n a_n \]

\[ N = a_0 + (1 + 9)a_1 + (1 + 99)a_2 \ldots + (1 + 99 \ldots 9)a_n \]

nines

\[ N = (a_0 + a_1 + \ldots + a_n) + (9a_1 + 99a_2 + \ldots + 99 \ldots 9a_n) \]

\[ N = (a_0 + a_1 + \ldots + a_n) + 9(a_1 + 11a_2 + \ldots + 11 \ldots 1a_n) \]

\[ \therefore N \text{ is divisible by 3 if, and only if, } a_0 + a_1 \ldots + a_n \text{ is divisible by 3.} \]

\[ N \text{ is divisible by 9 if, and only if, } a_0 + a_1 \ldots + a_n \text{ is divisible by 9.} \]

EXAMPLE 5. Prove the divisibility test for 4.

**Solution:**

\[ N = a_0 + 10a_1 + (2 \cdot 5)^2a_2 + (2 \cdot 5)^3a_3 \ldots + (2 \cdot 5)^n a_n \]

\[ N = a_0 + 10a_1 + 4 \cdot 25a_2 + 8 \cdot 125a_3 \ldots + 2^n \cdot 5^n a_n \]

\[ N = (a_0 + 10a_1) + 4(25a_2 + 250a_3 \ldots + 2^{n-2} \cdot 5^n a_n) \]

\[ \therefore N \text{ is divisible by 4 if, and only if, } 10a_1 + a_0 \text{ is divisible by 4.} \]

Questions

1. Test for integral quotients.
   a. 5208 ÷ 3  c. 5208 ÷ 5  e. 147,809 ÷ 7  g. 367,892 ÷ 11
   b. 5208 ÷ 4  d. 5208 ÷ 9  f. 9819 ÷ 9  h. 147,810 ÷ 13

2. Prove the rules for divisibility
   a. by 2  b. by 5  c. by 7  d. by 11  e. by 13

3. Explain why a number divisible by two different prime numbers is divisible by their product. Use this fact to devise a test for divisibility by 6.

4. Devise a rule to test for divisibility by 12, and test 1346 ÷ 12.

5. Devise a rule to test for divisibility by 25, and test 67,475 ÷ 25.

6. In this problem, the number $N$ is in its general form.
   a. Show that $N$ is divisible by 4 if $2a_1 + a_0$ is divisible by 4.
   b. Show that $N$ is divisible by 8 if $4a_2 + 2a_1 + a_0$ is divisible by 8.
"Salutation to the elephant-headed Being who infuses joy into the minds of his worshipers, ... whose feet are reverenced by the gods."

So begins a book written in India about the middle of the twelfth century. From the opening sentence you would scarcely suspect it to be a book about mathematics. But it is, and the man who wrote it, Bhaskara the Learned, was a great mathematician.

Bhaskara's most famous book is called The Lilivati. It is a great algebra book, named for the author's twelve-year-old daughter Lilivati. Bhaskara wrote it in an attempt to comfort his child in a great disappointment. The story is this:

Astrologers had found that there was only one moment in all Lilivati's life when she could be married safely, on one particular day when she was twelve years old. So Bhaskara had arranged her wedding for that day. As the favorable moment approached, the bride, adorned for the wedding, happily but anxiously watched the hour cup, which floated on a vessel of water near her. (A Hindu hour cup had a small hole in the bottom; water trickled in and caused the cup to sink at the end of the hour.) Unnoticed, a pearl fell from Lilivati's headdress into the cup, and stopped the trickle of water. The wedding party waited. But the ceremony was never performed. Before the accident was discovered, the time for the wedding had passed. Lilivati never could be married.

But Lilivati's name lives on. For her father fulfilled the promise he made to her that day: "I will write a book of your name which shall remain to the latest times; for a good name is a second life and the groundwork of eternal existence."

A page from The Lilivati, the algebra book that made a young girl famous.
Graphs

A picture is worth a thousand words. Reports on many subjects are illustrated by graphs, as you may know, but you may not be acquainted with the use of graphs in continuous recording devices.

The cardiogram, shown in the upper photo, is a vital tool in medical diagnosis because it is a record of a patient's heart action. A flight data recorder, lower left, notes the operations of vital parts of a jet aircraft. Although some devices, like the cardiograph, make ink traces on paper, the flight data recorder engraves its records in stainless steel tape with diamond styluses, assuring their survival in a crash.

Similar machine-made records report on the functioning of many parts of our industrial enterprises. Such graphs not only free men from dial watching and recording, but they can be filed as permanent records. The graphs are continuous, and they show immediately changes in variables which easily might be overlooked in long columns of figures. Designing machines to make graphs is an engineer's job; interpreting their reports is a part of the work of many people.

ORDERED PAIRS OF NUMBERS AND POINTS IN A PLANE

9–1 Open Sentences in Two Variables

Have you ever had this kind of problem? With $1.20 Nancy wishes to buy two kinds of cookies which cost 2¢ and 3¢ each.

If she buys some of each kind, many combinations are possible:

Let \( x \) = number of 2¢ cookies, \( y \) = number of 3¢ cookies. Then \( 2x \) = cost, in cents, of 2¢ cookies, and \( 3y \) = cost, in cents, of 3¢ cookies.

\[ 120 = \text{amount, in cents, available.} \]

Total cost of two purchases = total amount spent.

\[ 2x + 3y = 120 \]
Nancy may replace \( x \) and \( y \) with various nonnegative integers. The pair \((30, 20)\) makes the statement true, but the pair \((20, 30)\) makes it false. Each pair is taken only in the order written, and is called an ordered pair of numbers. The first number is the first coordinate, and the second is the second coordinate. Two ordered pairs of numbers are equal when their first coordinates are equal and their second coordinates are equal. Thus, \((1, 5) = \left(\frac{6}{5}, \frac{15}{3}\right)\) but \((1, 5) \neq (5, 1)\).

With the value of \( x \) given first, the ordered pair of numbers \((30, 20)\) is a root of the open sentence in two variables \(2x + 3y = 120\) because for this pair the sentence is true. The solution set of an open sentence in two variables is the set of all roots of the sentence. What are other roots of \(2x + 3y = 120\)?

**EXAMPLE 1.** Find the solution set of \(3x + y = 5\) when the replacement set for both \(x\) and \(y\) is \(A = \{0, 1, 2, 3, 4, 5\}\).

**Solution:**

1. Transform the given sentence into an equivalent one, solved for \(y\).

\[
3x + y = 5 \\
y = 5 - 3x
\]

2. Replace \(x\) by each member of its replacement set, in turn, and determine the corresponding value of \(y\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(5 - 3x)</th>
<th>(y = 5 - 3x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5 - 3(0)</td>
<td>5 (\in\ A)</td>
</tr>
<tr>
<td>1</td>
<td>5 - 3(1)</td>
<td>2 (\in\ A)</td>
</tr>
<tr>
<td>2</td>
<td>5 - 3(2)</td>
<td>-1 (\notin\ A)</td>
</tr>
<tr>
<td>3</td>
<td>5 - 3(3)</td>
<td>-4 (\notin\ A)</td>
</tr>
<tr>
<td>4</td>
<td>5 - 3(4)</td>
<td>-7 (\notin\ A)</td>
</tr>
<tr>
<td>5</td>
<td>5 - 3(5)</td>
<td>-10 (\notin\ A)</td>
</tr>
</tbody>
</table>

\(\therefore\) The solution set is \(\{(0, 5), (1, 2)\}\), Answer.
EXAMPLE 2. Find the solution set of \( 2y - x \geq 4 \) if \( x \in \{-2, 0, 2\} \) and \( y \in \{0, 1, 2, 3\} \).

Solution:

\[
2y - x \geq 4
\]

\[
2y \geq 4 + x
\]

\[
y \geq 2 + \frac{x}{2}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 2 + \frac{x}{2} )</th>
<th>( y \geq 2 + \frac{x}{2} )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( 2 + \frac{-2}{2} )</td>
<td>( y \geq 1 )</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>0</td>
<td>( 2 + \frac{0}{2} )</td>
<td>( y \geq 2 )</td>
<td>2, 3</td>
</tr>
<tr>
<td>2</td>
<td>( 2 + \frac{2}{2} )</td>
<td>( y \geq 3 )</td>
<td>3</td>
</tr>
</tbody>
</table>

\( \therefore \) The solution set is \( \{(-2, 1), (-2, 2), (-2, 3), (0, 2), (0, 3), (2, 3)\} \),

Answer.

ORAL EXERCISES

Is the given ordered pair of numbers a root of the open sentence? Why? Assume that the set of directed numbers is the replacement set of each variable.

SAMPLE. \( 5x - 2y = 7 \); \( (3, 11) \)

What you say: Not a solution, because \( 5 \cdot 3 - 2 \cdot 11 \neq 7 \)

1. \( x + y = 9 \); \( (5, 4) \)
2. \( 2x - y = 7 \); \( (4, 1) \)
3. \( 3x + y = 8 \); \( (4, -2) \)
4. \( x + 5y = 4 \); \( (-9, 1) \)
5. \( 2x - 3y = 8 \); \( (4, 0) \)
6. \( 7x - y > -2 \); \( (1, -9) \)
7. \( -2x + 5y \geq -7 \); \( (-1, -2) \)
8. \( -x - 4y \geq 0 \); \( (-2, -1) \)
9. \( x^2 - y = 7 \); \( (-3, 2) \)
10. \( xy - x = 2 \); \( (-3, 1) \)

Transform each open sentence into an equivalent one having \( y \) as one member.

SAMPLE. \( 2x + y = 3 \)

What you say: \( y = 3 - 2x \)

11. \( x + y \leq 1 \)
12. \( 3x + y \geq -4 \)
13. \( 5y = 6x - 10 \)
14. \( 6x - y = 0 \)
15. \( 7x - y = 0 \)
16. \( x - y = -2 \)
17. \( 5x + 2y = 8 \)
18. \( 12x + 4y = 6 \)
19. \( x - y > 2 \)
WRITTEN EXERCISES

Find all values of a and b for which these ordered pairs are equal.

**SAMPLE.** \((a + 3, 4b + 7) = (2a - 5, b + 4)\)

**Solution:**

\[
\begin{align*}
a + 3 &= 2a - 5 \quad \text{and} \quad 4b + 7 &= b + 4 \\
8 &= a \quad \quad \quad 3b &= -3 \\
\quad &= b = -1
\end{align*}
\]

**Check:**

\[
\begin{align*}
[8 + 3, 4(-1) + 7] &= [2(8) - 5, -1 + 4] \\
(11, -4 + 7) &= (16 - 5, 3) \\
(11, 3) &= (11, 3) \checkmark
\end{align*}
\]

\[
\therefore \quad a = 8, \quad b = -1, \quad \text{Answer.}
\]

**A**

1. \((a - 11, b + 8) = (12a, 3b)\)
2. \((3a, 2b - 1) = (a - 6, 3b)\)
3. \((1 - a, 3) = (5 - 7a, |b|)\)
4. \((2a + 4, b^2) = (3a + 7, -b)\)
5. \((2 - a^2, b^2) = (a, 36)\)
6. \((|a + 1|, b^3) = (2, 0)\)

Find the solution set of each sentence having \([-6, 0, 6]\) as the replacement set of \(x\) and \{directed numbers\} as that of \(y\).

7. \(y = x + 3\)
8. \(y = 4 - x\)
9. \(5x + y = 8\)
10. \(y - 3x = 7\)
11. \(7x - 3y = 30\)
12. \(3x + 2y = 12\)

Find the solution set.

13. \(-x + 6y = 10; x \in \{-10, 2\}, y \in \{\text{positive numbers}\}\)
14. \(x - 8y = 15; x \in \{-1, 15\}, y \in \{\text{negative numbers}\}\)
15. \(10 - 5y = 2x; x \in \{0, 5\}, y \in \{0\}\)
16. \(21 + 3y = 8x; x \in \{0, 3\}, y \in \{1\}\)

**B**

17. \(x + 5y = 7 - x; x \in \{-4, -1\}, y \in \{\text{integers}\}\)
18. \(1 - x = 3y + 4x; x \in \{1, 2\}, y \in \{\text{integers}\}\)
19. \(y + 1 \geq 2x; x \in \{0, 1\}, y \in \{-1, 1\}\)
20. \(y - 1 < 3x; x \in \{-1, 0\}, y \in \{-3, 0\}\)
21. \(|x| - y < 3; x \in \{-1, 3, 4\}, y \in \{0, 1, 2\}\)
22. \(|x| - y \geq 2; x \in \{-2, -1, 3\}, y \in \{0, 1, 6\}\)
Write an equation with two or more variables for each problem. Give appropriate replacement sets and solution sets.

27. If baseballs cost $1.50 apiece and bats cost $3.50 apiece, how many of each can a team purchase for $50?

28. Cupcakes cost 5 cents each, ice cream bars, 10 cents, and punch, 25 cents a quart. How can Jo spend her $7.50 for refreshments if she buys the same number of cupcakes as ice cream bars, at least 20 of each, and at least 12 quarts of punch?

9-2 Coordinates in a Plane

Each root of an open sentence in one variable is one number, which you can graph on a number line. Each root of an open sentence in two variables is a pair of numbers, which you graph on two number lines intersecting at right angles. Choose a horizontal line called the horizontal axis (x-axis) and a vertical line, the vertical axis (y-axis). Their point of intersection is the origin. Next, select a scale to make each axis a number line whose zero-point is the origin, and indicate the scale on the axes.

To locate the graph of the ordered pair \((3, -2)\) (Figure 9-1), draw a vertical line through the graph of 3 on the x-axis and a horizontal line through the graph of \(-2\) on the y-axis. The point of intersection of these lines is the graph of \((3, -2)\). Mark the point with a dot or cross; this is called plotting the point.
Positive numbers are paired with points on the $x$-axis to the right of the origin and with points on the $y$-axis above the origin (drawing above left). Negative numbers relate to points to the left of the origin on the $x$-axis and below the origin on the $y$-axis.

The surface on which the axes lie is a set of points called a plane. With every point in this plane you can associate a particular ordered pair of numbers.

From point $P$ in Figure 9–2 (above right) draw a vertical line to the $x$-axis; the coordinate of the point where it meets the axis is the abscissa (ab-sis-a) of $P$, $-4$.

Draw a horizontal line from $P$ to the $y$-axis; the coordinate of this meeting point is the ordinate (or-de-nate) of $P$, $3$.

The coordinates of $P$ always are written with the abscissa first, $(-4, 3)$.

In Figure 9–2, verify the coordinates of: $Q(-2, -2)$, $R(3, 0)$, $T(2, 4)$, and the origin $(0, 0)$.

The one-to-one correspondence between points and number pairs is called a plane coordinate system or a coordinate plane. Because of this correspondence you can think of a point as an ordered pair of numbers and can picture an ordered pair as a point.

The axes of the coordinate system divide the plane into four regions, called quadrants, numbered as shown in Figure 9–3.
ORAL EXERCISES

Give the coordinates of each numbered point.

21. The abscissa is 3.
22. The ordinate is 2.
23. The ordinate is \(-5\).
24. The abscissa is \(-3\).
25. The abscissa equals the ordinate.
26. The abscissa equals the negative of the ordinate.
27. The abscissa is \(-1\), and the ordinate is positive.
28. The abscissa is positive, and the ordinate is \(-4\).
29. The absolute value of the abscissa is 5.
30. The absolute value of the ordinate is 2.
31. What is the ordinate of every point on the x-axis?
32. What is the abscissa of every point on the y-axis?
WRITTEN EXERCISES

Plot the graph of each of the following.

A
1. (3, 4)  
2. (0, 9)  
3. (−8, 9)  
4. (−7, 6)  
5. (−4, 4)  
6. (9, 2)  
7. (12, −7)  
8. (7, −10)  
9. (−6, −10)  
10. (−11, −8)  
11. (8, 0)  
12. (−8, 0)

Exercises 13–18 list three vertices of a rectangle. Find the fourth vertex.

B
13. (0, 0), (0, −4), (6, 0)  
14. (0, 0), (−2, 0), (0, 3)  
15. (2, 2), (−3, −1), (2, −1)  
16. (3, 4), (−1, 1), (3, 1)  
17. (−8, 5), (−2, −3), (−2, 5)  
18. (−3, −3), (−7, −5), (−7, −3)

Plot three points in at least two quadrants whose coordinates are integers satisfying the given requirement.

19. The abscissa is twice the ordinate.  
20. The abscissa is two less than the ordinate.  
21. The ordinate is three more than half the abscissa.  
22. The ordinate is twice the absolute value of the abscissa.

LINEAR EQUATIONS AND STRAIGHT LINES

9–3 The Graph of a Linear Equation in Two Variables

Every root of \( y = 1 - 2x \) is an ordered pair of numbers represented by \((x, y)\). The graph of one such root, \((2, −3)\), is the point \(P\) in Figure 9–4. To find other roots of this equation, substitute values for \(x\) and obtain those for \(y\), as shown in the table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(1 - 2x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>1 − 2(−3)</td>
<td>7</td>
</tr>
<tr>
<td>−2</td>
<td>1 − 2(−2)</td>
<td>5</td>
</tr>
<tr>
<td>−1</td>
<td>1 − 2(−1)</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1 − 2(0)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1 − 2(1)</td>
<td>−1</td>
</tr>
<tr>
<td>2</td>
<td>1 − 2(2)</td>
<td>−3</td>
</tr>
<tr>
<td>3</td>
<td>1 − 2(3)</td>
<td>−5</td>
</tr>
</tbody>
</table>

\( \) Figure 9–4 \( \)
Figure 9–4 suggests that the points corresponding to these roots lie on an unending straight line. If \{directed numbers\} is the replacement set for both \(x\) and \(y\), each root of the equation gives the coordinates of a point on this line, and the coordinates of each point satisfy the given equation. Because this line is the set of all those points and only those points whose coordinates satisfy the equation, the line is called the **graph of the equation**, and \(y = 1 - 2x\) is an equation of the line.

Since its graph is a straight line, this is a **linear equation in two variables**. In a linear equation, each term is a constant or a monomial of degree 1. Thus, \(5x - 4y = 3\) is linear, but, \(\frac{1}{x} + y = 3\), \(x^2 - 4y = 3\), and \(xy = 6\) are not. Although you need plot only two points to graph a linear equation, it is good practice to plot a third point, as a check.

**EXAMPLE 1.** Graph \(2x + 3y = 12\).

**Solution:**

\[
2x + 3y = 12 \\
3y = 12 - 2x \\
y = 4 - \frac{2}{3}x
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(4 - \frac{2}{3}x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>(4 - \frac{2}{3}(-3))</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>(4 - \frac{2}{3}(0))</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>(4 - \frac{2}{3}(6))</td>
<td>0</td>
</tr>
</tbody>
</table>

**EXAMPLE 2.** Graph \(x = -3\).

**Solution:** Since the equation places no restrictions on \(y\), all those points having abscissa \(-3\) correspond to its roots, regardless of their second coordinate.

**ORAL EXERCISES**

State whether or not each equation is linear.

1. \(\frac{x}{2} - 5y = 1\)  
2. \(3x = 2y\)  
3. \(x^2 + y^2 = 9\)  
4. \(\frac{2x}{3} - \frac{y}{4} = 3\)  
5. \(xy = 6\)  
6. \(3x^2 - y = 4\)  
7. \(\frac{x}{2} = \frac{y}{8}\)  
8. \(y = 2x^2\)  
9. \(x^2 = 5x + 4\)
Solve each equation for \( y \) in terms of \( x \).

10. \( 2x - y = 6 \)  
13. \( x - y = 3 \)  
16. \( 2y - 4x = 0 \)
11. \( 2x + y = 7 \)  
14. \( x = 10y \)  
17. \( 4y - 2x = 0 \)
12. \( \frac{3x}{2} + y = 1 \)  
15. \( \frac{y}{6} = x \)  
18. \( \frac{y}{5} = 2x \)

State the relation between the ordinate and abscissa of points on the graph of each equation.

**SAMPLE.** \( y = 4x + 7 \)

*What you say:* The ordinate is seven more than four times the abscissa.

19. \( y = 3x \)  
22. \( y = 2x - 7 \)  
25. \( xy = 6 \)
20. \( y = -2x \)  
23. \( y = 3x + 4 \)  
26. \( y = x^2 - 3 \)
21. \( y = -7x \)  
24. \( 2x + 3y = 9 \)  
27. \( y = |x| - 2 \)

Tell which of the given pairs of points belong to the graph of the equation.

28. \( x - y = 9; (4, 13), (-4, -13) \)
29. \( x - 4y = 13; (3, 4), (1, -3) \)
30. \( 2y + 7x = 0; (0, 0), (-35, 10) \)
31. \( 4x - 3y - 1 = 0; (1, 1), (0, 0) \)

**WRITTEN EXERCISES**

Graph each equation.

\[ A \]
1. \( y = 3x \)  
4. \( y = -2x \)  
7. \( 2y = x \)  
10. \( 2x + y = 4 \)
2. \( y = 4x \)  
5. \( y = -3x \)  
8. \( 4y = -x \)  
11. \( 6x - 2y = 1 \)
3. \( x = 4 \)  
6. \( y = -2 \)  
9. \( 2x - y = 1 \)  
12. \( 2x + 3y = 6 \)

Find the coordinates of the point where the graph of each equation crosses (a) the \( x \)-axis and (b) the \( y \)-axis.

\[ B \]
13. \( 5x - 3y - 30 = 0 \)  
15. \( 12x - 60 = 5y \)  
17. \( 3y = 7x \)
14. \( 7y + 4x - 28 = 0 \)  
16. \( 9y - 36 = 2x \)  
18. \( y = |x| \)

Graph each pair of equations in the same coordinate plane. Name the coordinates of the point where the graphs intersect, and show by substitution that they satisfy both equations.

\[ C \]
19. \( x = y; 3x + y = 4 \)  
21. \( x + y = 4; 2x - y = 5 \)
20. \( y = -x; y - 4x = 5 \)  
22. \( x - y = 5; 7x + y = 3 \)
Graph the following equations.

23. $|x| = 2$  24. $y = |x|$  25. $|y| = 4$  26. $y = |x| - 2$

### 9–4 Slope of a Line

To describe the steepness, or grade, of a hill, you determine the vertical *rise* for every 100 feet of horizontal *run*. If a hill rises 20 feet for every 100 feet of horizontal distance, its grade is the ratio $\frac{20}{100}$, or 20%. To describe the steepness, or *slope*, of a straight line you choose two points on it, such as $P(2, 1)$ and $Q(4, 7)$ in Figure 9–5, and compute a similar quotient:

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{6}{2} = 3.
\]

Because the horizontal change in moving from $P$ to $Q$ is the change in the abscissa, the horizontal change is:

\[
\text{horizontal change} = \text{abscissa of } Q - \text{abscissa of } P = 4 - 2.
\]

The vertical change is the corresponding difference of ordinates:

\[
\text{vertical change} = \text{ordinate of } Q - \text{ordinate of } P = 7 - 1.
\]

Thus, using $m$ to denote slope, you have

\[
m = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{7 - 1}{4 - 2} = 3.
\]

To be consistent, you always move on a line from left to right. On a line with slope 3, each horizontal change of 1 unit produces a positive change of 3 units in the vertical direction. For the line joining $S$ and $T$:

\[
m = \frac{0 - 3}{4 - (-2)} = \frac{-3}{6} = -\frac{1}{2}.
\]

Check by counting boxes; from $S$ to $T$ are 6 units of horizontal change and $-3$ units of vertical change. For each positive
change of one horizontal unit, therefore, there is a negative change of half a vertical unit, a rate of change equal to $-\frac{1}{2}$.

Whenever a line falls from left to right, its slope is a negative number; when it rises from left to right, its slope is a positive number. Can the slope of a line be 0? The slope of the line joining $K(-2, -3)$ and $M(1, -3)$ in Figure 9-7 is

$$m = \frac{-3 - (-3)}{1 - (-2)} = \frac{-3 + 3}{1 + 2} = \frac{0}{3} = 0.$$  

Do you see that the slope of every horizontal line is 0?

![Figure 9-7](image)

The slope computation for Figure 9-8 is

$$\frac{5 - (-1)}{-2 - (-2)} = \frac{5 + 1}{-2 + 2} = \frac{6}{0}.$$  

Since you may not divide by 0, this line, like every vertical line, has no slope.

A basic property of a line is that its slope is constant. Thus, you may use any two of its points in computing its slope.

**ORAL EXERCISES**

Give the slope of each line in Exercises 1–8.
Do the points given in each table lie on a line? If so, tell its slope.

**SAMPLE.**

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-3</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

*What you say:* The points do not lie on a line, because equal increases in the value of $x$ do not produce equal changes in the value of $y$.

**WRITTEN EXERCISES**

Plot each pair of points, draw a straight line through them, and determine the slope of the line from the graph. Check by finding the slope algebraically.

1. $(-1, 5); (-1, -5)$
2. $(5, -1); (5, -1)$
3. $(2, 3); (4, 5)$
4. $(6, 1); (10, 3)$
5. $(1, -2); (4, -6)$
6. $(-2, 1); (2, -2)$
Through the given point, draw a line with the given slope.

**SAMPLE.** \((-2, -1); \text{slope } = -\frac{3}{5}\)

**Solution:** From \((-2, -1)\) measure 5 units to the right and 3 units down. The point reached together with the point \((-2, -1)\) determine the line.

7. \((1, 2); \text{slope } = 1\)
8. \((-2, 0); \text{slope } = \frac{3}{2}\)
9. \((3, -2); \text{slope } = -\frac{1}{3}\)
10. \((-1, -3); \text{slope } = -2\)
11. \((-3, 2); \text{slope } = 0\)
12. \((1, -4); \text{no slope}\)

Determine \(a\) so that the slope \(m\) of the line through each pair of points has the given value. Check your solution by graphing the points.

13. \((-3, 2a); (-1, 3a); m = -\frac{1}{2}\)
14. \((3, a); (-5, 5a); m = \frac{1}{2}\)
15. \((3, 5); (4, a); m = 2\)
16. \((2, 0); (7, a); m = -1\)
17. \((-2, 6); (1, 3a); m = -2\)
18. \((1, -4); (3, 2a); m = 5\)

**9–5 The Slope-Intercept Form of a Linear Equation**

The graph of \(y = 3x\) is the straight line containing the points whose coordinates are given in the table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Do you see that when the abscissas of two points on the line differ by 1 their ordinates differ by 3, the slope of the line? Notice that the line passes through the origin.

Can you guess the slope of the line whose equation is \(y = -3x\)? It is -3, because an increase of 1 in the abscissa produces a change of -3 in the ordinate. This line also contains the origin.
In general:

A line whose equation is of the form

\[ y = mx \]

has slope \( m \) and passes through the origin.

Do you see that the graph of \( y = 3x + 4 \) is a straight line of slope 3? In Figure 9-9, compare the graphs of \( y = 3x + 4 \) and \( y = 3x \). They have equal slope, but they cross the \( y \)-axis at different points. The ordinate of the point where a line crosses the \( y \)-axis is called the line’s \textit{y-intercept}. To determine the y-intercept, replace \( x \) by 0 in the equation of each line:

\[
\begin{align*}
\text{y} &= 3x \\
\text{y} &= 3 \cdot 0 \\
\text{y} &= 0 \\
\end{align*}
\]

If you write \( y = 3x \) as \( y = 3x + 0 \), you can see that the constant term is the y-intercept of each graph, \( y = 3x + 0 \) and \( y = 3x + 4 \).

An equation of the form \( y = mx + b \), called the \textit{slope-intercept form}, is the equation of a line whose slope is \( m \) and whose y-intercept is \( b \).

In describing a straight line, write its equation in the form \( y = mx + b \). Then read the values of the slope \( m \), and the y-intercept \( b \).

<table>
<thead>
<tr>
<th>Equation</th>
<th>Transforming to ( y = mx + b )</th>
<th>Describing the line</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3x - 3y = 7 )</td>
<td>( 3y = 3x - 7; y = 1x - 2\frac{1}{3} )</td>
<td>( 1 ) ( -2\frac{1}{3} )</td>
</tr>
<tr>
<td>( 4x + 2y = 0 )</td>
<td>( 2y = -4x; y = -2x + 0 )</td>
<td>(-2) 0</td>
</tr>
<tr>
<td>( 3y - 9 = 0 )</td>
<td>( 3y = 9; y = 0 \cdot x + 3 )</td>
<td>(0) 3</td>
</tr>
</tbody>
</table>
EXAMPLE. Draw the line with \( m = \frac{3}{4}, b = -2 \); then find its equation.

Solution: The \( y \)-intercept is \(-2\), so mark \((0, -2)\). As the slope is \( \frac{3}{4} \), from this point move \( 4 \) to the right and \( 3 \) up, to locate a second point on the line. Then draw a line through the points.

\[
y = mx + b
\]

\[
y = \frac{3}{4}x + (-2) \text{ or } 4y = 3x - 8, \text{ Answer.}
\]

ORAL EXERCISES

State the slope and \( y \)-intercept of each line.

1. \( y = 2x + 5 \)  
2. \( y = 3x + 4 \)  
3. \( y = -x + 10 \)  
4. \( y = -2x - 3 \)  
5. \( y + 8 = 0 \)  
6. \( y + 2 = 0 \)  
7. \( 2y = -6x + 1 \)  
8. \( 3y = -12x + 2 \)  
9. \( 7 - x = y \)  
10. \( x + y = 5 \)  
11. \( 3y - x = -9 \)  
12. \( 2x - 3y = 0 \)  
13. \( 3x + 2y = 0 \)  
14. \( x + 1 = 0 \)  
15. \( x - 3 = 0 \)

WRITTEN EXERCISES

Write a linear equation with integral coefficients and the given slope and \( y \)-intercept.

A

1. \( m = 3; b = 1 \)  
2. \( m = -2; b = 5 \)  
3. \( m = -\frac{1}{3}; b = -2 \)  
4. \( m = \frac{1}{3}; b = -3 \)  
5. \( m = -\frac{3}{4}; b = 0 \)  
6. \( m = 0; b = 7 \)  
7. \( m = 0; b = -10 \)  
8. \( m = -1; b = 2 \)  
9. \( m = 1; b = -1 \)  
10. \( m = 7; b = 0 \)

Use only the \( y \)-intercept and slope to graph each equation.

11. \( y - 3x = 4 \)  
12. \( 7x + y = 5 \)  
13. \( x + 2y - 3 = 0 \)  
14. \( x + 3y + 6 = 0 \)  
15. \( 2x - 5y = 0 \)  
16. \( 4x - 7y = 0 \)  
17. \( 15x + 3y + 4 = 0 \)  
18. \( 6y - 7x - 9 = 0 \)
**9-6 Determining the Equation of a Line**

The line in Figure 9-10 has slope $\frac{5}{2}$ and passes through the point $(-4, -3)$. The slope-intercept form of the equation of this line is

$$y = \frac{5}{2}x + b.$$

Since the point $(-4, -3)$ is on the line, its coordinates must satisfy the equation; that is,

$$-3 = \frac{5}{2}(-4) + b,$$

or $-3 = -10 + b$, or $7 = b$.

Thus, an equation of the line is

$$y = \frac{5}{2}x + 7 \text{ or } 2y = 5x + 14.$$

To determine an equation of a line containing two given points, find the slope of the line, and then find the $y$-intercept, as above. The following example illustrates the method.

**EXAMPLE.** Find an equation of the line which passes through the points whose coordinates are $(2, 4)$ and $(3, 7)$.

**Solution:**

1. Slope $= \frac{7 - 4}{3 - 2} = \frac{3}{1} = 3$

2. The slope-intercept form of the equation is

$$y = mx + b \text{ or } y = 3x + b$$

Choose one point, say $(2, 4)$. Since it lies on the line:

$$4 = 3 \cdot 2 + b \text{ or } 4 = 6 + b$$

$$\therefore -2 = b$$

3. To check, show that the coordinates of the other point $(3, 7)$ satisfy the equation:

$$y = 3x - 2$$

$$7 = 3 \cdot 3 - 2$$

$$7 = 9 - 2$$

$$7 = 7 \checkmark$$

$\therefore$ An equation of the line is: $y = 3x - 2$, Answer.
Find an equation of the line through the given point, having the given slope.

1. \((-3, -2); 3\)
2. \((-1, 0); 5\)
3. \((2, 8); -2\)
4. \((-5, 0); \frac{2}{3}\)
5. \((-2, -5); \frac{7}{2}\)
6. \((0, 0); -\frac{8}{3}\)
7. \((0, 0); -\frac{3}{8}\)
8. \((1, -5); 0\)
9. \((-2, 3); 0\)

Find an equation of the line through the given points.

10. \((2, 3); (5, 6)\)
11. \((1, 4); (3, 6)\)
12. \((0, 1); (0, 0)\)
13. \((0, 0); (0, -1)\)
14. \((0, 0); (-1, 2)\)
15. \((5, -3); (0, 0)\)
16. \((-4, -4); (-2, 1)\)
17. \((-5, 1); (0, -2)\)
18. \((0, -1); (4, -3)\)

Determine \(a\)'s value so that the equation's graph passes through the given point.

19. \(ax + 3y = 5; (1, 4)\)
20. \(2x + ay = 4; (3, 1)\)
21. \(5x - 2y + a = 0; (-1, 3)\)
22. \(4x - ay = 2; (-3, -2)\)

Find an equation of the line parallel to the given line, through the given point.

23. \(x + y = 5; (3, 4)\)
24. \(x - y = 3; (-2, 1)\)
25. \(x + 2y = 6; (2, -3)\)
26. \(x - 3y = 6; (-6, -2)\)

Determine the coordinates of the point where the lines intersect.

27. \(5x + 7y - 35 = 0; \) the \(x\)-axis
28. \(7y - 2x + 14 = 0; \) the \(x\)-axis
29. \(6x - 5y - 3 = 0; x = 3\)
30. \(3x - 8y + 27 = 0; x = -1\)

INEQUALITIES AND SPECIAL GRAPHS

9-7 Graph of an Inequality in Two Variables

In Figure 9-11 the graph (line \(l\)) of \(y = 3\) divides the coordinate plane into two regions. If you start at any point on line \(l\), say \((1, 3)\), and move vertically upward, the \(y\)-coordinate increases as you move. If you move vertically downward from this point, the value of \(y\) decreases. In either case, the value of \(x\) remains 1.
The equation \( y = 3 \) is the boundary of two half-planes. In Figure 9-12 the half-plane above the line consists of all points for which \( y > 3 \), and is the graph of that inequality. The half-plane below the line is the graph of \( y < 3 \). The half-plane above the line along with the boundary line forms the graph of \( y \geq 3 \), while the boundary line and the half-plane below it is the graph of \( y \leq 3 \).

These graphs are indicated by shading. If the boundary is part of the region, the line is a solid line; if it is not, a dashed line is used with appropriate shading to distinguish the half-planes above and below the dashed line.

The inequalities \( y > 3 \) and \( x > 2 \) are graphed on the same coordinate plane in Figure 9-13. The points in the upper right-hand section of the plane represent those points whose coordinates satisfy both inequalities. That is, these are the points for which \( y > 3 \) and \( x > 2 \).

Figure 9-14 shows the graph of \( y = 2x + 1 \) dividing the plane into two half-planes. For each \( x \), all the points on the line satisfy the equation \( y = 2x + 1 \); all the points in the region above the line satisfy the inequality \( y > 2x + 1 \); and all the points in the region below the line satisfy the inequality \( y < 2x + 1 \).
EXAMPLE.  Graph the inequality \( x - y < 2 \).

Solution:  
1. Transform the inequality into one having \( y \) as one member.

\[
x - y < 2 \\
y < 2 - x \\
y > x - 2
\]

2. Graph \( y = x - 2 \), and show it as a dashed line.

3. Shade the half-plane above the line.

ORAL EXERCISES

Transform each open sentence into one having \( y \) alone as one member.

1. \( x + y > 6 \)  
2. \( y - x < 2 \)  
3. \( y - 3x \leq 4 \)  
4. \( 5x + y \geq -1 \)  
5. \( 2x + 6y < 0 \)  
6. \( 9x + 3y \leq 0 \)  
7. \( x < 4y \)  
8. \( x > 7y \)  
9. \( 8x - 2y \geq 0 \)  
10. \( 10x - 5y < 0 \)  
11. \( x - y > -1 \)  
12. \( 2x - y \geq 3 \)

Which indicated points belong to the graph of the given inequality?

13. \( x - y \leq 0 \); (7, -3), (2, 2)  
14. \( y - x > 0 \); (1, 1), (-2, 5)  
15. \( 2y - x \geq 2 \); (2, 3), (6, 2)  
16. \( 2x - 3y > -1 \); (\( \frac{1}{2} \), 1), (0, 0)

WRITTEN EXERCISES

Graph each inequality in two variables.

A  
1. \( y \geq -3 \)  
2. \( y \leq 2 \)  
3. \( y < 0 \)  
4. \( x < 1 \)  
5. \( x > -3 \)  
6. \( y > x \)  
7. \( y \leq 2x \)  
8. \( y < -\frac{x}{4} \)  
9. \( y > -\frac{x}{2} \)  
10. \( x \geq 0 \)  
11. \( y + 5x \leq 0 \)  
12. \( y - 3x \geq -1 \)

In a coordinate plane indicate the region consisting of all points whose coordinates satisfy both inequalities.

B  
13. \( x \geq -2 \) and \( y \leq 5 \)  
14. \( x < 4 \) and \( y > -4 \)
15. \( x + y \leq -2 \) and \( y > 0 \)
16. \( x + y \geq 3 \) and \( y < 0 \)
17. \( x - y > -1 \) and \( x \leq 0 \)
18. \( x - y < 2 \) and \( x \geq 0 \)
19. \(|y| > 2\)
20. \(|y| \leq 1\)
21. \( y \leq |x|\)
22. \( y + |x| = 0\)
23. \(-1 < y \leq 3\)
24. \(-2 \leq x < 4\)

9–8 Graphs That Are Parabolas

Several points on the graph of the quadratic equation \( y = x^2 \) have been plotted in Figure 9–15, and connected by a smooth curve.

\[
\begin{array}{c|c|c}
 x & x^2 & y \\
 0 & 0^2 & 0 \\
 1 & 1^2 & 1 \\
 2 & 2^2 & 4 \\
 3 & 3^2 & 9 \\
 -1 & (-1)^2 & 1 \\
 -2 & (-2)^2 & 4 \\
 -3 & (-3)^2 & 9 \\
\end{array}
\]

\( \text{Figure 9–15} \)

In Figure 9–16 you see the graph of another quadratic equation, \( y = -2x^2 \).

\[
\begin{array}{c|c|c}
 x & -2x^2 & y \\
 0 & -2(0)^2 & 0 \\
 1 & -2(1)^2 & -2 \\
 \frac{3}{2} & -2(\frac{3}{2})^2 & -\frac{9}{2} \\
 2 & -2(2)^2 & -8 \\
 -1 & -2(-1)^2 & -2 \\
 -\frac{3}{2} & -2(-\frac{3}{2})^2 & -\frac{9}{2} \\
 -2 & -2(-2)^2 & -8 \\
\end{array}
\]

\( \text{Figure 9–16} \)

To construct such graphs, plot enough points to enable you to draw a smooth curve. A curve, such as those in Figures 9–15, 9–16, and 9–17 is called a parabola (pa-rab-oh-luh). The path of a projectile moving in a vacuum, the cable supporting a suspension bridge, and cer-
tain cross sections of the reflector on a searchlight are examples of parabolas.

The graph of every quadratic equation of the form $y = ax^2 + bx + c$ for $a \neq 0$ is a parabola.

<table>
<thead>
<tr>
<th>$y = x^2 + 2x - 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>-5</td>
</tr>
<tr>
<td>-4</td>
</tr>
<tr>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

**WRITTEN EXERCISES**

Graph the equations, and draw a smooth curve through the points.

**A**

1. $y = 2x^2$
2. $y = 3x^2$
3. $4y = x^2$
4. $3y = x^2$
5. $3y = -x^2$
6. $2y = -x^2$
7. $y = x^2 + 1$
8. $y = x^2 + 2$
9. $y = x^2 - 4$
10. $y = x^2 - 9$
11. $y = 2(x^2 - 1)$
12. $y = -2(x^2 - 1)$

**B**

13. $y = x^2 - 2x - 3$
14. $y = x^2 + 4x - 3$
15. $y = x^2 + 2x$
16. $y = x^2 - 4x$
17. $y = -x^2 - x$
18. $y = x^2 + x + 2$
19. $y = -1 - x^2$
20. $y = -1 + 4x - x^2$

**STATISTICAL GRAPHS**

**9-9 Broken-Line and Bar Graphs**

Pictures in the form of broken-line and bar graphs provide quick understanding of statistical data. The points at the breaks in
a broken-line graph and the points at the ends of the bars in a bar graph represent ordered pairs of numbers.

A broken-line graph results when you plot ordered pairs of numbers and join them by a series of straight line segments. In Figure 9–18 these line segments do not show the college enrollment for intermediate years, but help you visualize the changes from year to year. You can see readily, for example, that while enrollment was up from 1951 to 1959, the rise was very sharp from 1953 to 1957.

<table>
<thead>
<tr>
<th>Year</th>
<th>Public College Enrollment (nearest 100,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1949</td>
<td>1,200,000</td>
</tr>
<tr>
<td>1951</td>
<td>1,000,000</td>
</tr>
<tr>
<td>1953</td>
<td>1,200,000</td>
</tr>
<tr>
<td>1955</td>
<td>1,500,000</td>
</tr>
<tr>
<td>1957</td>
<td>1,800,000</td>
</tr>
<tr>
<td>1959</td>
<td>1,900,000</td>
</tr>
</tbody>
</table>

The graph in Figure 9–19 uses parallel bars of varying lengths to represent measurements of quantities. The scale of the bars must start at zero, so that their relative lengths will be correct.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Cars (nearest 100,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1955</td>
<td>7,900,000</td>
</tr>
<tr>
<td>1956</td>
<td>5,800,000</td>
</tr>
<tr>
<td>1957</td>
<td>6,100,000</td>
</tr>
<tr>
<td>1958</td>
<td>4,200,000</td>
</tr>
<tr>
<td>1959</td>
<td>5,600,000</td>
</tr>
<tr>
<td>1960</td>
<td>6,700,000</td>
</tr>
</tbody>
</table>
Such bar graphs compare facts clearly; here, you can see that more cars were produced in 1959 than in 1958, and that more were produced in 1955 than in any of the other years. Horizontal bars in Figure 9-20 show the positive or negative per cent change in population in several counties over a ten-year period.

_Pictographs_ or _pictorial graphs_ like Figure 9-21 are special bar graphs in which rows of uniform symbols replace the bars.

**Population Changes in Seven Counties**

<table>
<thead>
<tr>
<th>County</th>
<th>Per cent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gulf</td>
<td>38</td>
</tr>
<tr>
<td>Fox</td>
<td>36</td>
</tr>
<tr>
<td>Elm</td>
<td>28</td>
</tr>
<tr>
<td>Dill</td>
<td>12</td>
</tr>
<tr>
<td>Clay</td>
<td>4</td>
</tr>
<tr>
<td>Bigg</td>
<td>-5</td>
</tr>
<tr>
<td>Axe</td>
<td>-9</td>
</tr>
</tbody>
</table>

**United States Population, 1910–1960**

1910

1920

1930

1940

1950

1960

Key:  = 10,000,000 persons

---

**WRITTEN EXERCISES**

Solve, and give each graph a title.

1. Make a bar graph to show the oceans' areas, in millions of square miles: Pacific, 69; Atlantic, 41; Indian, 28; Antarctic, 8; Arctic, 5.
2. Make a bar graph showing these heights: Eiffel Tower, 1000 feet; Washington Monument, 550 feet; Empire State Building, 1250 feet; St. Peter’s Church, 450 feet; and the Great Pyramid, 480 feet.
3. Draw a bar graph of the per cent change in cost of these items in one year: food, 3; housing, 1; clothing, -2; medical care, 4; utilities, -1.
4. Show on a broken-line graph beginning with January the average monthly temperature in Galveston: 54°; 56°; 62°; 69°; 75°; 81°; 83°; 83°; 80°; 73°; 63°; 56°.

(continued on page 357)
Graphs of Inequalities

The following pages show the steps in graphing three separate sets of simultaneous inequalities. As you read the following paragraphs, refer to the diagrams in sequence.

The top sequence of diagrams graphs the set:

\[
\begin{align*}
y &\geq 2x \\
3x + 2y &\geq 6 \\
y &< -2
\end{align*}
\]

The line \( y = 2x \) drawn on page H separates the plane into two half-planes: the region above the line (hatched) and the region below (unhatched). The coordinates of the points above and on the line satisfy the inequality \( y \geq 2x \); below the line, \( y < 2x \). Since the graph of \( y \geq 2x \) includes the line \( y = 2x \), that line is drawn as a solid boundary.

On page I the line \( 3x + 2y = 6 \) or \( y = 3 - \frac{3}{2}x \) separates the plane into two half-planes: the red region above the line and the uncolored region below it. Above or on the line \( 3x + 2y \geq 6 \); below, \( 3x + 2y < 6 \). The set of points whose coordinates satisfy both of the inequalities \( y \geq 2x \) and \( 3x + 2y \geq 2 \) consists of the points of the region that is both hatched and red.

The line \( y = -2 \) on page J produces another separation of the plane: the green region below (\( y < -2 \)) and the region above (\( y > -2 \)). Since the graph of \( y < -2 \) does not include the line \( y = -2 \), that line is shown as a dashed boundary.

The points whose coordinates satisfy all three inequalities \( y \geq 2x \), \( 3x + 2y \geq 6 \), and \( y < -2 \) should be seen on page J in a region that is hatched and brown. But there is no such region! This means that the three inequalities have no common root; that is, the solution set of this simultaneous system is \( \emptyset \).

Can you use inequalities to describe the unhatched and uncolored region in the top diagram on page J? If you assume that this region includes its green boundary line, but not its blue or red boundaries, then the region is the graph of the simultaneous system: \( y < 2x \), \( 3x + 2y < 6 \) and \( y > -2 \). Can you similarly describe the other six regions shown in the plane? Be sure to decide whether or not you are to include the boundaries in describing the regions.

The second sequence of diagrams beginning on page H builds up the graph of the solution set of the simultaneous system: \( y \leq 2x + 4 \), \( x + y + 2 \geq 0 \), and \( y + 4x < 4 \). This system is equivalent to: \( y \leq 2x + 4 \), \( y \geq -x - 2 \), and \( y < 4 - 4x \). The graph is shown on page J as the region that is hatched and brown. Note that it includes its blue and red boundaries, but not its green one.

The final sequence of diagrams begins on page H with the graph of the inequality \( |x| > 2 \). Since a point belongs to the graph of this inequality if, and only if, its abscissa satisfies either of the inequalities \( x > 2 \) or \( x < -2 \), the hatched region consists of two parts: the region to the right of the vertical line \( x = 2 \) and the region to the left of the line \( x = -2 \).

Page I introduces the graph of \(-3 < y \leq 1\). To belong to the graph, a point must have its ordinate satisfy both of the inequalities \(-3 < y \) and \( y \leq 1 \). Hence, the graph consists of the red horizontal strip above the line \( y = -3 \), and on or below the line \( y = 1 \).
Do you see that together the inequalities $|x| > 2$ and $-3 < y \leq 1$ separate the plane into nine regions? By turning to page J you find each of these regions identified as the graph of a pair of inequalities. Note: (i) a hatched region does not include the points for which $|x| = 2$; (ii) a red region includes all points for which $y = 1$, but no points for which $y = -3$.

**CONVEX POLYGONS**

The regions into which a plane is separated by a finite number of straight lines are called convex regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.

Convex regions in the plane are important in studying the values of linear expressions such as $2x + 3y$. Let us call this expression $f$ and write $f = 2x + 3y$. Suppose you want to find the greatest and least values of $f$ over the set of ordered pairs $(x, y)$ satisfying the inequalities $y \leq 2x + 4$, $x + y + 2 \geq 0$, and $y + 4x \leq 4$. This means that you want to maximize and minimize $2x + 3y$ over the closed triangle shown as the hatched, brown region in the middle diagram of page J.

A remarkable fact is that over closed, convex polygons the maximum and minimum values of any linear expression occur at vertices (corner points) of the region. To see that this is true, you must first carefully examine the coordinates of points on the line segment joining any two points in the plane.

Let $P(a, b)$ and $Q(c, d)$ be two points in the plane, and let $T$ be a point on line segment $PO$. Suppose that the distance from $P$ to $T(PT)$ is one-third of the distance from $P$ to $Q(PQ)$. Can you guess what $x_T$, the abscissa of $T$, might be? It seems reasonable to guess that since $PT = \frac{1}{3}PQ$, $x_T$ might be the number one-third of the way from $a$ (the abscissa of $P$) to $c$ (the abscissa of $Q$). This amounts to guessing that in the right triangles $PTV$ and $PQR$ the following ratios of lengths of corresponding sides are equal: $\frac{PV}{PR} = \frac{PT}{PQ} = \frac{1}{3}$. Because these triangles are similar, these ratios are, indeed, equal, so your guess is correct. Thus, $\frac{x_T}{a} = \frac{y}{c - a}$. This means $x_T = (1 - \frac{1}{3})a + \frac{1}{3}c$. 

- $S < x$
- $x \geq y$
- $c - > x$
- $S \geq y$
Do you see that together the inequalities $|x| > 2$ and $-3 < y \leq 1$ separate the plane into nine regions? By turning to page J you find each of these regions identified as the graph of a pair of inequalities. Note: (i) a hatched region does not include the points for which $|x| = 2$; (ii) a red region includes all points for which $y = 1$, but no points for which $y = -3$.

**CONVEX POLYGONS**

The regions into which a plane is separated by a finite number of straight lines are called convex regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.

Convex regions in the plane are important in studying the values of linear expressions such as $2x + 3y$. Let us call this expression $f$ and write $f = 2x + 3y$. Suppose you want to find the greatest and least values of $f$ over the set of ordered pairs $(x, y)$ satisfying the inequalities $y \leq 2x + 4$, $x + y + 2 \geq 0$, and $y + 4x \leq 4$. This means that you want to maximize and minimize $2x + 3y$ over the closed triangle shown as the hatched, brown region in the middle diagram of page J.

A remarkable fact is that over closed, convex polygons the maximum and minimum values of any linear expression occur at vertices (corner points) of the region. To see that this is true, you must first carefully examine the coordinates of points on the line segment joining any two points in the plane.

Let $P(a, b)$ and $Q(c, d)$ be two points in the plane, and let $T$ be a point on line segment $PQ$. Suppose that the distance from $P$ to $T$ ($PT$) is one-third of the distance from $P$ to $Q(PQ)$. Can you guess what $x_T$, the abscissa of $T$, might be? It seems reasonable to guess that since $PT = \frac{1}{3}PQ$, $x_T$ might be the number one-third of the way from $a$ (the abscissa of $P$) to $c$ (the abscissa of $Q$). This amounts to guessing that in the right triangles $PTV$ and $PQR$ the following ratios of lengths of corresponding sides are equal: $\frac{PV}{PT} = \frac{PQ}{PQ} = \frac{1}{3}$. Because these triangles are similar, these ratios are, indeed, equal, so your guess is correct. Thus, $x_T$ is $\frac{1}{3}$ of the way from $a$ to $c$.

This means $x_T = \frac{1}{3}a + \frac{1}{3}c$. 

\[
\begin{align*}
S &< x \\
S &< x \\
\varepsilon &> y \\
\varepsilon &> y \\
\end{align*}
\]
Do you see that together the inequalities \(|x| > 2\) and \(-3 < y \leq 1\) separate the plane into nine regions? By turning to page J you find each of these regions identified as the graph of a pair of inequalities. Note: (i) a hatched region does not include the points for which \(|x| = 2\); (ii) a red region includes all points for which \(y = 1\), but no points for which \(y = -3\).

**CONVEX POLYGONS**

The regions into which a plane is separated by a finite number of straight lines are called *convex* regions. When a region is convex, it contains every line segment joining any two of its points. For example, in the following figure the circle and the triangle enclose convex regions, but the star does not.

Convex regions in the plane are important in studying the values of linear expressions such as \(2x + 3y\). Let us call this expression \(f\) and write \(f = 2x + 3y\). Suppose you want to find the greatest and least values of \(f\) over the set of ordered pairs \((x, y)\) satisfying the inequalities \(y \leq 2x + 4\), \(x + y + 2 > 0\), and \(y + 4x \leq 4\). This means that you want to maximize and minimize \(2x + 3y\) over the closed triangle shown as the hatched, brown region in the middle diagram of page J.

A remarkable fact is that over closed, convex polygons the maximum and minimum values of any linear expression occur at vertices (corner points) of the region. To see that this is true, you must first carefully examine the coordinates of points on the line segment joining any two points in the plane.

Let \(P(a, b)\) and \(Q(c, d)\) be two points in the plane, and let \(T\) be a point on line segment \(PQ\). Suppose that the distance from \(P\) to \(T\) (\(PT\)) is one-third of the distance from \(P\) to \(Q(PQ)\). Can you guess what \(x_T\), the abscissa of \(T\), might be? It seems reasonable to guess that since \(PT = \frac{1}{3}PQ\), \(x_T\) might be the number one-third of the way from \(a\) (the abscissa of \(P\)) to \(c\) (the abscissa of \(Q\)). This amounts to guessing that in the right triangles \(PTV\) and \(PQR\) the following ratios of lengths of corresponding sides are equal: \(\frac{PV}{PT} = \frac{PT}{PQ} = \frac{1}{3}\). Because these triangles are similar, these ratios are, indeed, equal, so your guess is correct. Thus,

\[
x_T = \frac{1}{3}(a + \frac{1}{3}(c - a))
\]

absissa of \(T\) is \(\frac{1}{3}\) of the way from \(a\) to \(c\).

This means

\[
x_T = (1 - \frac{1}{3})a + \frac{1}{3}c.
\]
By a similar argument you discover that

\[ y_T = (1 - \frac{1}{3})b + \frac{1}{3}d. \]

In general, if \( T \) is between \( P \) and \( Q \), there is a number \( t \) between 0 and 1 such that

\[ \frac{PT}{PQ} = t \quad \text{and} \quad x_T = (1 - t)a + tc \]

\[ y_T = (1 - t)b + td. \]

Now suppose that \( P \) and \( Q \) are any points on the boundary of a closed convex polygon.

Let \( f_P = 2a + 3b \) be the value of \( f \) at \( P \) and \( f_Q = 2c + 3d \) be the value of \( f \) at \( Q \). Then, \( f_T \) (the value of \( f \) at \( T \)) = \( 2x_T + 3y_T \).

\[ f_T = 2[(1 - t)a + tc] + 3[(1 - t)b + td] \]

\[ f_T = (1 - t)(2a + 3b) + t(2c + 3d) \]

\[ f_T = (1 - t)f_P + tf_Q \]

Thus, the value of \( f \) at \( T \) is between the values of \( f \) at \( P \) and \( Q \). Of course, if \( f_Q = f_P \) then \( f_T = f_P \).

Do you see the significance of this discovery? It means that at \( T \) the value of \( f \) can be neither greater nor less than it is at \( P \) or \( Q \).

But, \( P \) and \( Q \) are any points on the boundary of a closed convex polygon. Now, given any point \( T \) in the interior of such a region, there is a line segment containing \( T \) that intersects the boundary of the region in just two points and that otherwise lies wholly within the region. This means that the value of \( f \) at any interior point \( T \) can be neither larger nor smaller than the values of \( f \) on the boundary of the region.

Now suppose that \( P \) and \( Q \) are vertices of the polygon. It is still true that the value of \( f \) at \( T \), a nonvertex point on the boundary, can be neither greater nor less than its values at \( P \) and \( Q \).

We may, therefore, conclude that \( 2x + 3y \) (or any linear expression \( mx + ny \)) assumes its maximum and minimum values over a closed convex polygon at vertices of that region.

In particular, over the hatched, brown triangle in the middle diagram of page J, the greatest and least values of \( 2x + 3y \) must occur at one or another of the three vertices: \( A(2, -4), B(-2, 0), C(0, 4) \). Evaluating \( 2x + 3y \) at each of these points, you see that over the triangle the maximum 12 occurs at \( C \) and the minimum -8 occurs at \( A \).

It is possible that the maximum and minimum values occur at other points besides vertices. What matters is that by testing values at the vertices you can in a finite number of steps determine the greatest and least values of any linear expression over any closed convex polygon.
5. Make a broken-line graph of the number of scholarships awarded in each decade from 1920 to 1960: 700, 800, 1000, 1600, 7600.

6. Make a broken-line graph of the inches of rainfall in New York City in each month from January to December: 5.3; 2.2; 6.6; 5.1; 1.5; 3.4; 2.2; 3.1; 8.6; 3.4; 7.1; 3.2.

7. Make a pictograph of the membership distribution in the fifteenth assembly of the United Nations: Western bloc, 23; Latin American bloc, 20; Asian-African bloc, 41; Soviet bloc, 9; others, 3.

8. Make a pictograph of the median price of used cars in these years: 1951, $600; 1952, $850; 1953, $900; 1954, $700; 1955, $600; 1956, $650; 1957, $700; 1958, $600.

9-10 Circle Graphs

A circle graph is used to compare the parts of a whole with each other and with the total. The size of each sector depends on the angle at the center of the circle, so you must know the number of degrees in this central angle for your comparison.

Since the circle graph in Figure 9–22 shows that 10% of the drivers are under 20 years of age, the central angle of this sector contains 10% of the 360 degrees around the center of the circle, or \( \frac{10}{100} \times 360 = 36^\circ \).

In like manner, you can compute the number of degrees in the other central angles.

**Figure 9–22**

**WRITTEN EXERCISES**

1. Bonville’s taxes are spent thus: 33\( \frac{1}{3} \)% for schools, libraries, and museums; 25% for police, fire, and sanitation; 15% for streets, parks, and recreation; 16\( \frac{2}{3} \)% for jails, homes for the destitute, and hospitals; 10% for administration. (a) Make a circle graph. (b) If the third listed item costs $31,500, find Bonville’s total budget.
2. Of Midville Boys High School graduates, 40% join the army, 25% join the navy, 20% enter college, and 15% get jobs in town. (a) Make a circle graph. (b) If 80 boys join the army, how many graduate?

Illustrate each exercise with a circle graph.

3. The average American diet is: 46% carbohydrates, 14% proteins, 23% unsaturated fats, 17% saturated fats.

4. An ideal diet is: 69% carbohydrates, 16% proteins, 11% unsaturated fats, 4% saturated fats.

Coordinated Pictures

By drawing a silhouette on a sheet of graph paper and by noting the points at which the outline changes direction, you can make a list of coordinates which represent the picture. Someone else can take your list, plot your number pairs, and connect them with lines to reproduce your picture. If you plot and connect each of these points in turn, you will see such a picture develop:

\[ (-5, -10), (-5, -8), (-8, -4), (-8, 0), (-5, 4), (-3, 5), (1, 5), (3, 3), (2, 2), (5, -1), (4, -2), (3, -2), (2, -3), (2, -4), (4, -6), (4, -7), (3, -8), (1, -8), (1, -10) \]

You need not confine your artistry to human profiles; animals, plants, and cartoon characters also are suitable subjects for coordinated pictures.
Chapter Summary

Inventory of Structure and Method

1. To set up a rectangular coordinate system in a plane, choose a vertical and a horizontal line, and scale them as number lines intersecting at zero. The ordered pair \((a, b)\) corresponds to the point whose directed distance from the vertical axis is \(a\) and from the horizontal axis is \(b\).

2. A plane coordinate system enables you to picture the solution set of an open sentence in two variables as the set of points whose coordinates satisfy the open sentence. Thus, to graph a linear equation in two variables, each having the set of directed numbers as its replacement set, draw the straight line determined by plotting any two roots of the equation.

3. To measure the slope of a straight line, choose two different points on the line, and compute the ratio of the difference between the ordinates of the points to the corresponding difference between the abscissas of the points. It is a property of a straight line that this ratio is the same for every pair of distinct points on the line.

4. A line with slope \(m\) and \(y\)-intercept \(b\) is the graph of the equation \(y = mx + b\). This slope-intercept form of a linear equation can be used to find an equation for a line
   a. with given slope and passing through a given point;
   b. passing through two different points.

5. To graph an inequality in two variables graph the related equality, and then shade the half-plane defined by the inequality.

6. The graph of an equation of the form \(y = ax^2 + bx + c, a \neq 0\), is a curve called a parabola.

7. Broken-line, bar, and circle graphs are employed for the visual presentation of statistics to display comparisons and trends in data.

Vocabulary and Spelling

- ordered pair of numbers (p. 334)
- first coordinate (p. 334)
- second coordinate (p. 334)
- root of an open sentence in two variables (p. 334)
- solution set of an open sentence in two variables (p. 334)
- horizontal axis (p. 337)
- vertical axis (p. 337)
- origin (p. 337)
- graph of an ordered pair (p. 337)
- plotting a point (p. 337)
- abscissa (p. 338)
- ordinate (p. 338)
coordinates of a point (p. 338)  
plane coordinate system (p. 338)  
coordinate plane (p. 338)  
quadrant (p. 338)  
graph of an equation (p. 341)  
an equation of a line (p. 341)  
linear equation in two variables (p. 341)  
slope of a line (p. 343)  
y-intercept (p. 347)  
slope-intercept form (p. 347)  
graph of an inequality in two variables (p. 351)  
half-plane (p. 351)  
boundary line (p. 351)  
parabola (p. 353)  
broken-line graph (p. 354)  
bar graph (p. 354)  
pictograph (p. 356)  
circle graph (p. 357)

Chapter Test

9-1  
1. For what value of \( c \) does \((c, -4)\) belong to the solution set of \(3x + 4y = 17\)?
2. If \((3a - 1, b^2) = (5a - 7, 2b - 1)\), find the values of \(a\) and \(b\).
3. If \(x \in \{-1, 0, 1\}\) and \(y \in \{0, 1\}\), give the solution set of \(2x + y < 3\).

9-2  
4. Give the coordinates of each indicated point in the adjoining figure.
5. Graph these points in a coordinate plane:
   a. \((5, 2)\)
   b. \((-3, -4)\)
   c. \((2, -3)\)
   d. \((-2, 0)\)
   e. \((-4, 4)\)

9-3  
6. Graph the one linear equation:
   \(x^2 + y^2 = 4, \quad xy - 3x = 2, \quad x - 2y = 11\)
7. Do the points \((2, 7)\) and \((7, 2)\) belong to the graph of \(2y - 3x = 8\)? Justify your answer.
9-4  8. Find the slope of the line joining \((-7, 2)\) and \((1, 18)\).
     9. Draw a line with slope 3 through \((5, -2)\).
    10. If \((3, -1)\) is on the graph of \(2x + ay = 4\), find the value of \(a\).

9-5  11. Give the slope and \(y\)-intercept of the line \(5x - 3y = 24\).
     12. Write an equation of the line through \((0, -4)\) with slope 10.

9-6  13. Find an equation of the line containing \((8, 0)\) with slope \(-\frac{7}{4}\).
     14. Write an equation of the line joining \((5, 1)\) and \((7, -3)\).

9-7  15. Graph \(y + 2x < 6\).  16. Graph \(2y \geq 3x + 4\).

9-8  17. Graph \(y = 2x^2 - 4\).

9-9  18. Make a bar graph of the sale of twenty-eight $25 bonds, thirty-five $50 bonds, fourteen $100 bonds, and seven $500 bonds.

19. Construct a broken-line graph of:

<table>
<thead>
<tr>
<th>Daily Calorie Needs of Children</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age in Years</td>
</tr>
<tr>
<td>Calories</td>
</tr>
</tbody>
</table>

9-10 20. On a circle graph show how each dollar was used in a business if 30¢ was spent for labor, 10¢ for operations, and 60¢ for materials.

Chapter Review

9-1 Open Sentences in Two Variables  Pages 333–337

1. A root of an open sentence in two variables is an ___ ? ___ of numbers for which the sentence is true.

2. \((a, b) = (c, d)\) if ___ ? ___ and ___ ? ___.

3. \((3, ___ ? ___)\) is a root of the equation \(10x + y = 7\).

4. If \(x \in \{-1, 0, 1\}\) and \(y \in \{\text{integers}\}\), the solution set of \(5x - 2y = 11\) is ___ ? ___.

9-2 Coordinates in a Plane  Pages 337–340

5. To associate a point with each ordered pair of numbers, draw two ___ ? ___ at right angles whose point of intersection is the ___ ? ___.
6. The first coordinate of \( P(3, -2) \) is the \(?\) of \( P \); the second coordinate is the \(?\) of \( P \).

7. A point whose ordinate is \(-3\) lies \(?\) units below the \(?\) axis.

8. Points within the \(?\) and \(?\) have positive abscissas.

9. When both coordinates are negative, a point lies in the \(?\) quadrant.

In a coordinate plane plot the points in Exercises 10–13.

10. (3, 1)  
11. (–4, 3)  
12. (–3, –5)  
13. (4, –2)

9-3 The Graph of a Linear Equation in Two Variables  Pages 340–343

14. The graph of an equation is the set of points whose coordinates \(?\) the equation.

15. The point \((5, \_\_)\) belongs to the graph of \( x + 2y = 7 \).

16. In a linear equation each term is a \(?\) or a monomial of degree \(?\).

17. The graph of a linear equation in two variables, each having the set of directed numbers as replacement set, is a \(?\) \(?\).

18. Graph \( y = 2x - 4 \) and \( 5x + 3y = 10 \).

9-4 Slope of a Line  Pages 343–346

19. The slope of a line measures its \(?\).

20. Given two points on a line, the slope of the line is the ratio of the difference of their \(?\) to the corresponding difference of their \(?\).

21. The slope of the line containing \((5, 4)\) and \((7, -6)\) is \(?\).

22. Every horizontal line has slope \(?\).

23. A vertical line has slope \(?\).

24. When a line rises from left to right, its slope is a \(?\) number.

25. A line whose slope is a \(?\) number falls from left to right.

9-5 The Slope-Intercept Form of a Linear Equation  Pages 346–348

26. The line with equation \( y = mx + b \) has slope \(?\) and \( y \)-intercept \(?\).

27. The \( y \)-intercept of a line is the \(?\) of the point where the line intersects the \(?\) -axis.

28. The line \( y - 4x + 5 = 0 \) crosses the \( y \)-axis at \((0, \_\_\_\_\_\_)\); its slope is \(?\).

29. An equation of the line with slope \(-7\) and \( y \)-intercept 1 is \(?\).
9-6 Determining the Equation of a Line Pages 349–350
30. If \( y = 2x + b \) passes through the point \((-5, 1)\), then \( b = \) ?.
31. Give an equation of the line through \((2, -6)\) with slope \(-4\).
32. Give an equation of the line through \((-4, -2)\) and \((2, 1)\).
33. Give an equation of the line parallel to \(4x - 2y = 1\) and passing through \((-1, -4)\).

9-7 Graph of an Inequality in Two Variables Pages 350–353
34. The graph of \( y = 3x - 1 \) divides the coordinate plane into two _?_. Points above the line belong to the graph of \( y \) _?_ \( 3x - 1 \); points below the line belong to the graph of \( y \) _?_ \( 3x - 1 \).
35. To graph \( y \leq 5 - 2x \), draw the graph of \( y \) _?_ \( 5 - 2x \). Then shade the region _?_ that line and show the boundary as a _?_ line.
36. To graph \( 5x + 3y > 6 \), make _?_ the left member of the inequality.
37. Graph \( 5x + 3y > 6 \) and \( 5x - 3y > 6 \).

9-8 Graphs That Are Parabolas Pages 353–354
38. Calculate the values of \( y \) in this table if \( y = -x^2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>9</td>
<td>?</td>
<td>?</td>
<td>0</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
39. Graph \( y = -x^2 \) from the preceding table.
40. On the parabola \( y = x^2 + 2 \) name the coordinates of the point having the least ordinate.
41. On the parabola \( y = 3 - x^2 \) name the coordinates of the point having the greatest ordinate.

9-9 Broken-Line and Bar Graphs Pages 354–357
42. On a broken-line graph, the only significant points are those that have been _?_.
43. Make a broken-line graph, and make a bar graph of the average price of admission to movies over forty years: 1919, 30€; 1929, 75€; 1939, 40€; 1949, 75€; 1959, 95€.

9-10 Circle Graphs Pages 357–358
44. A circle graph is used to compare the _?_ of a whole with each other and with the _?_.
45. The size of a sector of a circle depends on the size of the \( \angle \) angle.

46. Make a circle graph of Jane's budget: savings, 20%; school expenses, 25%; personal needs, 20%; entertainment and gifts, 25%; charity, 10%.

---

**Extra for Experts**

The Sieve of Eratosthenes

Eratosthenes (air-a-tos-tha-neez) was a Greek scholar who lived about 200 B.C.; his sieve was not a kitchen utensil, but a scheme for identifying prime numbers. You, like Eratosthenes, will find it useful to be able to sift prime numbers from products; so, in order to learn how, write all the numbers from 1 to 100. (For convenience, arrange them in ten rows.)

Do you remember the definition of a prime — an integer greater than 1, divisible only by itself and 1? By definition, then, 2 is a prime number. But every second number after 2 is divisible by 2; so, beginning with 4, cross out every second number in your table of 100. The number 3 is divisible only by itself and 1; so leave it untouched. Every third number after 3 is divisible by 3. Some of these numbers, such as 6 and 12, already have been crossed out; cross out all the rest of them.

The number 4 is a product, and so is every fourth number thereafter. Go on to 5, which is a prime. Cross out every fifth number not already crossed out after it.

By continuing in this manner, you can exhaust the nonprimes for any given range of integers.

**Questions**

1. Construct a sieve of Eratosthenes for the integers from 1 to 200.
2. What is the largest prime number you need to consider as a factor before you eliminate all the nonprimes (a) from 1 to 100 (b) from 1 to 200? Explain your answer.
3. How many primes did you find between (a) 1 and 50, (b) 50 and 100, (c) 100 and 150, (d) 150 and 200?
4. Do you see any regular pattern in the distribution of prime numbers?
5. Are any even numbers prime? Explain.
"My advice," said the school principal to frail, young René Descartes, "is to lie in bed as late as you like each morning." Descartes acted on this advice, not only while he was a sickly boy in school, but all the rest of his life.

As a young man, Descartes drifted. After several years in Paris, he entered the Dutch army — a fashionable and not very strenuous career for a seventeenth-century French gentleman.

But in Holland something happened. One day, Descartes saw a poster in Flemish, a language he could not read. Curious, he asked a passer-by to translate it to him. And he received a reply like this: "The poster bears a challenge," said the man, who happened to be a college president. "It asks whether anyone can solve a certain problem in mathematics." He stopped. "I'll read you the problem on one condition. You must promise me you will try to solve it."

Descartes was intrigued. He promised to try. And he found a solution. It was work, and took time, but he enjoyed it.

When Descartes discovered the pleasure in study, he left the army, and for the rest of his life devoted himself to learning. He became a great scientist, a great philosopher, and a great mathematician. He still stayed in bed in the morning. There he was undisturbed, and he could use this time to think. He watched a fly crawl across the ceiling, and figured out how to describe its path by an equation. He thought of ways to apply algebra to geometry, and of how to apply geometry to algebra. He invented a branch of mathematics: coordinate geometry.

It is not where a person is that counts so much as what he does!

René Descartes after he had gained renown as scientist, philosopher, and mathematician.
Sentences in Two Variables

Although computers solve many problems for scientific research, their use to predict presidential election results (top picture) is more familiar to you. From previous election patterns, statisticians develop prediction equations containing variables whose values are determined by the early returns. Substituting these values in the equations it has been given, the machine makes its predictions. As later returns come in (bottom picture), new values are substituted and revised predictions made. Without the machine, the calculation of one prediction would take longer than counting the ballots.

In this chapter, you will learn methods used by some digital computers for solving two equations containing two variables. However, the machines are capable of solving thirty-six equations in thirty-six variables in less time than it may take you to solve one of the problems here.

SOLVING SYSTEMS OF LINEAR OPEN SENTENCES

10–1 The Graphic Method

The graph of a linear equation in two variables is a straight line. When the graphs of two such equations in the same variables are drawn on the same axes, the resulting lines may have in common:

A. no point — the lines are parallel;
B. all their points — the lines coincide;
C. just one point — the lines intersect.

\[
\begin{align*}
A &: y = x + 4 \\
y & = x - 1 \\
B &: y = x + 4 \\
2y & = 2x + 8 \\
C &: y = x + 4 \\
y & = 3x \\
\end{align*}
\]

\textit{Figure 10–1}
Because two equations impose two conditions on the variables at the same time, they are called a system of simultaneous equations. To solve such a system, you seek the ordered pairs of numbers that satisfy both equations of the system.

The graphs in Figure 10-1 show that:

A. The system \( y = x + 4 \) has no solution; the graphs do not intersect.
\( y = x - 1 \)

B. The system \( y = x + 4 \) has an unlimited number of solutions; \( 2y = 2x + 8 \) the graphs coincide.

C. The system \( y = x + 4 \) has just one solution, \((2, 6)\); the graphs intersect at one point.

\( y = 3x \)

To understand why the equations of System A cannot have a common root, notice that if \( y = x + 4 \) and \( y = x - 1 \) were both true statements for some ordered pair \((x, y)\), then by substitution

\[
\begin{align*}
x + 4 &= x - 1, \\
4 &= -1, \text{ a false statement.}
\end{align*}
\]

Simultaneous equations having no common root are called inconsistent.

Because the equations of Systems B and C do have common roots, they are called consistent equations. In System B the equations are equivalent: multiply each member of \( y = x + 4 \) by 2, and you obtain \( 2y = 2x + 8 \). Such equivalent equations are also said to be dependent.

Neither equation of System C can be obtained from the other by multiplication. They are independent equations and have just one common root.

A pair of linear equations can be solved by graphing both on the same axes and determining the coordinates of the point of intersection.

Examine Figure 10-2. Do you see that \((2, 6)\) is the common root of any pair of linear equations whose graphs pass through that point? In particular, the pair of red lines in the figure pass through it. One of the red lines is the horizontal line whose equation is \( y = 6 \), and the other is the vertical line \( x = 2 \).

\[ \cdot \text{ Figure 10-2 } \cdot \]
Because the system of equations \( x = 2, \ y = 6 \) has the same solution set as the system \( y = x + 4, \ y = 3x \) the systems are said to be equivalent to each other.

### Oral Exercises

Determine the coordinates of the point of intersection.

1. \( x - y = 3 \)
   \( 2x - 2y = 6 \)
2. \( x - y = 4 \)
   \( 2x + y = 5 \)
3. \( x + y = 1 \)
   \( x + y = 2 \)
4. \( x + 2y = 12 \)
   \( x - 2y = 0 \)
5. \( x + 2y = 6 \)
   \( 2y - x = 4 \)
6. \( x + 2y = 6 \)
   \( x - 4y = 2 \)

Do the equations in each pair have only one common root? Explain.

**Sample.** \( x + y = 1 \)
\( x + y = 2 \)  

**What you say:** No common root, because they are inconsistent; the sum of two numbers cannot be both 1 and 2.

7. \( x + y = 4 \)
   \( 2x + y = 5 \)
8. \( 2x + y = 5 \)
   \( 2x + y = 8 \)
9. \( x - y = 3 \)
   \( 2x - 2y = 6 \)
10. \( x - y = 7 \)
    \( 2x - y = 16 \)
11. \( x - 2y = 5 \)
    \( x - 2y = 10 \)
12. \( x + 2y = 13 \)
    \( 2x + 4y = 26 \)
13. \( x + y = 4 \)
    \( 2x + 2y = 6 \)
14. \( 3x - 6y = 3 \)
    \( 4x - 8y = 4 \)
15. \( 2x - 4y = 10 \)
    \( 3x - 6y = 15 \)
Graph the equations. When the equations are consistent and independent, find their common root.

A 1. \( y = 3 - x \)  
   \( y = 1 + x \)  
2. \( y = 5 - x \)  
   \( y = 3 + x \)  
3. \( y = 2x + 4 \)  
   \( x + y = 1 \)  
4. \( x - y = 1 \)  
   \( y = 2x + 1 \)  
5. \( -x - y = 0 \)  
   \( x + y = -4 \)  
6. \( x + 2y = 3 \)  
   \( 2x + 4y = 6 \)  
7. \( 2y - 3x = 6 \)  
   \( y = 3x \)  
8. \( x + 2y = 5 \)  
   \( x = 3y \)  
9. \( x = 1 - y \)  
   \( y = -1 - x \)  
10. \( x = y \)  
    \( 3y = 0 + 3x \)  
11. \( y = 2x - 4 \)  
    \( 6x + y = 0 \)  
12. \( y = \frac{3}{2}x + 4 \)  
    \( y = -\frac{5}{2}x - 5 \)  

Solve graphically, and estimate the answer to the nearest tenth.

B 13. \( 2x - y = 7 \)  
    \( x + 3y = 2 \)  
14. \( x + 2y = 6 \)  
    \( y - 2x = 5 \)  
15. \( 3x + 2y = 5 \)  
    \(-6y - 9x = 7 \)  

16. Where on the graph of \( 2x - 5y = 3 \) is the abscissa twice the ordinate?  
17. Find the area of a triangle whose vertices are determined by the graphs of \( y = 2x + 4, x + y = 10, \) and \( y = -2. \)

10–2 The Addition and Subtraction Method

The algebraic method of solving simultaneous equations avoids inaccuracies which can occur in the drawing and reading of graphs.

**EXAMPLE 1.** Solve \( x + 4y = 27 \) and \( x + 2y = 21. \)

**Solution:**

The algebraic solution provides a systematic method for finding the equivalent system made up of the equations of the horizontal and vertical lines through the point of intersection.
You can use the addition or subtraction property of equality to obtain such equations.

\[
\begin{align*}
\hspace{1cm} x + 4y &= 27 \\
\hspace{1cm} x + 2y &= 21 \\
\hspace{1cm} 2y &= 6 \\
\hspace{1cm} y &= 3 \\
\end{align*}
\]

(horizontal line through the intersection)

Now, substitute 3 for \( y \) in either of the original equations:

\[
\begin{align*}
\hspace{1cm} x + 4y &= 27 \\
\hspace{1cm} x + 4(3) &= 27 \\
\hspace{1cm} x + 12 &= 27 \\
\hspace{1cm} x &= 15 \quad \text{(vertical line through the intersection)}
\end{align*}
\]

Check: Substitute 15 for \( x \) and 3 for \( y \) in both original equations.

\[
\begin{align*}
\text{The solution set is } \{(15, 3)\}, \text{ Answer.}
\end{align*}
\]

**EXAMPLE 2.** Solve \( 5m + 2n = 20 \) and \( 3m - 2n = 4 \) simultaneously.

Solution: \[
\begin{align*}
\hspace{1cm} 5m + 2n &= 20 \\
\hspace{1cm} 3m - 2n &= 4 \\
\hspace{1cm} 8m &= 24 \\
\hspace{1cm} m &= 3 \\
\hspace{1cm} 5m &= 20 \\
\hspace{1cm} 5 \cdot 3 + 2n &= 20 \\
\hspace{1cm} 15 + 2n &= 20 \\
\hspace{1cm} 2n &= 5 \\
\hspace{1cm} n &= \frac{5}{2}
\end{align*}
\]

Check: \[
\begin{align*}
\hspace{1cm} 5m + 2n &= 20 \\
\hspace{1cm} 5(3) + 2(\frac{5}{2}) &= 20 \\
\hspace{1cm} 15 + 5 &= 20 \\
\hspace{1cm} 20 &= 20 \checkmark \\
\hspace{1cm} 3m - 2n &= 4 \\
\hspace{1cm} 3(3) - 2(\frac{5}{2}) &= 4 \\
\hspace{1cm} 9 - 5 &= 4 \\
\hspace{1cm} 4 &= 4 \checkmark
\end{align*}
\]

\( \therefore \) The solution set is \( \{(3, \frac{5}{2})\} \), Answer.
Solve by addition or subtraction.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(x + y = 74)</td>
<td>(x - y = 16)</td>
</tr>
<tr>
<td>2.</td>
<td>(w + z = 122)</td>
<td>(w - z = 28)</td>
</tr>
<tr>
<td>3.</td>
<td>(2r - s = 20)</td>
<td>(2r + s = 48)</td>
</tr>
<tr>
<td>4.</td>
<td>(3m - n = 18)</td>
<td>(3m + n = 60)</td>
</tr>
<tr>
<td>5.</td>
<td>(3p - q = 10)</td>
<td>(2p - q = 7)</td>
</tr>
<tr>
<td>6.</td>
<td>(5A - B = 7)</td>
<td>(3A - B = 5)</td>
</tr>
<tr>
<td>7.</td>
<td>(3s + 2t = 17)</td>
<td>(s + 2t = 5)</td>
</tr>
<tr>
<td>8.</td>
<td>(4w - 5z = 10)</td>
<td>(2w + 5z = -10)</td>
</tr>
<tr>
<td>9.</td>
<td>(5m - 3n = 19)</td>
<td>(2m + 3n = -5)</td>
</tr>
<tr>
<td>10.</td>
<td>(2p - 5t = 12)</td>
<td>(2p - 3t = 12)</td>
</tr>
<tr>
<td>11.</td>
<td>(3r - 4s = 1)</td>
<td>(3r - 2s = -1)</td>
</tr>
<tr>
<td>12.</td>
<td>(2x - 3y = -4)</td>
<td>(4x - 3y = -6)</td>
</tr>
<tr>
<td>13.</td>
<td>(2 = p - 3n)</td>
<td>(12 = p - n)</td>
</tr>
<tr>
<td>14.</td>
<td>(6 = s - 3t)</td>
<td>(20 = s - t)</td>
</tr>
<tr>
<td>15.</td>
<td>(3s - 5t = -38)</td>
<td>(3s + 5t = 2)</td>
</tr>
<tr>
<td>16.</td>
<td>(11x - y = 42)</td>
<td>(3x + y = 0)</td>
</tr>
<tr>
<td>17.</td>
<td>(3w + 8z = 158)</td>
<td>(7w + 8z = 230)</td>
</tr>
<tr>
<td>18.</td>
<td>(2m + 11n = 265)</td>
<td>(5m + 11n = 316)</td>
</tr>
</tbody>
</table>

Clear the equations of fractions before adding or subtracting.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>19.</td>
<td>(y + x = 8)</td>
<td>(\frac{1}{4}(x - y) = 1)</td>
</tr>
<tr>
<td>20.</td>
<td>(2x - 3y = 16)</td>
<td>(\frac{1}{2}(3y + 2x) = -4)</td>
</tr>
<tr>
<td>21.</td>
<td>(\frac{r}{5} - \frac{s}{3} = 0)</td>
<td>(\frac{s}{6} + \frac{r}{10} = -2)</td>
</tr>
<tr>
<td>22.</td>
<td>(\frac{m}{2} - \frac{n}{3} = 3)</td>
<td>(\frac{n}{6} - \frac{m}{8} = -1)</td>
</tr>
</tbody>
</table>

### 10-3 Problems with Two Variables

Problems concerning two numbers can be solved by using one or two variables. A solution using two variables to form two open sentences is usually the more direct.
A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Tom paid 27 cents for a book he kept seven days, while Sue paid 21 cents for one she kept five days. Find the fixed charge and the charge for each extra day.

**Solution:**

<table>
<thead>
<tr>
<th>Choose two variables to represent the desired numbers.</th>
<th>Let ( x ) = the fixed charge and ( y ) = the extra daily charge.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form two open sentences using the facts in the problem.</td>
<td>( x + 4y = 27 ) (Ted's fee) ( x + 2y = 21 ) (Sue's fee)</td>
</tr>
<tr>
<td>Solve the equations.</td>
<td>( x + 4y = 27 ) ( x + 4(3) = 27 ) ( x + 2y = 21 ) ( x + 12 = 27 ) ( 2y = 6 ) ( x = 15 ) ( y = 3 )</td>
</tr>
<tr>
<td>Check your results in the words of the problem.</td>
<td>Ted's fee: fixed charge of 15¢ plus 4 days at 3¢ a day is 27¢. ( \checkmark )</td>
</tr>
<tr>
<td></td>
<td>Sue's fee: fixed charge of 15¢ plus 2 days at 3¢ a day is 21¢. ( \checkmark )</td>
</tr>
<tr>
<td></td>
<td>Fixed charge is 15 cents</td>
</tr>
<tr>
<td></td>
<td>Extra daily charge is 3 cents ( } ) Answer.</td>
</tr>
</tbody>
</table>

**ORAL EXERCISES**

Translate into a pair of equations with two variables.

1. One number is three times another. Their sum is 48.
2. The sum of two numbers is 42. One number is five times the other.
3. One number is three times the other. Their difference is 26.
4. One number is five times another. Their difference is 28.
5. A number is 8 more than another. Their sum is 164.
6. A number is 7 less than another. Their sum is 75.
7. One of two complementary angles exceeds the smaller by 18°.
8. One of two supplementary angles exceeds twice the smaller by 36°.
9. The length of a rectangle is 4 times its width. Its perimeter is 280 cm.
10. A melon costs 25¢ more than half a pineapple, and both cost $1.60.
Use two variables and two equations to solve each problem.

1. Half the perimeter of a house, which is 12 feet longer than it is wide, is 72 feet. What are the dimensions of the house?

2. Mr. Alden’s farm is 82 acres larger than Mr. Bradford’s. The two farms together contain 276 acres. How large is each?

3. The difference between twice one number and a smaller number is 21. The sum of the smaller and twice the larger is 27. Name the numbers.

4. If Mary were twice as old, she would be 19 years older than Ann, and their combined ages would be 41. How old are the girls?

5. If a baker orders three times as much whole-wheat flour as usual, he orders 11 pounds more of it than of white flour. But if he had ordered twice as much whole-wheat, he would have ordered 2 pounds less of it than of white. How much of each does he order?

6. Robert buys five magazines at the same price and a sixth at a higher price, paying 25¢ more for all five than for the sixth. If he buys only 3 of the cheaper magazines, they will cost, in all, a nickel less than the more expensive one. Find each price.

7. For $15.50, a grocer buys two grades of eggs, at 53¢ and 59¢ per dozen. If the cheaper eggs cost $2.52 more than the more costly, how many dozen of each does he buy?

8. Two styles of men’s hats cost $40 and $52 a dozen. If an order totals $224 and the cheaper line costs $16 more than the costlier line, how many dozen hats of each style are ordered?

9. One catcher’s mitt and 8 fielder’s gloves cost $41.60. One catcher’s mitt and 12 fielder’s gloves cost $59. What is the cost of each?

10. After a sale a store had taken in $10.50 more on its $5.95 dresses than on its $3.95 dresses. The total sale was $287. How many dresses were sold at each price?

10–4 Multiplication in the Addition and Subtraction Method

Sometimes adding or subtracting the members will not eliminate either variable because the coefficients of corresponding terms do not have the same absolute value. You then can use the multiplication property of equality to make the coefficient of a variable in one equation equal to the corresponding coefficient in the other.
EXAMPLE 1. Solve. \[ 3x + 4y = 45 \]
\[ 6x - 2y = 30 \]

Solution:

1. Multiply both members of the first equation by 2. \[ 6x + 8y = 90 \]
\[ 6x - 2y = 30 \]

2. Subtract the equations, and solve for \( y \).
\[ 10y = 60 \]
\[ y = 6 \]

3. Find the value of \( x \) by substituting 6 for \( y \) in one of the given equations.
\[ 3x + 4(6) = 45 \]
\[ 3x = 21 \]
\[ x = 7 \]

4. Check: Substitute in original equations. (This is left to you.)

EXAMPLE 2. Solve \[ 2x + 5y = 28 \] and \[ 3x + 2y = 31 \].

Solution:
\[ -2(2x + 5y) = -2(28) \rightarrow -4x - 10y = -56 \]
\[ 5(3x + 2y) = 5(31) \rightarrow 15x + 10y = 155 \]

\[ 11x = 99 \]
\[ x = 9 \]

Check:
\[ 2 \cdot 9 + 5 \cdot 2 = 28 \]
\[ 3 \cdot 9 + 2 \cdot 2 = 31 \]
\[ 28 = 28 \sqrt{ } \]
\[ 31 = 31 \sqrt{ } \]
\[ 3(9) + 2y = 31 \]
\[ 2y = 4 \]
\[ y = 2 \]

\[ \therefore \text{The solution set is } \{(9, 2)\}, \text{ Answer.} \]

ORAL EXERCISES

By what number would you multiply one equation in each pair to eliminate one variable? Give the transformed equation.

1. \[ w + z = 10 \]
\[ 2w + 3z = 23 \]

2. \[ r + t = 7 \]
\[ 4r - 5t = 1 \]

3. \[ 3m + 2n = 8 \]
\[ 6m - n = 11 \]

4. \[ m - 5n = 16 \]
\[ 2m + 3n = 6 \]

5. \[ 5x - 7y = -5 \]
\[ 2x - y = 7 \]

6. \[ 7x + 10y = 0 \]
\[ 3x + y = 23 \]

7. \[ 7w - 2z = 3 \]
\[ 3w + 8z = -43 \]

8. \[ 4p - 9q = 10 \]
\[ 8p - 15q = 26 \]

9. \[ 2u - 5t = 6 \]
\[ 6u + 3t = -4 \]

10. \[ 6x - 5y = -18 \]
\[ 4x + 3y = 7 \]

11. \[ 12A + 5B = 31 \]
\[ 8A + 7B = 50 \]

12. \[ 2r - 9s = 0 \]
\[ 3r - 6s = -5 \]
Solve each pair of equations algebraically.

13. \[ \frac{9s}{6s} - \frac{14t}{12t} = \frac{8}{0} \]

14. \[ 4u - 7v = 8 \\
    5u + 9v = 81 \]

15. \[ 5x + 4y = 22 \\
    3x + 6y = 24 \]

**WRITTEN EXERCISES**

**A**
1. \[ \frac{x}{2} - \frac{y}{7} = \frac{73}{21} \]
2. \[ m - n = 52 \\
    3m - 8n = 6 \]
3. \[ \frac{2m}{3} + 2n = 1 \\
    \frac{4m}{9} - \frac{n}{3} = \frac{1}{9} \]
4. \[ \frac{c}{5} + \frac{d}{2} = -1 \\
    \frac{d}{4} - \frac{c}{3} = -\frac{1}{2} \]
5. \[ 2s - 3t = 1 \\
    3s - 4t = 7 \]
6. \[ 2a + 5b = 18 \\
    3a + 4b = 27 \]
7. \[ c - 2d = 5 \\
    4d - 5c = -22 \]
8. \[ \frac{x}{4} - \frac{y}{6} = 0 \]
9. \[ 9q - 8r = 1 \\
    6q + 12r = 5 \]
10. \[ 4c - d = -10 \\
    3c + 5d = 4 \]
11. \[ 5p - 3w = 6 \\
    7w + p = -52 \]
12. \[ \frac{x}{2} + \frac{y}{3} = 2 \]

**B**
13. \[ \frac{2}{x} + \frac{1}{y} = 3 \\
    \frac{3}{x} - \frac{2}{y} = 8 \]

**Hint:** Rewrite: \[ 2 \left( \frac{1}{x} \right) + \frac{1}{y} = 3 \]
\[ 3 \left( \frac{1}{x} \right) - 2 \left( \frac{1}{y} \right) = 8 \]

Let \[ a = \frac{1}{x} \] and \[ b = \frac{1}{y} \]. Solve for \( a \) and \( b \).

Then \[ x = \frac{1}{a} \] and \[ y = \frac{1}{b} \].

14. \[ \frac{3}{2x} - \frac{2}{y} = -7 \]
\[ \frac{9}{2x} + \frac{5}{y} = 12 \]

15. \[ \frac{4}{m} + \frac{3}{2n} = 0 \]
\[ \frac{2}{3m} + \frac{13}{4n} = 1 \]

16. \[ \frac{3}{2m} - \frac{3}{4n} = 1 \]
\[ \frac{4}{3m} - \frac{11}{9n} = 2 \]

17. \[ \frac{x + 2y}{3} - \frac{2x - y}{2} = -3 \]
\[ \frac{3x - 2y}{14} - \frac{3x + 2y}{4} = \frac{3}{4} \]

18. \[ \frac{3x - 2y}{6} + \frac{9x + y}{2} = -1 \frac{2}{3} \]
\[ \frac{2x + 3y}{2} - \frac{7x - 4y}{3} = \frac{4}{9} \]
Solve for \( x \) and \( y \).

19. \[ 2ax - 3by = 13ab \]
   \[ 5ax + 2by = 4ab \]

20. \[ 3rx + 2sy = -19rs \]
   \[ 4rx - 7sy = 23rs \]

PROBLEMS

Solve, using two equations in two variables.

1. Mr. Bates’s rent and savings each month total $100. If he saves $5 more monthly, his savings will be half his rent. What is his rent?

2. A dealer has 30 cars and trucks. When 2 more cars are delivered, he has 3 times as many cars as trucks. How many of each has he?

3. The ages of Eleanor’s uncle and aunt total 68 years, and if her uncle’s age were doubled the difference between their ages would be 40 years. How old is Eleanor’s aunt?

4. The town’s share of a $10,000 legacy was $4,000 less than three times a charity’s share. Find the town’s share.

5. A garden 45 meters square is in the center of a rectangular park 150 meters longer than it is wide. The perimeter of the park is 20 meters less than 4 times that of the garden. Find the park’s dimensions.

6. A frame 30 inches square encloses two matching rectangular prints, each 4 inches longer than twice its width. The perimeter of the frame is 16 inches less than the combined perimeters of the prints. Find the dimensions of each print.

7. When 3 pads and 2 pencils cost $.34, and 4 pads and 5 pencils cost $.57, what is the price of a pad?

8. Alice buys 2 pounds of butter and 3 dozen eggs for $.29. She then buys an extra pound of butter and 2 dozen eggs, for $.17. Find the unit price of butter and eggs.

9. Mr. Kasner buys 3 golf balls and a soft drink for $.21. Later he buys one more golf ball and soft drinks for himself and his three friends for $.12. What is the cost of each ball and each drink?

10. The team buys 7 bats and 5 balls for $.16.95. Later, they buy 3 bats and 6 balls for $.13.05. What is the price of each item?

11. A man wishes to gain $33 each year, so he invests $600, part at 4% and the rest at 6%. How much does he invest at each rate?

12. Mr. Russo needs $113. He invests a sum at 6% and another, $500 more than the first, at 5%. Find the two sums.
10–5 The Substitution Method

You can solve either of a pair of equations for one variable in terms of the other, and use the substitution principle to obtain a third equation with only one variable. This method is sometimes easier to use than the addition and subtraction method.

EXAMPLE. Solve \(x + 4y = 3\) and \(2x - 3y = 17\).

Solution:

1. Solve for \(x\) in the first equation.
   \(x + 4y = 3\)
   \(x = 3 - 4y\)

2. Substitute this expression for \(x\) in the other equation.
   \(2x - 3y = 17\)
   \(2(3 - 4y) - 3y = 17\)

3. Solve for \(y\).
   \(6 - 8y - 3y = 17\)
   \(-11y = 11\)
   \(y = -1\)

Solving for \(x\) and checking are left to you.

The solution set is \{(7, -1)\}, Answer.

WRITTEN EXERCISES

Solve each pair of equations by substitution.

A 1. \(x + 2y = 5\)
   \(x = 3y\)

2. \(y = 5x\)
   \(2x + y = 7\)

3. \(2r - s = 7\)
   \(r - s = 1\)

4. \(3r - s = 20\)
   \(r - s = 2\)

5. \(2r + 3s = 19\)
   \(r - s = 12\)

6. \(3r + 2s = 5\)
   \(r - s = 5\)

7. \(2m - 3n = 5\)
   \(3m - n = 18\)

8. \(3m - 2n = 11\)
   \(2m - n = 8\)

9. \(3x + 2y = 33\)
   \(y - 2x = -8\)

10. \(2x + 3y = 22\)
    \(y - 2x = -6\)

11. \(2r - 3s = 0\)
    \(3r + 3s = 33\)

12. \(3r - 4s = 0\)
    \(2r + s = 33\)

13. \(w + z = 9\)
    \(w - \frac{z}{3} = 5\)

14. \(\frac{1}{3}(r - c) = 4\)
    \(\frac{1}{3}(r + c) = 4\)

15. \(s - t = 8\)
    \(\frac{s}{3} - \frac{t}{2} = 1\)

16. \(x - y = 2\)
    \(\frac{1}{2}(x + y) = 3\)

17. \(w + z = 9\)
    \(w - \frac{z}{4} = 4\)

18. \(\frac{1}{3}(r - c) = 2\)
    \(\frac{1}{3}(r + c) = 2\)
Solve, using two equations in two variables and the substitution method.

1. Half the sum of two numbers is $-\frac{1}{3}$. Half their difference is $\frac{3}{2}$. Find the two numbers.

2. Two men traveled toward each other from points 300 miles apart. When they met, Mr. Wright had traveled 12 miles less than twice as far as Mr. Black. How far had each traveled?

3. Frank and James walked in opposite directions. When they were 9 blocks apart, Frank had gone 1.5 blocks less than twice the distance James had gone. Find the distance traveled by each.

4. The senior play cost students 35¢ and adults 60¢. Five hundred tickets were sold for $254.50. How many adults attended?

5. Mr. Simpson invested $3000, some at 3\%\%\%\%\%\%, the rest at 6\% per year. The return from the 3\%\%\%\%\%\% investment exceeded that from the 6\% investment by $10. How much was invested at each rate?

6. How fast does each operate if two sorters process 1200 cards in one minute, or 1200 cards in two minutes when one breaks down after operating one-half minute?

7. The average of two numbers is $\frac{5}{8}$. One-fourth of their difference is $\frac{1}{15}$. Find the two numbers.

8. One sum at 6\% and another at 4\% yield $57. Were the investments interchanged, their income would increase by $6. Name the sums.

9. A 6\% investment over some years yields an income $150 in excess of itself. If invested at 5\% for half the time, the income is $375 less than the investment. How large is the investment?

10. A grocer blends teas worth 66¢ and 48¢ a pound. If he interchanges the amounts, he saves $6 in a blend of 100 pounds. Find the ratio of the weight of the two teas in the original blend.

**10-6 Graphs of Pairs of Linear Inequalities (Optional)**

Not only can graphs be used to solve systems of equations; they can also be employed in determining the solution set of simultaneous inequalities such as:

\[
y > 3x - 2
\]
\[
x + y < 6
\]

First draw the graphs of \( y = 3x - 2 \) and \( x + y = 6 \) (Figure 10-3) on the same axes. Just as the solution set of \( y > 3x - 2 \)
consists of all points in the half-plane above the graph of \( y = 3x - 2 \), the solution set of \( x + y < 6 \) consists of all points in the half-plane below the graph of \( x + y = 6 \). The intersection (common points) of the two half-planes (black-red) represents the intersection of the two solution sets and contains all, and only those, points satisfying both inequalities. Some points in the common solution set are \((1, 4), (-1, 0),\) and \((-2, -1)\).

**WRITTEN EXERCISES**

Graph each pair of inequalities, indicating their solution set with cross-hatching.

### A

1. \( y < 2x \)
   \( x > 1 \)
   3. \( y \geq 3x \)
   \( y \leq 3 \)
   5. \( y < x - 1 \)
   \( y > 1 - x \)
   7. \( y \geq 3x - 6 \)
   \( y \leq 2x + 4 \)

2. \( y > 2x \)
   \( x > 1 \)
   4. \( y \geq x \)
   \( y \geq -1 \)
   6. \( y < 2x + 1 \)
   \( y > x - 2 \)
   8. \( y \leq 3x - 3 \)
   \( y \geq 2x + 2 \)

### B

9. \( x \geq 1 \)
   \( x \leq 3 \)
   11. \( x + 2y \geq 2 \)
   \( x + 2y \leq 6 \)
   13. \( x + 2y \geq 4 \)
   \( 2x + 4y \leq 8 \)

10. \( y - 3 \leq 0 \)
    \( y + 1 > 0 \)
12. \( 2x - y \leq 2 \)
   \( 2x - y \geq 0 \)
14. \( y \geq 3x - 6 \)
   \( 2y \leq 6x - 12 \)

Graph each inequality in each simultaneous system. Show the solution set of each system as points in a three-way shaded region.

**SAMPLE 1.** \( y \leq x, y \geq 0, x \leq 4 \)

**Solution:**

a. The solution set of \( y \leq x \) consists of points on \( y = x \) and in the diagonally shaded region below it.

b. The solution set of \( y \geq 0 \) consists of points on the x-axis and in the entire region above it.

c. The solution set of \( x \leq 4 \) consists of points on the line \( x = 4 \) and in the entire region to the left of it.

d. The intersection of these three sets is the three-way shaded region (small triangle), including points on its boundaries as well as in its interior.
### Sentences in Two Variables

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<th>15.</th>
<th>( y \geq x )</th>
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<td>( x \geq 0 )</td>
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<td>( y \leq 4 )</td>
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<tr>
<th>16.</th>
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<td>( x &gt; -2 )</td>
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<td>( y &gt; -2 )</td>
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<td>( y &lt; 3 - x )</td>
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<td>( y &gt; -1 )</td>
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<td>( x &gt; 0 )</td>
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<td></td>
<td>( y \geq 0 )</td>
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Solve each pair graphically, and check by solving algebraically.

#### Sample 2.

\[ y = x \]
\[ x + y \geq 2 \]

**Solution:**

a. The heavy red line, including (1, 1), is the graph of the solution set, Answer.

b. Substitute \( x \) for \( y \) in

\[ x + y \geq 2: \quad x + x \geq 2 \]

Solve for \( x \):

\[ 2x \geq 2 \]
\[ x \geq 1 \]

Since \( y = x \):

\[ y \geq 1 \]

\( \therefore \) \( x \geq 1 \) and \( y \geq 1 \)

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<thead>
<tr>
<th>21.</th>
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<td>( y \geq 0 )</td>
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### Additional Problems

#### 10–7 Digit Problems

The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 not only differ in value among themselves, but each one represents different values in different positions within a numeral (76 = 7 \( \cdot \) 10 + 6 \( \cdot \) 1; 67 = 6 \( \cdot \) 10 + 7 \( \cdot \) 1).

All two-digit decimal numerals have the same form: 76 = 7 \( \cdot \) 10 + 6, 67 = 6 \( \cdot \) 10 + 7. In general, 10\( t \) + \( u \), where \( t \) is the tens digit and \( u \) is the ones (units) digit, and \( t \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and \( u \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). If you wish to represent a number with the same digits in reverse order, you write 10\( u \) + \( t \). In either case, the sum of the digits is represented by \( t + u \).
EXAMPLE. The sum of the digits in a two-digit numeral is 10. When the digits are interchanged, the number designated is 18 more than the original number. Find the original number.

Solution:

Let \( t \) = the tens digit in the original numeral
and \( u \) = the units digit in the original numeral.

Then \( 10t + u \) = the original number
and \( 10u + t \) = the new number.

The sum of the digits is 10. \( t + u = 10 \)

The new number is 18 more than the original. \( 10u + t = 18 + 10t + u \)

\( (10u + t) - (10t + u) = 18 \)

\( 9u - 9t = 18 \)

\( u - t = 2 \)

\( 2u = 12 \) \( \Rightarrow u = 6 \)

\( 6 - t = 2 \) \( \Rightarrow t = 4 \)

Is the sum of the digits 10? \( 4 + 6 = 10 \checkmark \)

Is 64 eighteen more than 46? \( 64 - 46 = 18 \checkmark \)

\( \therefore \) The number is 46, Answer.

PROBLEMS

1. The sum of the digits of a two-digit numeral is 13. If 27 is added to the number, the result is the number with its digits interchanged. Find the original number.

2. The sum of the digits of a two-digit numeral is 12. If the order of the digits is reversed, the result names a number exceeding the original by 36. Find the original number.

3. The sum of the digits of a two-digit numeral is 9. The value of the number is 12 times the tens digit. Find the number.

4. The sum of the digits of a two-digit numeral is 12. The number with its digits interchanged is 13 times the original units digit. Find the original number.

5. The units digit of a two-digit numeral exceeds twice the tens digit by 1. The sum of the digits is 7. Find the number.

6. The units digit of a two-digit numeral is twice the tens digit. The sum of the digits is 12. Find the number.
7. The sum of the digits of a two-digit numeral is 8. The number with the digits interchanged is 7 times the tens digit of the original number. Find the original number.

8. The sum of the digits of a two-digit numeral is 6. The number with the digits in reverse order is 12 times the original units digit. Find the original number.

9. A teller mistakenly reversed the two digits in the face amount of a check, overpaying $9. If the sum of the digits was 9, determine the amount for which the check was drawn.

10. Find a number less than 100 whose tens digit exceeds twice its units digit by 1 and whose digits in reverse order give a number 4 more than 3 times their sum.

11. Find a three-digit number whose tens digit is 3 times its hundreds digit and twice its units digit, and whose digits total 11.

12. A three-digit number is 198 more than itself reversed. The hundreds digit is 3 times the tens digit, and the sum of the digits is 19. Find the original number.

13. If a two-digit number is divided by its tens digit, the quotient is 11 and the remainder is 4. If the number with its digits interchanged is divided by its original units digit, the quotient is 10 and the remainder is 5. Find the original number.

14. Show that the difference between a three-digit number and the number with the order of the digits reversed is always divisible by 99.

Problems 15–18 refer to two-place decimal fractions between 0 and 1.

15. The sum of the digits of a two-place fraction is 9. When its digits are reversed, the new fraction exceeds the original by .09. Find the original fraction.

16. When the digits of a two-place fraction are reversed, the new fraction is \( \frac{4}{9} \) the original fraction. If the sum of the digits is 9, find the original fraction.

17. The sum of the digits of a two-place fraction is 13. The fraction with its digits reversed is .04 less than twice itself. Find the original fraction.

18. The tenths digit of a two-place fraction exceeds twice the hundredths digit by 1. If the digits are reversed, the original is .01 less than twice the new fraction. Find the original fraction.

10–8 Motion Problems

You can solve some motion problems conveniently by using two equations with two variables.
EXAMPLE. With a tail wind, a plane flew 180 miles in half an hour. With no change in the wind, the return trip took 40 minutes. Find the speed of the wind and the plane's rate in still air.

Solution:

Let \( s \) = the rate, in m.p.h., of the plane in still air

and \( v \) = the speed, in m.p.h., of the wind.

\[
\begin{array}{c|c|c|c}
 r & t & d \\
\hline
\text{With tail wind} & s + v & \frac{1}{2} & 180 \\
\text{Against head wind} & s - v & \frac{3}{2} & 180 \\
\end{array}
\]

\[ \frac{1}{2}(s + v) = 180 \]

\[ \frac{3}{2}(s - v) = 180 \]

Solve the equations.

Check the roots.

The rate of the plane is 315 miles per hour.

The speed of the wind is 45 miles per hour.

Answer.

PROBLEMS

1. A motorboat covers 6 miles in 45 minutes. The return trip takes 1\( \frac{1}{2} \) hours. Find the boat's speed in still water.

2. A cyclist rode 1 mile in 3 minutes with the wind, and returned in 4 minutes against the wind. Find his speed without a wind.

3. Mr. Hanson flew 450 miles against the wind in 1\( \frac{3}{4} \) hours. The return trip took 1\( \frac{1}{4} \) hours, with no wind change. What was the speed of the wind?

4. It required 1\( \frac{1}{2} \) hours for a 540-mile plane trip and 1\( \frac{1}{8} \) hours for the return, in unchanged wind. What would have been the speed of the plane without wind?

5. Larry took 36 minutes to row 3 miles. When he returned, he took 90 minutes. What was the river's current?

6. A canoeist paddles 6 miles downstream in 40 minutes and returns in 3 hours. At the same rate, how fast does he go in still water?

7. A man rows 4 miles upstream and back in 2\( \frac{3}{4} \) hours. He rows 1 mile against the current in the time he rows 3 miles with it. At what rate does he row? What was his average rate of travel?

8. A cyclist takes 3\( \frac{1}{2} \) hours on a 30-mile round trip. On the return trip against the wind he did 2 miles in the time that he did 5 miles on the trip out. Find his average rate. What was the wind's speed?
9. A round-trip flight of 1105 miles takes $7\frac{1}{2}$ hours. The part of the flight with the wind takes one hour less than the other half of the trip. Find the speed of the plane in still air and the speed of the wind.

10. A steamer sails a distance up a river in the time it sails twice that distance downstream. If the speed of the steamer is $s$ and that of the current is $c$, find the relationship between $s$ and $c$.

### 10–9 Age Problems

You can simplify the solution of age problems by using two variables and by organizing the facts in chart form.

**Example.** Four years ago, Polly was $\frac{2}{3}$ as old as Paul. Four years from now, she will be $\frac{4}{5}$ as old as he. How old are both now?

**Solution:**

<table>
<thead>
<tr>
<th>Time</th>
<th>Polly</th>
<th>Paul</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 years ago</td>
<td>$x - 4$</td>
<td>$y - 4$</td>
</tr>
<tr>
<td>This year</td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>4 years hence</td>
<td>$x + 4$</td>
<td>$y + 4$</td>
</tr>
</tbody>
</table>

\[ x - 4 = \frac{2}{3}(y - 4) \]

\[ x + 4 = \frac{4}{5}(y + 4) \]

Solve these equations and check.

Polly is 12 years old.  
Paul is 16 years old.  

**Answer.**

**Problems**

1. A man is 5 times as old as his son. In five years he will be 3 times as old as his son will be. How old is the son now?

2. Ruth's father is 7 times as old as Ruth is. One year ago he was 9 times as old as Ruth was. Find Ruth's present age.

3. Three years ago, Joe's age was 1 year more than twice Jill's. Six years from now, his will be 10 years more than half her age. How old is Joe?

4. Five years ago, Jerry was $\frac{2}{3}$ as old as Jeff. Ten years from now, he will be $\frac{5}{6}$ as old as Jeff. How old is each now?

5. Janet is $\frac{3}{5}$ as old as Phil. Four years ago she was $\frac{3}{4}$ as old as he. How old is each?

6. A man said, "My son is twice as old as my daughter. My wife is 3 times as old as the combined ages of both, and I am as old as my wife and son together. My mother, who is as old as all of us together, is 69." How old is the son?
7. Ray said, “If I were $\frac{3}{5}$ as old as I am and Joyce were $\frac{2}{3}$ as old as she is, we would be 3 years older together than I am alone. But if I were $\frac{2}{5}$ as old as I am and Joyce were $\frac{1}{2}$ as old as she is, together we would be 3 years younger than I am alone.” How old is Ray?

8. Mary is twice as old as Jane was when Mary was as old as Jane is now. Find the relationship between Mary’s present age ($M$) and Jane’s ($J$).

9. Mary is twice as old as Jane was when Mary was as old as Jane is now. In 3 years Mary will be 3 times as old as Jane was 4 years ago. Find their present ages.

10. Bob is twice as old as his brother will be when Bob is 8 times as old as his brother is now. Find the relationship between Bob’s present age ($B$) and his brother’s ($b$).

10–10 Problems about Fractions

Among the problems you can solve by using two variables are those about fractions, like this one:

**EXAMPLE.** A fraction has a value of $\frac{4}{5}$. If the numerator is increased by 10 and the denominator is decreased by 13, the resulting fraction is equal to twice the reciprocal of the original fraction. Find the original fraction.

**Solution:**

1. Let $\frac{n}{d}$ = the original fraction.

2. $\frac{n + 10}{d - 13} = 2 \cdot \frac{5}{4}$ or $\frac{n + 10}{d - 13} = \frac{5}{2}$

3. $2(d - 13) \cdot \frac{n + 10}{d - 13} = 2(d - 13) \cdot \frac{5}{2}$

   $5n = 4d$

   $n = \frac{4d}{5}$

   $2n + 20 = 5d - 65$

   $2 \left( \frac{4d}{5} \right) + 20 = 5d - 65$

   $8d + 100 = 25d - 325$

   $-17d = -425$

   $d = 25$

4. $n = \frac{4(25)}{5}$

   $n = 20$

Check is left to you.

$\therefore$ The original fraction is $\frac{20}{25}$, Answer.
Using two variables, find the original fraction.

1. The denominator is 3 more than the numerator. If each is increased by 1, the value of the resulting fraction is \( \frac{2}{3} \).

2. The denominator is 5 more than the numerator. If 1 is added to each, the value of the resulting fraction is \( \frac{1}{2} \).

3. The denominator exceeds the numerator by 3. If 1 is subtracted from the numerator, and the denominator is unchanged, the resulting fraction has a value of \( \frac{1}{2} \).

4. The denominator exceeds the numerator by 5. If 1 is subtracted from the numerator, a fraction is obtained whose value is \( \frac{1}{3} \).

5. A fraction has a value of \( \frac{3}{4} \). When 7 is added to its numerator, the resulting fraction equals the reciprocal of the original fraction.

6. A fraction's value is \( \frac{3}{5} \). When its numerator is increased by 10, the new fraction equals the reciprocal of the value of the original fraction.

7. The two digits in the numerator of a fraction whose value is \( \frac{3}{4} \) are reversed in its denominator. The reciprocal of the fraction is the value of the fraction obtained when 11 is added to the original numerator and 22 is subtracted from the original denominator.

8. The numerator equals the sum of the two digits in the denominator. The value of the fraction is \( \frac{1}{4} \). When both numerator and denominator are increased by 3, the resulting fraction has a value of \( \frac{1}{4} \).

9. The two digits in the numerator of a fraction are reversed in its denominator. If 11 is added to the numerator and 7 to the denominator, the value of the resulting fraction is \( \frac{1}{2} \). The fraction whose numerator is the sum and whose denominator is the difference of the units and tens digits equals 2.

10. The numerator is a three-digit number whose hundreds digit is 2. The denominator is the number with the digits reversed. If 16 is added to the denominator, the value of the fraction is \( \frac{1}{2} \). If 111 is subtracted from the numerator, the resulting fraction equals \( \frac{1}{4} \).

List all possible members of each solution set.

11. The numerator is a two-digit number and the denominator is that number with the digits reversed. The value of the fraction is \( \frac{3}{7} \).

12. The numerator of a fraction whose value is \( \frac{3}{4} \) is a three-digit number whose tens digit is 0. The denominator contains the same digits in reverse order.
You have heard of the Great Wall of China. Have you heard of the emperor who used it as a concentration camp for scholars? His name was Shih Huang Ti, and he came to the throne in 221 B.C.

Shih Huang Ti had delusions of grandeur. He was determined to be remembered as the greatest of all emperors. Among other things, he wished to be famous as the ruler in whose reign knowledge increased beyond measure.

He took a strange way to get this last wish. He ordered all books on certain topics — including mathematics and related subjects — to be burned! Apparently he reasoned this way: “If in years to come, there is no mathematics book in all China that was written before my reign, but many books written during my reign, then people will think that mathematics began with me.”

Shih Huang Ti knew that scholars did not willingly burn books; so he fixed a penalty for failure to obey his order: branding and four years hard labor on the Great Wall. Even so, 460 scholars banded together to defy the emperor. But Shih Huang Ti was more powerful than they; he had them buried alive.

So the books were burned. And the emperor called for new books. Of course, new books were written. Mathematicians who were neither slaving on the Wall nor buried in the sod worked feverishly to record their knowledge for the use of future generations. One of the books they rewrote is called Arithmetic in Nine Sections. There is reason to believe that this book originally was written before 1000 B.C. It is more algebra than arithmetic; it includes many topics that you will study this year and some that are more advanced than a first course in algebra.

This impression from a wall carving shows an attempt to assassinate Shih Huang Ti, the emperor who tried to advance knowledge by destroying books.
Chapter Summary

Inventory of Structure and Method

1. The graphs of consistent and independent equations intersect in one point, which can be determined by graphic or algebraic methods. To solve such a pair of simultaneous linear equations, graph them and find the point of intersection of the lines.

2. When the coefficients of one variable have the same absolute value, use the addition or subtraction property of equality to eliminate that variable; then solve for the other variable. When the coefficients of both variables have different absolute values in the two equations, use the multiplication property of equality before adding or subtracting.

3. A pair of simultaneous linear equations in the same variables can be solved by applying the substitution principle.

4. To solve a pair of linear inequalities, graph both on one set of axes; the intersection of their regions contains all points which satisfy both.

5. By using two variables to form two equations, you can solve digit problems, motion problems, age problems, and problems about fractions.

Vocabulary and Spelling

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>intersection</td>
<td>367</td>
</tr>
<tr>
<td>simultaneous equations</td>
<td>368</td>
</tr>
<tr>
<td>inconsistent equations</td>
<td>368</td>
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<tr>
<td>consistent equations</td>
<td>368</td>
</tr>
<tr>
<td>dependent equations</td>
<td>368</td>
</tr>
<tr>
<td>independent equations</td>
<td>368</td>
</tr>
<tr>
<td>addition and subtraction</td>
<td>370</td>
</tr>
<tr>
<td>substitution method</td>
<td>378</td>
</tr>
<tr>
<td>decimal numeral</td>
<td>381</td>
</tr>
</tbody>
</table>

Chapter Test

10-1 1. Solve $3x + y = 10$ and $x + 2y = 0$ graphically.

10-2 Use addition or subtraction to solve each pair of equations.

2. $x + y = 37$  
   $x - y = 8$

3. $5r - 4w = 14$

4. $2s + 3t = 122$
   $2s + t = 78$

5. Marie scored 40 on a test, receiving 3 points for each right answer and losing 1 point for each wrong answer. Had 4
points been awarded for each correct answer and 2 points been deducted for each incorrect answer, Marie would have scored 50. How many questions were in the test?

Solve by using multiplication with addition or subtraction.

10-4

6. \( x - y = 26 \)

7. \( 8w + 5z = 31 \)

3\(x - 8y = 3 \)

3\(w - 2z = 0 \)

10-5

Solve by substitution.

8. \( 3m - 2n = 1 \)

9. \( 2r + s = 3; 3r - 2s = 57 \)

\( m - n = 2 \)

10. \( r + 2s = 30 \)

\( 2r - s = -45 \)

10-6

11. Solve \( y < 2x + 3 \) and \( x + y > 3 \) graphically.

10-7

12. The sum of the two digits of a number is 15. The digits are reversed when 9 is added to the number. Find the number.

10-8

13. Walter's boat goes 48 miles upstream in 4 hours and returns in 3 hours. How fast is the current?

10-9

14. Henry is 20 years older than George. In 5 years George will be half as old as Henry is then. Use two variables to find their present ages.

10-10

15. If 3 is added to the numerator and denominator of a fraction with a value of \( \frac{2}{3} \), the resulting fraction equals \( \frac{3}{5} \). Use two variables to find the original fraction.

Chapter Review

10-1 The Graphic Method

1. In solving a pair of linear equations having the same variables, both graphs are drawn on the ___ ___.

2. The point at which the two lines cross represents the ___ ___ which satisfies both equations.

3. The coordinates of the point of intersection of a pair of linear equations are a ___ of any pair of linear equations whose graphs pass through this point.
Questions 4–6 refer to the graphs of $x + y = 8$ and $x - y = -2$.

4. Give the coordinates of a point which satisfies $x + y = 8$ but not $x - y = -2$.

5. Name a number pair satisfying $x - y = -2$, but not $x + y = 8$.

6. The point $\_\_\_\_$ satisfies both equations.

10-2 The Addition and Subtraction Method

Solve each pair of equations.

7. $2x + y = 11$  
8. $3r - t = 10$  
9. $5m - 3n = 11$

$2x - y = 9$  
$2r - t = 40$  
$5m + 3n = 29$

10. If the variables in a pair of equations are replaced by numbers for which both equations are true, the difference of the left members $\_\_\_\_$ the difference of the right members.

10-3 Problems with Two Variables

Translate each statement into a pair of equations.

11. One number is 5 less than twice another, and their sum is 100.

12. The length of a rectangle is 5 times its width. The perimeter is 72 feet.

Solve, using two variables and two equations.

13. A 7-foot board is cut in two. Three times the longer part would be 9 feet more than five times the shorter. Find both lengths.

10-4 Multiplication in the Addition and Subtraction Method

Solve by using multiplication with addition or subtraction.

14. $x + 3y = 26$  
$3x + 2y = 29$  

15. $7r - 10s = 1$  
$56r - 40s = 28$

16. $9w + 5z = 33$  
$6w - 7z = -9$
10-5 The Substitution Method

Solve by substitution.

17. \(2u + 3v = 7\) \quad 19. \(6p + 3q = -3\)
   \(u - v = 11\) \quad \(p + 2q = 10\)

18. \(3w - 2z = 38\) \quad \(2w - z = 18\)

10-6 Graphs of Pairs of Linear Inequalities (Optional)

20. The graph of a linear inequality in two variables is represented by a ____.

Questions 21–22 refer to the graphs of \(x + y \geq 5\) and \(y \leq 2x - 4\).

21. Give the coordinates of a point which satisfies \(x + y \geq 5\) but not the other inequality.

22. Name a number pair satisfying \(y \leq 2x - 4\) but not \(x + y \geq 5\).

10-7 Digit Problems

23. The value represented by a digit depends on its ___ in a number.

24. In 31, the tens digit is ___; the units digit is ___.

25. The tens digit of a two-digit numeral exceeds the units digit by 2. The sum of the digits is 16. What is the number?

26. The sum of the two digits of a numeral is 12. With its digits reversed, the number is 36 less than it was. Find the original number.

10-8 Motion Problems

27. If you row downstream, the rate is ___ than in still water.

28. An airplane cruises at 250 m.p.h. It flies in a head wind of 45 m.p.h. at ___ m.p.h.
29. Norman cycled 15 miles in 1\(\frac{1}{4}\) hours into the wind. With the wind behind him, he made the trip in \(\frac{3}{4}\) hour. What would Norman’s speed be without wind?

30. A steamer goes 91 miles up a river in 6\(\frac{1}{2}\) hours. The return trip is scheduled for 3\(\frac{3}{5}\) hours. What is the rate in still water?

10-9 Age Problems

Solve, using two variables and two equations.

31. Allen is \(\frac{3}{5}\) as old as Bertrand. In 4 years he will be \(\frac{5}{6}\) as old as Bertrand. How old is each boy now?

32. Three years ago, June was \(\frac{3}{7}\) Jack’s age. In two years, June will be \(\frac{2}{3}\) his age. How old are they?

33. Mr. Granger is 10 years more than twice as old as Tim. Last year he was 4 years less than 3 times as old as Tim. How old is each?

10-10 Problems about Fractions

34. If you add 1 to its denominator, a certain fraction becomes equal to \(\frac{5}{8}\). If you subtract 4 from its numerator, that fraction becomes equal to \(\frac{2}{3}\). Find the fraction.

35. A fraction’s value is \(\frac{3}{7}\). If its numerator loses 1 and its denominator gains 5, the reciprocal of the result equals 5. Find the fraction.

Extra for Experts

Diophantine Equations

Can you find a solution of \(x + 2y = 3\) in the set of directed numbers? Of course you can; \((0, 1.5)\) is a solution, as is \((-1, 2)\). In fact, the number of solutions is unlimited. Can you see, however, that if the replacement set for \(x\) and for \(y\) is \{positive integers\}, the only solution is \((1, 1)\) since \(y = \frac{3 - x}{2}\) ?

Because a system of one or more equations with more variables than equations may have an infinite solution set, the equations are called indeterminate. When the replacement sets of the variables are restricted to subsets of \{integers\}, the equations are called Diophantine (di’o-fan-tin) equations, after the Greek algebraist Diophantus. The solutions must be so restricted in many practical situations:
EXAMPLE. How many quarters and how many dimes together make up $1.60?

Solution: Let $d$ = the number of dimes and $q$ = the number of quarters, where $d$ and $q \in \{\text{nonnegative integers}\}$.

$$10d + 25q = 160 \rightarrow 2d + 5q = 32 \rightarrow q = \frac{2(16 - d)}{5}$$

For $q$ to be a nonnegative integer, $16 - d$ must represent a nonnegative multiple of 5:

$$16 - d = 5t, \text{ where } t \in \{\text{nonnegative integers}\}.$$ 

Substituting $5t$ in the expression for $q$,

$$q = \frac{2 \cdot 5t}{5} = 2t.$$ 

Thus, $d$ and $q$ are expressed in terms of $t$ by the equations, $d = 16 - 5t$ and $q = 2t$, where $t \in \{\text{nonnegative integers}\}$.

Specific values for $t$ give corresponding pairs $(d, q)$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>16</td>
<td>11</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$q$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Additional choices of $t$ result in negative values of $d$, so that the chart gives all the acceptable pairs, $(d, q)$. Check these values in the problem.

Questions

1. A 55-yard cloth is cut into pieces $3\frac{1}{2}$ and $4\frac{1}{2}$ yards long, without remnants. How many pieces of each length are there?

2. If a baker arranges his almost 6 dozen loaves of bread in groups of 4, 1 is left over; in groups of 5, 4 are left over. How many loaves has he?

3. Find the smallest positive integer which when divided by 2, 3, and 7 leaves remainders of 1, 2, and 6, respectively.

4. This “Chinese Problem of a Hundred Fowl,” dates at least to the sixth century: If a rooster is worth 5 yuan, a hen is worth 3 yuan, and 3 chicks are worth 1 yuan, how many of each, 100 in all, would be worth 100 yuan? Assume that at least 5 roosters are required.
5. If at least one coin of each kind is used, how can you change a dollar into 15 coins, each less than a quarter in value?

6. Solve in positive integers:
   \[
   \begin{align*}
   4a - 11b + 12c &= 22 \\
   a + 5b - 4c &= 17
   \end{align*}
   \]

7. How many combinations of three-cent and five-cent stamps can you buy with exactly fifty cents, getting at least one of each kind?

8. A compound of carbon, hydrogen, and oxygen has a molecular weight of 46. Using atomic weights of 12 for carbon, 1 for hydrogen, and 16 for oxygen, what are the possible formulas for the compound in the form \(C_xH_yO_z\)?

### Just for Fun

#### Make a Friend Repeat Himself

You can make a person repeat himself, whether or not he wishes to do so.

<table>
<thead>
<tr>
<th>You say</th>
<th>If he selects 28</th>
<th>If he selects 63</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply any two-digit number by 15.</td>
<td>(15 \times 28 = 420)</td>
<td>(15 \times 63 = 945)</td>
</tr>
<tr>
<td>Now multiply by 7.</td>
<td>(7 \times 420 = 2940)</td>
<td>(7 \times 945 = 6615)</td>
</tr>
<tr>
<td>Subtract 4 times the original number from the product. Why are you repeating yourself?</td>
<td>(4 \times 28 = 112) (2940 - 112 = 2828)</td>
<td>(4 \times 63 = 252) (6615 - 252 = 6363)</td>
</tr>
<tr>
<td>Let's start over: Multiply the original number by 13.</td>
<td>(13 \times 28 = 364)</td>
<td>(13 \times 63 = 819)</td>
</tr>
<tr>
<td>Now multiply by 8.</td>
<td>(8 \times 364 = 2912)</td>
<td>(8 \times 819 = 6552)</td>
</tr>
<tr>
<td>Subtract three times the original number — You're repeating yourself again!</td>
<td>(3 \times 28 = 84) (2912 - 84 = 2828)</td>
<td>(3 \times 63 = 189) (6552 - 189 = 6363)</td>
</tr>
</tbody>
</table>

The person repeats himself because he is multiplying by 101. Any two-digit number multiplied by 101 gives a four-digit number containing the same digits in the same order.
The Real Numbers

The ancient Greeks thought that all lengths should be measurable in terms of integers or common fractions. However, they found several examples in which they were not. You already know about one of these.

If the diameter of a circle is one foot, what is the circumference? Is it $3\frac{1}{7}$, 3.14, or 3.1416? No matter what common fraction you use, it will not measure the circumference exactly. Likewise, the diagonal of a one-foot square can be given only as a decimal approximation.

The architect who designed the public auditorium of Pittsburgh (top picture) used an approximation of $\pi$. A space navigator needs even greater accuracy in his approximation of $\pi$, so he takes advantage of a computer's talents (bottom picture).

Many of the problems of mathematics and science cannot be solved with only common fractions. You need a different kind of number about which you will learn in this chapter.

THE SYSTEM OF RATIONAL NUMBERS

11–1 The Nature of Rational Numbers

You define a number system by giving a set of numbers and telling how to add and multiply the members. Thus, the numbers you first met in arithmetic, \{1, 2, 3, 4, \ldots\}, form the system of positive integers, which is closed under addition and multiplication.

The next numbers you met were elements of the set of common fractions. The system of positive fractions and integers is closed under addition, multiplication, and division. When you extended your idea of number to include zero and the negative integers and fractions, you had a system of numbers closed under subtraction, too.

The resulting set of positive and negative integers and fractions, together with zero, is the set of rational numbers. A rational number is any number which can be expressed as the ratio of two integers. Because the system of rational numbers is closed under addition, subtraction, multiplication, and division (except by zero), these are called the rational operations.
A rational number can be expressed in an unlimited number of ways:

\[
\begin{align*}
0 = \frac{0}{7} &= \frac{0}{5} \\
6 = \frac{6}{1} &= \frac{-18}{-3} \\
3 = \frac{3}{4} &= \frac{-3}{-8} \\
\frac{5}{17} &= \frac{-5}{-17} = \frac{20}{68} \\
1.8 = \frac{18}{10} &= \frac{27}{15} \\
23\% = \frac{23}{100} &= \frac{115}{500}
\end{align*}
\]

Furthermore, you always can tell which of two rational numbers is the greater, by writing them with the same positive denominator and comparing their numerators.

\[-\frac{1}{2} > -\frac{3}{2} \text{ because } -1 > -3; \quad \frac{3}{2} > \frac{5}{4} \text{ because } \frac{6}{5} > \frac{5}{4} \text{ since } 6 > 5\]

This test can be developed in another form:

Let \(a\) and \(b\) be integers and \(c\) and \(d\) be positive integers; if one of the following statements is true, the others are true.

\[
\frac{a}{c} > \frac{b}{d} \quad \quad \frac{a}{c}(cd) > \frac{b}{d}(cd) \quad \quad ad > bc
\]

\[
\frac{5}{6} > \frac{3}{4} \text{ because } 5(4) > 3(6) \quad \quad -\frac{1}{2} > -\frac{3}{2} \text{ because } -1(2) > -3(2)
\]

For each integer there is a next larger one. This is not true for the set of rational numbers; it possesses the property of density:

Between every pair of different rational numbers there is another rational number.

**EXAMPLE.** Find a rational number between \(\frac{3}{4}\) and \(\frac{5}{6}\).

**Solution:**

1. Find the difference of the numbers. \(\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}\)

2. Add half this difference to the smaller. \(\frac{3}{4} + \frac{1}{2}(\frac{1}{12}) = \frac{19}{24}\)

3. Check. Is \(\frac{3}{4} < \frac{19}{24} < \frac{5}{6}\)?

\[
3(24) < 19(4) \text{ and } 19(6) < 5(24) \quad \quad 72 < 76 \checkmark \quad 114 < 120 \checkmark
\]

\[\therefore \text{ A rational number between (it is halfway between) } \frac{3}{4} \text{ and } \frac{5}{6} \text{ is } \frac{19}{24}, \text{ Answer.}\]
Another rational number between \( \frac{3}{4} \) and \( \frac{5}{6} \) is \( \frac{7}{9} \):

\[
\frac{3}{4} + \frac{1}{3} \left( \frac{1}{12} \right) = \frac{27}{36} + \frac{1}{36} = \frac{28}{36} = \frac{7}{9}.
\]

Check that \( \frac{3}{4} < \frac{7}{9} < \frac{5}{6} \).

\[
\begin{array}{cccc}
\frac{3}{4} & \frac{7}{9} & \frac{19}{24} & \frac{5}{6} \\
\frac{27}{36} & \frac{28}{36} & \frac{29}{36} & \frac{30}{36}
\end{array}
\]

Do you see that the number of rational numbers between \( \frac{3}{4} \) and \( \frac{5}{6} \) is unlimited? The property of density implies that between every pair of rational numbers there is an infinite set of rational numbers.

**Oral Exercises**

Express each number as a quotient of integers.

**Sample.** 4.3

| 1. 2.2          | 5. .07       | 9. -7      | 13. \((-3)(-\frac{3}{4})\) |
| 2. 32.5        | 6. .005      | 10. -2     | 14. \(\frac{1}{4} \div \frac{7}{3}\) |
| 3. \(-4\frac{1}{2}\) | 7. 12%   | 11. \(\frac{3}{5} + (-\frac{5}{6})\) | 15. \(1\frac{1}{4} + \frac{3}{4}\) |
| 4. \(-3\frac{1}{5}\) | 8. 15%   | 12. \(\frac{4}{7} - \frac{6}{7}\) | 16. \(0 \cdot (-\frac{7}{2})\) |

State which number is the greater.

| 17. \(\frac{4}{5}, \frac{8}{9}\) | 19. \(-5, \frac{1}{3}\) | 21. \(-\frac{4}{7}, -\frac{9}{7}\) | 23. 5, \(\frac{3}{4}\) |
| 18. \(\frac{17}{6}, \frac{13}{8}\) | 20. \(\frac{1}{2}, -3\) | 22. \(-\frac{5}{2}, -\frac{3}{2}\) | 24. 9, \(\frac{55}{6}\) |

If \(a > 0\), and \(x \in \{1, 2, 3, 4\}\), taking each value in succession, which fractions grow larger and which, smaller?

| 25. \(\frac{x}{1}\) | 27. \(\frac{a + 1}{x + a}\) | 29. \(\frac{a}{2x - 1}\) | 31. \(\frac{3a}{3x - 2}\) |
| 26. \(\frac{1}{x}\) | 28. \(\frac{x + 3}{a}\) | 30. \(\frac{a}{5 - x}\) | 32. \(\frac{2a}{9 - 2x}\) |

**Written Exercises**

Replace the ? by =, <, or > to make a true statement.

| 1. \(\frac{8}{15}\) ? \(\frac{7}{15}\) | 3. \(\frac{19}{12}\) ? \(\frac{56}{35}\) | 5. \(\frac{200}{105}\) ? \(\frac{240}{120}\) | 7. \(-17\frac{1}{3}\) ? \(-\frac{400}{12}\) |
| 2. \(-\frac{23}{45}\) ? \(-1\frac{2}{25}\) | 4. \(-\frac{4}{49}\) ? \(-\frac{53}{56}\) | 6. \(\frac{18\frac{8}{16}}{10\frac{2}{16}}\) ? \(\frac{18\frac{4}{16}}{21\frac{6}{16}}\) | 8. \(\frac{3\frac{1}{16}}{21\frac{1}{8}}\) |
Arrange the members of each set in increasing order.

9. \( \left\{ \frac{1}{2}, -\frac{4}{5}, \frac{10}{11} \right\} \)  
10. \( \left\{ \frac{2}{3}, -\frac{3}{5}, \frac{9}{14} \right\} \)  
11. \( \left\{ \frac{3}{5}, \frac{2}{3}, \frac{4}{5} \right\} \)  
12. \( \left\{ \frac{2}{7}, \frac{10}{3}, \frac{3}{5}, \frac{5}{2} \right\} \)  
13. \( \left\{ -\frac{11}{4}, -\frac{8}{3}, -\frac{19}{6} \right\} \)  
14. \( \left\{ -\frac{2}{3}, -\frac{9}{6}, -\frac{7}{6}, -\frac{17}{12} \right\} \)

Find the number halfway between these numbers.

15. \( \frac{1}{12}, 1 \)  
16. \( \frac{9}{14}, \frac{5}{7} \)  
17. \( \frac{1}{100}, \frac{1}{10} \)  
18. \( \frac{1}{100}, -\frac{1}{1000} \)  
19. \( -\frac{6}{5}, -\frac{7}{5} \)  
20. \( 4\frac{2}{3}, 5\frac{1}{8} \)

21. Find the number one-third of the way from \( \frac{7}{8} \) to \( 1\frac{1}{4} \).
22. Find the number one-fifth of the way from \( -\frac{3}{8} \) to \( -\frac{1}{4} \).
23. Show that the number halfway between \( a \) and \( b \) is \( \frac{a + b}{2} \).
24. What number is one-third of the way from \( p \) to \( q \) (a) if \( p < q \), and (b) if \( p > q \)?

Explain why each statement is true.

25. Three is the smallest integer greater than 2.
26. There is no smallest rational number greater than 2.

In each case, tell whether the statement is true with the blanks filled as indicated.

27. If \( m \) and \( n \) are different \((x)\), there are as many \((y)\) between \( m \) and \( n \) as you please.
   a. \( (x) \) rational numbers; \( (y) \) rational numbers
   b. \( (x) \) integers; \( (y) \) integers
   c. \( (x) \) rational numbers; \( (y) \) integers
   d. \( (x) \) integers; \( (y) \) rational numbers

In Exercises 28–31 \( a, b, c, \) and \( d \) are nonzero integers. Explain why these expressions represent rational numbers.

28. \( \frac{a}{c} + \frac{b}{d} \)  
29. \( \frac{a}{c} - \frac{b}{d} \)  
30. \( \frac{a}{c} \cdot \frac{b}{d} \)  
31. \( \frac{a}{c} \div \frac{b}{d} \)

11–2 Decimal Forms of Rational Numbers

In arithmetic you learned to change common fractions to decimals and decimals to common fractions. To change a common fraction to a decimal, you carry out the indicated division.

\[ \frac{3}{16} = 3 \div 16 \quad \frac{1}{12} = 1 \div 12 \quad \frac{7}{11} = 7 \div 11 \]
A decimal with a finite number of places, like \( .1875 \), above, is called terminating, ending, or finite. Such a decimal represents a rational number; for example, \( .1875 \) equals \( \frac{1875}{10000} \) or \( \frac{3}{16} \).

In the division of 1 by 12, however, you never have a remainder of 0, but the remainder 4 repeats step after step, and 3 repeats in the quotient. A decimal which continues indefinitely is called nonterminating or unending. Moreover, the decimal for \( \frac{1}{12} \) is repeating or periodic, because the same digit (or block of digits) repeats unendingly. You may write

\[
\frac{1}{12} = .08333 \ldots \quad \text{or} \quad \frac{1}{12} = .0\overline{83},
\]

where the dots and the bar indicate continue unendingly.

When 7 is divided by 11, the successive remainders are 4, 7, 4, 7, \ldots , and the quotient is a repeating decimal.

\[
\frac{7}{11} = .636363 \ldots , \quad \text{or} \quad \frac{7}{11} = .\overline{63}.
\]

When you divide an integer by 11, the remainder at each step is from \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Within no more than ten steps after only zeros are left in the dividend, either the remainder is 0 and the division terminates, or a sequence of other remainders repeats unendingly. This sort of reasoning leads to the following result.

The decimal form of any rational number \( \frac{r}{s} \) either terminates or eventually repeats in a block of fewer than \( s \) digits.

Conversely, you can show that this statement is true.

All terminating decimals and all repeating decimals represent rational numbers which can be written in the form \( \frac{r}{s} \).
The preceding conversion of .1875 to $\frac{3}{16}$ shows that a terminating decimal can be written as a common fraction. The following examples show that repeating decimals can also be written as common fractions.

**EXAMPLE 1.** Write $.3\overline{24}$ as a common fraction.

**Solution:**

\[
100N = 32.42424
\]
\[
N = .32424
\]
\[
99N = 32.10000
\]
\[
N = \frac{32.1}{99} = \frac{107}{330}
\]

**EXAMPLE 2.** Write $.1\overline{25}$ as a common fraction.

**Solution:**

\[
1000N = 125.125
\]
\[
N = .125
\]
\[
999N = 125.000
\]
\[
N = \frac{125}{999}
\]

If the number of digits in the block of repeating digits is $p$, multiply the given number $N$ by $10^p$, producing a number with the same repeating block as the given number, so that subtracting the given number from it yields a terminating decimal.

It often is convenient to break off a lengthy decimal, leaving an approximation of the number represented. You may write

\[\frac{1}{12} \doteq .08333 \quad \text{or} \quad \frac{1}{12} \doteq .083 \quad \text{or} \quad \frac{1}{12} \doteq .08,\]

where $\doteq$ is read *equals approximately*, using this rule:

**ORAL EXERCISES**

Approximate each number to the nearest tenth.

1. 3.42 4. -2.745 7. -74.35 10. -0.94
2. 5.61 5. .372 8. -68.55... 11. $\frac{1}{4}$
3. -1.36 6. .836 9. .047 12. $\frac{3}{4}$

Give a second form for each number.

13. $\frac{5}{3}$ 16. 1.35 19. $-\frac{6}{27}$ 22. 16.6
14. 8 17. 1.099... 20. $-\frac{5}{2}$ 23. -0.082
15. .78 18. 23.9 21. 4.333 24. -7.024
WRITTEN EXERCISES

Write as terminating or repeating decimals.

1. \(\frac{2}{2} \)  

2. \(\frac{3}{5} \)  

3. \(\frac{7}{3} \)  

4. \(\frac{9}{8} \)  

5. \(-\frac{10}{11} \)  

6. \(-\frac{5}{3} \)  

7. \(-\frac{2}{7} \)  

8. \(-\frac{17}{80} \)  

9. \(\frac{81}{40} \)  

10. \(\frac{52}{70} \)  

Write as common fractions.

13. .666  

14. .4  

15. \(\frac{3}{3} \)  

16. \(\frac{3}{3} \)  

17. .27272727...  

18. .303303303...  

19. \(-2.567 \)  

20. \(-1.202 \)  

Find the difference of these numbers, and give a number between them.

21. .27 and \(\frac{27}{100} \)  

22. .33 and \(\frac{33}{100} \)  

23. .9 and \(\frac{9}{1} \)  

24. .16 and \(\frac{16}{100} \)  

25. .0833 and \(\frac{1}{12} \)  

26. .0909 and \(\frac{1}{11} \)  

27. \(\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{8}{9}\right\} \)  

28. \(\left\{\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \ldots, \frac{10}{11}\right\} \)  

29. \(\left\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}\right\} \)  

30. \(\left\{\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots, \frac{12}{13}\right\} \)  

IRRATIONAL NUMBERS

11–3 Roots of Numbers

The **power of a number** is the product of factors each equal to that number: \(5^2 = 5(5)\); \(5^3 = 5(5)(5)\); and \(5^n = 5(5)\ldots(5)\), taking 5 as a factor \(n\) times. This operation of *raising to a power* is called *involution*.

Just as addition and multiplication have inverse operations, so has raising to a power. Its inverse operation is called *extracting a root* (evolution). For any positive integer \(n\), a number \(x\) is an \(n^{th}\) root of the number \(a\) if it satisfies \(x^n = a\). That is, since \(3^4 = 81\), 3 is a fourth root of 81.

To indicate the \(n^{th}\) root of \(a\), use the expression \(\sqrt[n]{a}\), which is called a *radical* (in Latin *radix* means root). The symbol \(\sqrt{}\) indicates that a root is to be extracted; \(n\) is the *root index*, signifying the root to be taken; the bar, usually incorporated in the radical symbol, covers the *radicand* (rad'i-cand), the expression whose root is to be extracted. With no root index, \(\sqrt{}\) indicates square root: \(\sqrt{81} = 9\); \(\sqrt[3]{125} = 5\); \(\sqrt[4]{8} = 3\).
When you square a positive or a negative number, you get a positive result. That is, $5^2 = 25$, and $(-5)^2 = 25$. Thus, every positive number has two square roots, one positive and the other negative. Zero, however, has only one square root, zero. You use the expression $\sqrt{25}$ to indicate the positive root 5 (the principal root), $-\sqrt{25}$ to indicate the negative root $-5$, and $\pm\sqrt{25}$ (read positive and negative square root of 25) to represent both roots. Thus, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $-\sqrt{\frac{4}{9}} = -\frac{2}{3}$, and $\pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$. Since the square of every directed number is either positive or zero, negative numbers do not have square roots in the set of directed numbers.

One method of finding the square root of a large number is to determine its factors, and then to express it as a product of powers, and take the square roots of the powers.

**EXAMPLE 1.** Evaluate $\sqrt{2025}$.

**Solution:**

$$2025 = 5(405) = 5(5)(81) = 5^2(9^2)$$

$$\sqrt{2025} = \sqrt{5^2(9^2)} = 5(9) = 45$$

**Check:**

$$45(45) = 2025 \checkmark \quad \sqrt{2025} = 45, \text{ Answer.}$$

This method of solution is based on the **product property of square roots**.

If $a \geq 0$ and $b \geq 0$, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

To prove this property, we show that $\sqrt{a} \cdot \sqrt{b}$ is a nonnegative number and that its square is $ab$. Notice that only principal roots are used.

To show that $\sqrt{a} \cdot \sqrt{b} \geq 0$:

If $a \geq 0$, and $b \geq 0$,

then $\sqrt{a} \geq 0$, $\sqrt{b} \geq 0$,

and $\sqrt{a} \cdot \sqrt{b} \geq 0$.

To show that $(\sqrt{a} \cdot \sqrt{b})^2 = ab$:

$$(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a} \cdot \sqrt{b})(\sqrt{a} \cdot \sqrt{b})$$

$= (\sqrt{a} \cdot \sqrt{a})(\sqrt{b} \cdot \sqrt{b})$

$= a \cdot b$

$\therefore (\sqrt{a} \cdot \sqrt{b})^2 = ab$

Can you guess the **quotient property of square roots**?

If $a \geq 0$ and $b > 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.
EXAMPLE 2. Evaluate \( \sqrt{\frac{3600}{2401}} \).

Solution: 

\[ 3600 = (36)(100) = 6^2(10^2) \]
\[ 2401 = 7(343) = 7(7)(49) = 7^2(7^2) = 7^4 \]

\[ \sqrt{\frac{3600}{2401}} = \sqrt{\frac{6^2(10^2)}{7^4}} = \frac{6(10)}{7^2} = \frac{60}{49} \]

Check: 

\[ \frac{60}{49} \cdot \frac{60}{49} = \frac{3600}{2401} \checkmark \quad \sqrt{\frac{3600}{2401}} = \frac{60}{49} \text{, Answer.} \]

ORAL EXERCISES

Give the principal square root.

1. 144  
2. 121  
3. \( \frac{81}{64} \)  
4. \( \frac{4}{49} \)  
5. 100\( z^2 \)  
6. 144\( w^4 \)  
7. \( 81a^2b^2 \)  
8. \( 36x^2y^2 \)  
9. \( 0.25r^6t^4 \)  
10. \( 0.04k^4t^2 \)  
11. \( \frac{4d^4}{g^8} \)  
12. \( \frac{9p^2}{q^{10}} \)

Simplify each expression.

13. \( \sqrt{1} \)  
14. \( \sqrt{81} \)  
15. \( \sqrt{16 \over 49} \)  
16. \( \sqrt{9 \over 9} \)  
17. \( \sqrt{(-2)^2} \)  
18. \( \sqrt{(-3)^2} \)  
19. \( -\sqrt{36x^4} \)  
20. \( -\sqrt{64y^8} \)  
21. \( -\sqrt{9m^2n^4} \)  
22. \( -\sqrt{49m^2n^6} \)  
23. \( \pm \sqrt{5^2 \over 121} \)  
24. \( (\sqrt{81})^2 \)  
25. \( (\sqrt{49})^2 \)  
26. \( \sqrt{4^2 + 3^2} \)  
27. \( -\sqrt{13^2 - 12^2} \)  
28. \( \pm \sqrt{10^2 - 8^2} \)  
29. \( \pm \sqrt{5^2 - 4^2} \)  
30. \( \pm \sqrt{r^4} \)

WRITTEN EXERCISES

Evaluate each expression.

1. \( \sqrt{484} \)  
2. \( \sqrt{441} \)  
3. \( \sqrt{1296} \)  
4. \( \sqrt{2304} \)  
5. \( -\sqrt{1156} \)  
6. \( -\sqrt{784} \)  
7. \( -\sqrt{\frac{1089}{25}} \)  
8. \( -\sqrt{\frac{484}{9}} \)  
9. \( \pm \sqrt{\frac{1}{324}} \)  
10. \( \pm \sqrt{\frac{1}{1521}} \)  
11. \( \pm \sqrt{\frac{16}{1225}} \)  
12. \( \pm \sqrt{\frac{49}{1296}} \)
Solve. Use \{rational numbers\} as the replacement set of the variable.

**SAMPLE.**  \( y^2 = 9 \)

**Solution:**

\[
\begin{align*}
    y^2 &= 9 \\
    \therefore y &= \pm \sqrt{9} = \pm 3 \\
    \text{Check:} & \\
    (3)^2 &= 9 \\
    (-3)^2 &= 9
\end{align*}
\]

\( \{3, -3\} \), Answer.

13.  \( x^2 = 36 \)
14.  \( y^2 = 49 \)
15.  \( 4a^2 - 1 = 0 \)
16.  \( 9b^2 - 4 = 0 \)
17.  \( 3t^2 - 27 = 0 \)
18.  \( 5k^2 - 125 = 0 \)

19. If \( r^2 + s^2 = 12 \) and \( rs = 2 \), find the positive value of \( r + s \).
20. If \( r - s = 5 \) and \( rs = 3 \), find the value of \( r^2 + s^2 \).
21. If \((x - y)^2 = 5\) and \( xy = 1 \), find the negative value of \( x + y \).
22. If \( x + y = 8 \) and \( x^2 + y^2 = 40 \), find the positive value of \( x - y \).
23. Find the fallacy in the following argument.

Let \( a \) be any rational number.

\[
\begin{align*}
    a^2 &= (-a)^2 \\
    \therefore \sqrt{a^2} &= \sqrt{(-a)^2} \\
    \text{But} \quad \sqrt{a^2} &= a \quad \text{and} \quad \sqrt{(-a)^2} = -a \\
    \therefore a &= -a
\end{align*}
\]

24. Find the fallacy in this proof that every rational number is 1.

Let \( x \) be any rational number.

\[
\begin{align*}
    x - 1 &= -(1 - x) \\
    \therefore (x - 1)^2 &= (1 - x)^2 \\
    \therefore x - 1 &= 1 - x \\
    \therefore 2x &= 2 \\
    \therefore x &= 1
\end{align*}
\]

25. Solve the equation \( \sqrt{x^2} - x = 18 \).
26. Solve the equation \( y - \sqrt{y^2} = 50 \).
27. Prove the quotient property of square roots.
28. Prove that \( \sqrt{a^3} = (\sqrt{a})^3 \), if \( a \geq 0 \).
29. Prove that if \( a > b > 0 \), then \( a^2 > b^2 \). (Hint: Show that \( a^2 > ab \) and that \( ab > b^2 \).) Use this result to explain why a positive number cannot have two different positive square roots.
30. Prove that if \( b < a < 0 \), then \( a^2 < b^2 \). Use this result to explain why a positive number cannot have two different negative square roots.

11-4 Properties of Irrational Numbers

Rational numbers like 25, 36, and \( \frac{13}{6} \) which are squares of rational numbers are called perfect squares. However, not every positive rational number is a perfect square.

Do those rational numbers which are not perfect squares have rational square roots? Consider some positive integer \( n \). Assume that its square root is a fraction \( \frac{a}{b} \) in lowest terms. That is, \( \sqrt{n} = \frac{a}{b} \), where \( a, b, \) and \( n \) are positive integers, and \( a \) and \( b \) have no common factors.

If \( \sqrt{n} = \frac{a}{b} \), then \( n = \frac{a^2}{b^2} \). Since \( a^2 \) has the same prime factors as \( a \), and \( b^2 \) has the same prime factors as \( b \), if \( a \) and \( b \) have no factors in common, neither do \( a^2 \) and \( b^2 \), and \( \frac{a^2}{b^2} \) is in lowest terms. If a fraction in lowest terms is equal to an integer, the denominator of the fraction must be 1. Thus, since \( n \) is an integer, and \( \frac{a^2}{b^2} \) is in lowest terms, \( b^2 = 1 \), and \( b = 1 \), which means that \( \frac{a}{b} = \frac{a}{1} = a \). Therefore, if the square root of a positive integer is a rational number, \( \frac{a}{b} \), the root is in fact an integer, \( \frac{a}{1} \). Thus, only integers which are squares of integers can have rational square roots.

As there are positive integers which are not squares of integers, if symbols like \( \sqrt{2}, \sqrt{3}, \) and \( \sqrt{5} \) are to have any meaning, your concept of numbers must be extended to include irrational numbers, numbers which cannot be expressed in the form \( \frac{r}{s} \), where \( r \) and \( s \) are integers. Together, the rational and irrational numbers form the real number system, which has all the properties you have studied, including the property of density. In addition, the real numbers have the property of completeness:

On the number line, each point corresponds to one real number; and each real number corresponds to one point on the line.

Terminating and repeating decimals represent rational numbers; therefore, the decimals for irrational numbers must neither terminate
nor repeat. One method of finding successive digits in the decimals of irrational numbers which are square roots is based on the property of pairs of divisors of any number:

If you divide a number by a divisor which is smaller in absolute value than the square root of that number, the quotient will be larger in absolute value than the square root.

Consider 100 and its square root 10. \(100 \div 10 = 10\); but \(100 \div 25 = 4\), \(100 \div 5 = 20\), and \(100 \div 50 = 2\). If the divisor is greater than 10, the quotient is less than 10, and vice versa.

**EXAMPLE 1.** Approximate \(\sqrt{17}\).

**Solution:**

1. As your first approximation select the integer whose square is nearest 17.

Let \(a\) be a number such that \(a^2 = 17\)

First \(a = 4\)

\(17 \div 4 = 4.2\)\(\Rightarrow\) 2)8.2

Second \(a = 4.1\)

\(17 \div 4.1 = 4.146\)\(\Rightarrow\) 2)8.246

Third \(a = 4.123\)

\(17 \div 4.123 = 4.1232112\)

Fourth \(a = 4.1231056\)

Check:

\(4.12310565 \div 4.1231056)17.00000000\)

\(\therefore \sqrt{17} = 4.1231056\),

Answer.

The approximation found by this method represents a rational number which differs from the true root usually by no more than 2 in the last digit retained.
To obtain a good approximation rapidly, start with an approximation even closer than the nearest integer. The tables of squares and square roots (Appendix) help in determining the first approximations for your roots.

The product and quotient properties of square roots, too, may be useful in finding the decimal approximation for roots of numbers less than 1 and numbers greater than 100. Notice how even powers of 10 are used in this example.

**EXAMPLE 2.** Evaluate (a) $\sqrt{12996}$ and (b) $\sqrt{.012996}$.

**Solution:**

a. $12996 = 10^4(1.2996)$; $\sqrt{12996} = 100\sqrt{1.2996}$

b. $.012996 = \left(\frac{1}{100}\right)(1.2996)$; $\sqrt{.012996} = \frac{1}{10} \sqrt{1.2996}$

Both solutions require the determination of $\sqrt{1.2996}$. To find this:

1. First approx. $= 1.1$
2. $1.2996 \div 1.1 = 1.181$
3. Second approx. $= \frac{2.281}{2} = 1.14$

$\therefore \sqrt{1.2996} = 1.14$, the true root.

Substituting this value in the expressions in (a) and (b) gives:

a. $\sqrt{12996} = 100(1.14) = 114$, Answer.

b. $\sqrt{.012996} = \frac{1}{10}(1.14) = .114$, Answer.

The next example shows how you can use the values in the table of square roots in solving some square root problems.

**EXAMPLE 3.** Approximate the square roots of (a) .57 and (b) 570.

**Solution:** Make use of the table of square roots.

- **a.** $\frac{.57}{100} = \frac{57}{100}$
  
  $\sqrt{\frac{57}{100}} = \frac{7.550}{10}$

  $\therefore \sqrt{.57} = .7550$,

- **b.** $570 = 57(10)$
  
  $\sqrt{57(10)} = 7.550(3.162)$

  $\therefore \sqrt{570} = 23.87$, Answer.
Identify each expression as rational or irrational. Give the numbers represented by the rational ones.

**SAMPLE.** \(\sqrt{\frac{3}{2} + \frac{7}{8}}\)

<table>
<thead>
<tr>
<th>Expression</th>
<th>What you say:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (\sqrt{4})</td>
<td>Rational; (\frac{3}{2})</td>
</tr>
<tr>
<td>2. (-\sqrt{25})</td>
<td></td>
</tr>
<tr>
<td>3. (\sqrt{25})</td>
<td></td>
</tr>
<tr>
<td>4. (3\sqrt{18})</td>
<td></td>
</tr>
<tr>
<td>5. (\frac{3}{8})</td>
<td></td>
</tr>
<tr>
<td>6. (\sqrt{\frac{3}{2} + \frac{7}{8}})</td>
<td></td>
</tr>
<tr>
<td>7. (3\sqrt{81})</td>
<td></td>
</tr>
<tr>
<td>8. (\frac{3}{\sqrt{9}})</td>
<td></td>
</tr>
<tr>
<td>9. (\sqrt{\frac{3}{2} + \frac{7}{8}})</td>
<td></td>
</tr>
<tr>
<td>10. (\sqrt{1.44})</td>
<td></td>
</tr>
<tr>
<td>11. (-\sqrt{.64})</td>
<td></td>
</tr>
<tr>
<td>12. (\pm\sqrt{\frac{2}{18}})</td>
<td></td>
</tr>
</tbody>
</table>

13. \(\sqrt{49} - \sqrt{81}\)
14. \(\frac{\sqrt{13}}{\sqrt{13}}\)
15. \(\sqrt{18} - \sqrt{18}\)
16. \(-\sqrt{35} + \sqrt{35}\)
17. \(\frac{\sqrt{28}}{\sqrt{7}}\)

Name the integer closest to the square root of each number.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Integer Closest</th>
</tr>
</thead>
<tbody>
<tr>
<td>19. 30</td>
<td>5</td>
</tr>
<tr>
<td>20. 59</td>
<td>7</td>
</tr>
<tr>
<td>21. 11.2</td>
<td>11</td>
</tr>
<tr>
<td>22. 27.28</td>
<td>27</td>
</tr>
<tr>
<td>23. 3.61</td>
<td>3</td>
</tr>
<tr>
<td>24. 1.69</td>
<td>1</td>
</tr>
<tr>
<td>25. 14.4</td>
<td>14</td>
</tr>
<tr>
<td>26. 42.398</td>
<td>42</td>
</tr>
<tr>
<td>27. 800</td>
<td>89</td>
</tr>
<tr>
<td>28. 1200</td>
<td>120</td>
</tr>
<tr>
<td>29. 625.2</td>
<td>625</td>
</tr>
<tr>
<td>30. 397.4</td>
<td>397</td>
</tr>
<tr>
<td>31. 7734</td>
<td>773</td>
</tr>
<tr>
<td>32. 8296</td>
<td>829</td>
</tr>
<tr>
<td>33. 2500</td>
<td>250</td>
</tr>
<tr>
<td>34. 3600</td>
<td>360</td>
</tr>
<tr>
<td>35. .32</td>
<td>0</td>
</tr>
<tr>
<td>36. .68</td>
<td>0</td>
</tr>
<tr>
<td>37. .1391</td>
<td>0</td>
</tr>
<tr>
<td>38. .467</td>
<td>0</td>
</tr>
<tr>
<td>39. .5</td>
<td>0</td>
</tr>
<tr>
<td>40. .7</td>
<td>0</td>
</tr>
<tr>
<td>41. .000428</td>
<td>0</td>
</tr>
<tr>
<td>42. .000331</td>
<td>0</td>
</tr>
</tbody>
</table>

Express as the product of a number between 1 and 100 and a power of 100.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>27. 800</td>
<td>8 \times 10^2</td>
</tr>
<tr>
<td>28. 1200</td>
<td>12 \times 10^2</td>
</tr>
<tr>
<td>29. 625.2</td>
<td>62 \times 10</td>
</tr>
<tr>
<td>30. 397.4</td>
<td>39 \times 10</td>
</tr>
<tr>
<td>31. 7734</td>
<td>77 \times 10</td>
</tr>
<tr>
<td>32. 8296</td>
<td>83 \times 10</td>
</tr>
<tr>
<td>33. 2500</td>
<td>25 \times 100</td>
</tr>
<tr>
<td>34. 3600</td>
<td>36 \times 100</td>
</tr>
</tbody>
</table>

Express as the quotient of a number between 1 and 100 and a power of 100.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>35. .32</td>
<td>.32 \times 10^{-1}</td>
</tr>
<tr>
<td>36. .68</td>
<td>.68 \times 10^{-1}</td>
</tr>
<tr>
<td>37. .1391</td>
<td>.1391 \times 10^{-1}</td>
</tr>
<tr>
<td>38. .467</td>
<td>.467 \times 10^{-1}</td>
</tr>
<tr>
<td>39. .5</td>
<td>.5 \times 10^{-1}</td>
</tr>
<tr>
<td>40. .7</td>
<td>.7 \times 10^{-1}</td>
</tr>
<tr>
<td>41. .000428</td>
<td>.428 \times 10^{-4}</td>
</tr>
<tr>
<td>42. .000331</td>
<td>.331 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Find the indicated square roots.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Square Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (\sqrt{3.61})</td>
<td>1.90</td>
</tr>
<tr>
<td>2. (\sqrt{4.41})</td>
<td>2.1</td>
</tr>
<tr>
<td>3. (\sqrt{10.24})</td>
<td>3.2</td>
</tr>
<tr>
<td>4. (\sqrt{30.25})</td>
<td>5.5</td>
</tr>
<tr>
<td>5. (\sqrt{1764})</td>
<td>42</td>
</tr>
<tr>
<td>6. (\sqrt{5329})</td>
<td>73</td>
</tr>
<tr>
<td>7. (-\sqrt{316.84})</td>
<td>-17.8</td>
</tr>
<tr>
<td>8. (-\sqrt{408.04})</td>
<td>-20.2</td>
</tr>
<tr>
<td>9. (\sqrt{24,336})</td>
<td>156</td>
</tr>
<tr>
<td>10. (\sqrt{110,889})</td>
<td>333</td>
</tr>
</tbody>
</table>

Find each square root to the nearest hundredth.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Square Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. (\sqrt{2})</td>
<td>1.41</td>
</tr>
<tr>
<td>12. (\sqrt{3})</td>
<td>1.73</td>
</tr>
<tr>
<td>13. (\sqrt{50.3})</td>
<td>7.09</td>
</tr>
<tr>
<td>14. (\sqrt{10.7})</td>
<td>3.27</td>
</tr>
<tr>
<td>15. (-\sqrt{356})</td>
<td>-18.8</td>
</tr>
<tr>
<td>16. (-\sqrt{279})</td>
<td>-16.7</td>
</tr>
<tr>
<td>17. (\sqrt{27})</td>
<td>5.2</td>
</tr>
<tr>
<td>18. (\sqrt{35})</td>
<td>5.9</td>
</tr>
</tbody>
</table>

Find both roots to the nearest tenth.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Square Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>19. (x^2 = 960)</td>
<td>(\pm 31.0)</td>
</tr>
<tr>
<td>20. (y^2 = 405)</td>
<td>(\pm 20.1)</td>
</tr>
<tr>
<td>21. (t^2 - 3.2 = 0)</td>
<td>(\pm 0.6)</td>
</tr>
<tr>
<td>22. (u^2 - 12.6 = 0)</td>
<td>(\pm 3.6)</td>
</tr>
<tr>
<td>23. (800 = 5z^2)</td>
<td>(\pm 17.3)</td>
</tr>
<tr>
<td>24. (1000 = 8w^2)</td>
<td>(\pm 14.1)</td>
</tr>
</tbody>
</table>
25. Find $\sqrt{21}$ by taking $a = 4$ and by taking $a = 5$.
26. Find $\sqrt{90}$ by taking $a = 9$ and by taking $a = 10$.

Solve to the nearest tenth.

27. $0.9p^2 = 1.062$
28. $(x + 1)^2 + (x - 1)^2 = 55$
29. $11y^2 - 36 = 0$
30. $9z^2 - 11 = 0$
31. If $a$ is rational and $\sqrt{b}$ is irrational, prove that $a + \sqrt{b}$ is irrational.
32. Prove that $3\sqrt{n}$ may be irrational by an argument like that for $\sqrt{n}$.

**PROBLEMS**

Find each answer to the nearest tenth, unless otherwise directed.

1. Find the side of a square whose area is 75 square inches.
2. The area of a square is 46 square feet. How long is one side?
3. The area of a circle is $A = 3.14r^2$. Find the radius, correct to hundredths, of a circle whose area is 87.92 square centimeters.
4. The area of a circle is $A = .7854d^2$. Find the diameter, correct to hundredths, of a circle whose area is 15.708 square centimeters.
5. Find the side of a square whose area is $\frac{3}{5}$ square meters.
6. The area of a square is $\frac{6}{9}$ square meters. Find its side.
7. A rectangle whose area is 225 square meters has a length three times its width. Find the length and width of this rectangle.
8. The length of a rectangle is twice its width. Its area is 1814 square centimeters. Find the dimensions of the rectangle.

11-5 Geometric Interpretation of Square Roots

How can you locate irrational square roots on the number line without using approximations? Pythagoras proved the existence of distances which could not be measured by rational numbers in the Pythagorean theorem (pith-ag'uh-re-an the-uh-rem). (A theorem is a statement which can be proved.)

In any right triangle the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

The hypotenuse of a right triangle is the longest side and is opposite the right angle.
Figure 11–1 illustrates the Pythagorean theorem: \( c^2 = a^2 + b^2 \), where \( c \) is the length of the hypotenuse, and \( a \) and \( b \) are the lengths of the other two sides.

To find a length equal to \( \sqrt{2} \), draw a square whose sides are 1 (Figure 11–2). The diagonal \( OP \) divides it into two right triangles in which \( a = 1 \) and \( b = 1 \).

\[
\begin{align*}
c^2 &= a^2 + b^2 \\
c^2 &= 1^2 + 1^2 \\
c^2 &= 1 + 1 \\
c^2 &= 2 \\
c &= \sqrt{2}
\end{align*}
\]

Figure 11–3 combines this square with the number axes. The semicircle has the origin as its center and \( \sqrt{2} \) as its radius.
Points \( Q, R, \) and \( S \) are on the line \( y = 1 \) at distances of \( \sqrt{3}, \sqrt{4}, \) and \( \sqrt{5} \) from the origin. Each can be found by using the previously constructed square root and drawing perpendiculars to form new right triangles. If \( OQ, OR, \) and \( OS \) represent lengths, then:

\[
(OQ)^2 = (\sqrt{2})^2 + 1^2 \quad \quad (OR)^2 = (\sqrt{3})^2 + 1^2 \quad \quad (OS)^2 = (\sqrt{4})^2 + 1^2
\]

\[
(OQ)^2 = 2 + 1 \quad \quad (OR)^2 = 3 + 1 \quad \quad (OS)^2 = 4 + 1
\]

\[
(OQ) = \sqrt{3} \quad \quad (OR) = \sqrt{4} = 2 \quad \quad (OS) = \sqrt{5}
\]

**EXAMPLE.** Is a triangle whose sides are 3, 4, and 5 a right triangle?

**Solution:**

\[
c^2 = a^2 + b^2
\]

\[
5^2 = 3^2 + 4^2
\]

\[
25 = 25 \checkmark \quad \text{A 3-4-5 triangle is a right triangle, Answer.}
\]

**WRITTEN EXERCISES**

Determine whether or not each is a right triangle.

1. The three sides are 6, 8, and 10 inches.
2. The sides are 9 feet, 12 feet, and 15 feet.
3. The sides are 48, 52, and 20 centimeters.
4. The sides are 45, 51, and 24 centimeters.

In each right triangle find the missing dimension to the nearest hundredth.

5. \( a = 15 \) inches; \( b = 20 \) inches
6. \( a = 32 \) meters; \( b = 24 \) meters
7. \( a = 6 \) miles; \( b = 1\frac{3}{4} \) miles
8. \( a = 2\frac{1}{4} \text{ yards}; b = 3 \text{ yards}\)
9. \( b = 9 \) meters; \( c = 41 \) meters
10. \( a = 10 \) feet; \( c = 25 \) feet

**PROBLEMS**

Make a sketch for each problem. Work to 2 decimal places.

1. If the bottom of a 17-foot ladder is 8 feet from a wall, how high on the wall does it reach?
2. A wire from the top of a telephone pole to a point on the ground 16 feet from the pole is thirty-four feet long. How high is the pole?
3. A rectangular lot is 80 feet long and 50 feet wide. How long is a straight line from one corner to the corner diagonally opposite?
4. Two sides of a triangular bracket are 16 inches long. How long is the third side?

5. A right triangle has sides whose lengths in inches are expressed by consecutive even integers. Find the length of each side.

6. The sides of one right triangle, when expressed in feet, are given by consecutive integers. Find the length of each side.

7. The sides of one right triangle have a ratio of 3:4. The hypotenuse is 35 feet in length. Find the length of each side.

8. A man drives 30 miles north, then 5 west, then south to a point 15 miles northwest of his starting point. How far south does he drive?

9. A one-way tunnel entrance is a rectangle 10' w. X 14' h. surmounted by a semicircular arch. How high a truck, 7 ft. wide, can enter?

10. Find the length of the diagonal of a box 24'' X 18'' X 15'' deep.

### RADICAL EXPRESSIONS

**11–6 Multiplication, Division, and Simplification of Radicals**

The product and quotient properties of square roots (page 404) together with the commutative and associative properties enable you to multiply, divide, and simplify square-root radicals quickly.

\[
\sqrt{7} \cdot \sqrt{3} = \sqrt{21} \quad \sqrt{3} \cdot \sqrt{27} = \sqrt{81} = 9
\]

\[
(2\sqrt{5})(4\sqrt{2}) = (2 \cdot 4)(\sqrt{5} \cdot \sqrt{2}) = 8\sqrt{10}
\]

\[
\frac{\sqrt{10}}{\sqrt{2}} = \sqrt{\frac{10}{2}} = \sqrt{5} \quad \frac{\sqrt{11}}{\sqrt{75}} = \sqrt{\frac{11}{75}} \cdot \frac{3}{3} = \sqrt{\frac{33}{225}} = \frac{\sqrt{33}}{15}
\]

An expression having a square-root radical is in **simplest form** when

1. no integral radicand has a square factor other than 1,
2. no fractions are under a radical sign, and
3. no radicals are in a denominator.

\[
\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5} \quad \text{and} \quad 3\sqrt{96} = 3\sqrt{16 \cdot 6} = 12\sqrt{6}
\]

\[
\sqrt{5} \quad \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5} \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{\sqrt{30}}{6} \quad ; \quad \frac{1}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}
\]

\[
\frac{2\sqrt{3}}{3\sqrt{8}} \quad = \frac{2\sqrt{3} \cdot \sqrt{2}}{3\sqrt{8} \cdot \sqrt{2}} = \frac{2\sqrt{6}}{3\sqrt{16}} = \frac{2\sqrt{6}}{3 \cdot 4} = \frac{\sqrt{6}}{6}
\]
The process of changing the form of a fraction with an irrational denominator to an equal fraction with a rational denominator is called rationalizing the denominator. Rationalizing the denominator of a radical expression helps in approximating its value.

**Oral Exercises**

Express in simplest form.

1. $\sqrt{5} \cdot \sqrt{2}$
2. $\sqrt{5} \cdot \sqrt{3}$
3. $\frac{\sqrt{28}}{\sqrt{7}}$
4. $\frac{\sqrt{27}}{\sqrt{3}}$
5. $\sqrt{5} \cdot 8\sqrt{3}$
6. $4\sqrt{3} \cdot \sqrt{7}$
7. $\frac{\sqrt{50}}{\sqrt{2}}$
8. $\frac{\sqrt{80}}{\sqrt{5}}$
9. $3\sqrt{2} \cdot 2\sqrt{8}$
10. $4\sqrt{2} \cdot 7\sqrt{3}$
11. $\frac{15}{\sqrt{3}}$
12. $\frac{21}{\sqrt{3}}$
13. $(3\sqrt{10})^2$
14. $(2\sqrt{3})^2$
15. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}}$

16. $\sqrt{\frac{1}{3}} \cdot \sqrt{\frac{1}{3}}$
17. $\sqrt{32}$
18. $\sqrt{500}$
19. $\sqrt{\frac{2}{3}}$
20. $\frac{1}{\sqrt{3}}$
21. $\sqrt{27}$
22. $\sqrt{12}$
23. $\sqrt{\frac{3}{2}}$
24. $\frac{2}{\sqrt{14}}$
25. $\sqrt{72}$
26. $\sqrt{24}$
27. $\sqrt{\frac{1}{8}}$
28. $\sqrt{\frac{1}{x}}$
29. $\sqrt{45}$
30. $\sqrt{48}$
31. $\sqrt{\frac{7}{18}}$
32. $\sqrt{\frac{1}{x^3}}$
33. $\sqrt{28}$
34. $\sqrt{80}$
35. $\sqrt{\frac{3}{40}}$
36. $\frac{5}{\sqrt{a}}$

Give two factors, one of which is the largest possible perfect square.

**Sample.**

What you say: 25 times 3

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>300</td>
<td>39</td>
<td>250</td>
<td>41</td>
</tr>
<tr>
<td>38</td>
<td>50</td>
<td>40</td>
<td>90</td>
<td>42</td>
</tr>
</tbody>
</table>

**Written Exercises**

Express in simplest form. (Assume that all given radicands are nonnegative real numbers.)

1. $3\sqrt{17} \cdot 8\sqrt{17}$
2. $5\sqrt{3} \cdot 13\sqrt{26}$
3. $\sqrt{3} \cdot \sqrt{2} \cdot \sqrt{6}$
4. $2\sqrt{3} \cdot \sqrt{7} \cdot \sqrt{2}$
5. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{5}{3}}$
6. $2\sqrt{\frac{3}{5}} \cdot 3\sqrt{\frac{5}{4}}$
7. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{3}{5}}$
8. $\frac{3}{8}\sqrt{\frac{5}{7}} \cdot \frac{3}{8}\sqrt{\frac{1}{20}}$
9. $\sqrt{\frac{8}{5}} \cdot \sqrt{\frac{2}{5}}$
10. $\sqrt{\frac{7}{2}} \cdot \sqrt{\frac{3}{5}}$
11. $(-5\sqrt{a})(2\sqrt{a})$
12. $(3\sqrt{b})(-4\sqrt{b})$
33. \((-5\sqrt{ab^2})(-3\sqrt{a})\)

34. \((-7\sqrt{c^2d})(2\sqrt{d})\)

35. \(\sqrt{x}(\sqrt{x} - 3)\)

36. \(\sqrt{y}(2 - \sqrt{y})\)

37. \((\sqrt{6})(-2\sqrt{3})(\sqrt{8})\)

38. \((-5\sqrt{14})(-\sqrt{7})(\sqrt{8})\)

39. \((\sqrt{21y})(\sqrt{y})(\sqrt{7})\)

40. \((\sqrt{x})(\sqrt{3x})(\sqrt{24})\)

41. \((2a\sqrt{5a})^2\)

42. \((3b\sqrt{2b})^2\)

43. \(2\sqrt{6}(3\sqrt{2} + \sqrt{3})\)

44. \(5\sqrt{10}(2\sqrt{5} - \sqrt{15})\)

45. \(-6\sqrt{10}\)

46. \(5\sqrt{12}\)

47. \(-5\sqrt{2.4}\)

48. \(6.47\sqrt{.35}\)

49. \(\frac{2\sqrt{7} + \sqrt{14}}{\sqrt{7}}\)

50. \(\frac{50\sqrt{34}}{5\sqrt{17} - 3\sqrt{17}}\)

51. \(\frac{5\sqrt{2} + \sqrt{8}}{\sqrt{2}}\)

52. \(\frac{\sqrt{3} - \sqrt{12}}{\sqrt{3}}\)

53. \(\frac{7\sqrt{a^3} + \sqrt{a}}{\sqrt{a}}\)

54. \(\frac{\sqrt{b} - 4\sqrt{b^3}}{\sqrt{b}}\)

**PROBLEMS**

Find answers to the nearest tenth, unless otherwise directed.

**B**

1. One number is three times another. The difference of their squares is \(\frac{3}{4}\). Find the numbers.

2. Find two numbers in the ratio 4 to 3, whose squares differ by 12.

3. The dimensions of a rectangle are in the ratio 3 to 2. Its area is 128 square inches. Find its dimensions.

4. The area of a square is 70 square inches. How long is its diagonal?

5. Find the area of a triangle whose sides are 30, 60, and 70 centimeters long, by the rule \(A = \sqrt{s(s - a)(s - b)(s - c)}\), where \(a\), \(b\), and \(c\) are the lengths of the sides and \(s\) is half the perimeter.

6. Find the radius \(r\) of the circle inscribed in the right triangle shown, by the rule \(r = \frac{1}{2}(a + b - c)\). The shorter sides of the triangle have lengths of .5 meter and .8 meter. (Sketch p. 417 top left)
7. A square of side $s$ is inscribed in a circle of diameter $d$, as shown above. If $d = 4.6$, find $s$.

8. A square whose area is 25 square inches is inscribed in a circle of diameter $d$. Find the radius of the circle.

9. Will a square whose area is 25 square inches fit inside a circle whose area is 77 square inches? Support your answer by calculations.

10. Show that an equilateral triangle, whose side is 4 inches, is smaller in area than a square inscribed in a circle whose diameter is 4 inches.

11. The altitude $h$ divides the equilateral triangle of side $s$ into two identical right triangles, as shown. Find $h$, if $s = 2$.

12. Find the length of the side $s$ of an equilateral triangle of altitude $h$, if $h = 6$.

### 11-7 Addition and Subtraction of Radicals

Because $6\sqrt{2}$ and $5\sqrt{2}$ have the common factor $\sqrt{2}$, you can simplify their sum by using the distributive property.

$$6\sqrt{2} + 5\sqrt{2} = (6 + 5)\sqrt{2} = 11\sqrt{2}$$

The sum or difference of square-root radicals having the same radicand is the sum or difference of the coefficients of the radicals, multiplied by the common radical. On the other hand, the addition or subtraction of radicals having unlike radicands can only be indicated.

$$5\sqrt{6} - 8\sqrt{5} + 2\sqrt{6} + 6\sqrt{5} = 7\sqrt{6} - 2\sqrt{5}$$
By reducing each radical to simplest form, you sometimes can combine terms in a sum of radicals.

**EXAMPLE.** Simplify \(-2\sqrt{48} + 7\sqrt{3} + 2\sqrt{147}\).

*Solution:*

\[
-2\sqrt{48} + 7\sqrt{3} + 2\sqrt{147} = -2\sqrt{16 \cdot 3} + 7\sqrt{3} + 2\sqrt{49 \cdot 3}
\]

\[
= -2(4\sqrt{3}) + 7\sqrt{3} + 2(7\sqrt{3})
\]

\[
= -8\sqrt{3} + 7\sqrt{3} + 14\sqrt{3}
\]

\[
= 13\sqrt{3}, \text{Answer.}
\]

To add or subtract square-root radicals:

1. Express each radical in simplest form.
2. By the distributive property, combine radicals with like radicands.
3. Indicate the sum or difference of radicals with unlike radicands.

**ORAL EXERCISES**

From each group select the radicals having the same radicand.

1. \(\sqrt{2}, 3\sqrt{2}, 2\sqrt{3}\)
2. \(\sqrt{10}, 2\sqrt{10}, \sqrt{20}\)
3. \(\sqrt{5}, \sqrt{17}, 5\sqrt{17}\)
4. \(\sqrt{11}, 11\sqrt{3}, 3\sqrt{11}\)
5. \(6\sqrt{5}, 5\sqrt{7}, 3\sqrt{7}, 5\sqrt{6}\)
6. \(2\sqrt{10}, 7\sqrt{6}, 10\sqrt{2}, 2\sqrt{6}\)

Combine these radicals.

7. \(2\sqrt{2} + 3\sqrt{2}\)
8. \(5\sqrt{3} - 2\sqrt{3}\)
9. \(4\sqrt{5} + \sqrt{5}\)
10. \(\sqrt{7} + 8\sqrt{7}\)
11. \(6\sqrt{11} - 3\sqrt{11} - 4\sqrt{11}\)
12. \(2\sqrt{13} + 4\sqrt{13} - 9\sqrt{13}\)
13. \(6\sqrt{15} - 8\sqrt{15} + \sqrt{15}\)
14. \(3\sqrt{10} - 9\sqrt{10} + 2\sqrt{10}\)
15. \(2\sqrt{a} + \sqrt{a} - 3\sqrt{a}\)
16. \(\sqrt{c} - 4\sqrt{c} + \sqrt{c}\)

**WRITTEN EXERCISES**

Combine these radicals. (Assume that all given radicands are nonnegative real numbers.)

**A**

1. \(3\sqrt{5} + 5\sqrt{5} - 2\sqrt{5}\)
2. \(2\sqrt{11} - 6\sqrt{11} - 5\sqrt{11}\)
3. \(\sqrt{2} - \sqrt{3} + 5\sqrt{2} + 5\sqrt{3}\)
4. \(\sqrt{5} + \sqrt{2} - 3\sqrt{5} + 2\sqrt{2}\)
11-8 Multiplication of Binomials Containing Radicals

Sometimes in dealing with radicals, you may wish to find a product like $(3 + \sqrt{7})(3 - \sqrt{7})$. Do you see that this indicated product resembles $(a + b)(a - b) = a^2 - b^2$? Two binomials of the form $x + \sqrt{y}$ and $x - \sqrt{y}$ are called conjugates of each other. They differ only in the sign between the two terms, and their product is a rational number, as this example shows.

**EXAMPLE 1.** $(3 + \sqrt{7})(3 - \sqrt{7})$

**Solution:**

$$(3 + \sqrt{7})(3 - \sqrt{7}) = 3^2 - (\sqrt{7})^2$$

$$= 9 - 7$$

$$= 2, \text{ Answer.}$$
Consider the product \((2 + \sqrt{5})(2 + \sqrt{5})\), whose form is \((a + b)(a + b) = a^2 + 2ab + b^2\).

**EXAMPLE 2.** \((2 + \sqrt{5})(2 + \sqrt{5})\)

**Solution:**
\[
(2 + \sqrt{5})(2 + \sqrt{5}) = 2^2 + 2(2\sqrt{5}) + (\sqrt{5})^2 = 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}, \text{ Answer.}
\]

Binomial multiplication by the conjugate of the denominator can help you rationalize a denominator.

**EXAMPLE 3.** Rationalize the denominator: \(\frac{1}{2\sqrt{3} + 1}\).

**Solution:** Multiply the numerator and denominator by \(2\sqrt{3} - 1\).
\[
\frac{1}{2\sqrt{3} + 1} = \frac{1(2\sqrt{3} - 1)}{(2\sqrt{3} + 1)(2\sqrt{3} - 1)} = \frac{2\sqrt{3} - 1}{(2\sqrt{3})^2 - 1^2} = \frac{2\sqrt{3} - 1}{12 - 1} = \frac{2\sqrt{3} - 1}{11}, \text{ Answer.}
\]

**WRITTEN EXERCISES**

Express in simplest form.

**A**

1. \((2 + \sqrt{3})(2 - \sqrt{3})\)
2. \((4 - \sqrt{5})(4 + \sqrt{5})\)
3. \((\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})\)
4. \((\sqrt{7} + \sqrt{6})(\sqrt{7} - \sqrt{6})\)
5. \((2\sqrt{3} - 5)(2\sqrt{3} + 3)\)
6. \((5\sqrt{2} - 4)(5\sqrt{2} + 1)\)

**B**

7. \((1 + \sqrt{7})^2\)
8. \((5 - \sqrt{10})^2\)
9. \((5\sqrt{2} - 1)^2\)
10. \((3\sqrt{7} + 2)^2\)
11. \(2\sqrt{6}(3\sqrt{2} + \sqrt{3})\)
12. \(5\sqrt{10}(2\sqrt{5} - \sqrt{15})\)
13. \((4\sqrt{3} + 1)(2\sqrt{3} - 3)\)
14. \((5\sqrt{7} - 2)(\sqrt{7} + 2)\)
15. \((2\sqrt{6} - \sqrt{3})(\sqrt{6} + 3\sqrt{3})\)
16. \((6\sqrt{15} + \sqrt{5})(2\sqrt{15} - 3\sqrt{5})\)

Rationalize the denominator of each fraction.

17. \(\frac{1}{\sqrt{5} - 1}\)
18. \(\frac{1}{\sqrt{7} + 1}\)
19. \(\frac{\sqrt{2}}{\sqrt{2} + 3}\)
20. \(\frac{\sqrt{6}}{5 - \sqrt{6}}\)
21. \(\frac{3 - \sqrt{5}}{2 - \sqrt{5}}\)
22. \(\frac{\sqrt{6} - 1}{3 + \sqrt{6}}\)
23. \(\frac{2}{3\sqrt{2} - 2}\)
24. \(\frac{6}{3 + 2\sqrt{3}}\)

(continued on page 421)
The Real Number System

In the system of rational numbers you can add, subtract, multiply, and divide, except by zero. Furthermore, these operations satisfy all the familiar rules of arithmetic. Therefore, since you can use rational numbers for counting and for measuring to any desired degree of accuracy, it might seem that these numbers are entirely adequate for all possible applications.

As a matter of fact, the system of rational numbers is insufficient for many uses in mathematics. Can you solve the equation $x^2 = 2$? You cannot if the replacement set of $x$ is just the set of rational numbers! The Greek mathematician Pythagoras is credited with having made the remarkable discovery that there is no rational number whose square is 2.

This means that there is no rational number that measures the length of the diagonal of a square having sides one unit in length. By using a scale to measure the diagonal, you may find a rational number such as 1.4 or 1.41 to approximate the length, but you will find no rational number that measures the length exactly. Thus, on the number line the point $P$ cannot be paired with a rational number.

Just as the omission of this point would cause a gap in the line, the absence of a positive number whose square is 2 (symbolized by $\sqrt{2}$) produces a gap in the set of rational numbers. To fill such gaps in the number system, the irrational numbers were invented. A formal way of stating that the set made up of the rational and irrational numbers has no gaps in it is to say that it possesses the property of completeness.

The familiar long division process enables you to express any rational number as a terminating or a repeating decimal; for example, $\frac{3}{4} = 1.25$ and $\frac{2}{15} = .133\ldots = .1\overline{3}$.

Since terminating and repeating decimals always represent rational numbers, the decimal for an irrational number can neither terminate nor repeat. A process for finding successive digits in the decimal representation of $\sqrt{2}$ is illustrated in the sequence of drawings on the upper halves of the following pages. The drawings on the lower halves of these pages are a separate sequence illustrating methods of finding points on the number line corresponding to some other irrational numbers.
On the number line the point associated with $\sqrt{2}$ lies between 1 and 2 because $1^2 < 2 < 2^2$. By subdividing this unit interval into ten equal parts each of length .1, you can see that $1.4 < \sqrt{2} < 1.5$.

The diagram on the left shows the construction for locating on the axes the points associated with the square roots of consecutive integers. Both of the points marked $\sqrt{2}$ are on the circumference of the circle with center at the origin and radius equal to the diagonal of the unit square.

To locate $\sqrt{3}$ draw the circle whose radius coincides with the diagonal of the rectangle whose horizontal side is $\sqrt{2}$ units long, and whose vertical side is 1 unit long. The successive square roots are located by repeating this process with a rectangle of unit height based on the line segment whose length is given by the preceding square root.

$$\sqrt{2} = \sqrt{1^2 + 1^2}$$
$$\sqrt{3} = \sqrt{(\sqrt{2})^2 + 1^2}$$
$$\sqrt{4} = \sqrt{(\sqrt{3})^2 + 1^2}$$
$$\ldots$$
The geometric process of finding \( \sqrt{2} \) corresponds to the algebraic process of finding successively better approximations of rational numbers between which \( \sqrt{2} \) lies.

You find the first pair by knowing that \( 1 < 2 < 1.41 \). You find the second pair by computing \( (1.4)^2 = 1.96 \) and knowing that \( 1.4 < \sqrt{2} < 1.5 \).

You find the third pair by computing \( (1.41)^2 = 1.9881 \) and knowing that \( 1.41 < \sqrt{2} < 1.415 \).

You find the fourth pair by computing \( (1.4142)^2 = 1.99993684 \) and knowing that \( 1.4142 < \sqrt{2} < 1.41421 \).

A succession of intervals of this type is called a sequence of nested intervals. This states that there is one and only one number between which \( \sqrt{2} \) lies. This means that \( \sqrt{2} \) is a number whose decimal representation is nonterminating, nonrepeating, and a nonterminating, nonrepeating decimal is a number that can be expressed as a repeating decimal. For example, 1 = 0.999 and \( 1 \neq 0.9999 \).

You approximate \( \sqrt{2} \) as you wish. You will always come closer to \( \sqrt{2} \) by knowing that \( 1 < \sqrt{2} < 1.41 \). You will never exact because \( \sqrt{2} \) is a nonterminating, nonrepeating decimal. By extending the number system to include nonterminating, nonrepeating decimals, we have the irrational number system. This system consists of all nonterminating, nonrepeating decimals. Each such number corresponds to the set of intervals: \([1.4, 1.5], [1.49, 1.50], [1.499, 1.500], 

\[0.3, 0.33, 0.333, 0.3333, 0.33333, \ldots, \] 

\[0.6, 0.66, 0.666, 0.6666, 0.66666, \ldots, \] 

\[0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots, \] 

\[1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots, \] 

\[2.1, 2.12, 2.123, 2.1234, 2.12345, \ldots, \] 

\[3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots, \] 

\[\text{The digits in the addends, you obtain as the sum with} \] 

\[\frac{1}{3} + \frac{2}{3} = 1. \]
To pinch in more closely on $\sqrt{2}$, subdivide the interval from 1.4 to 1.5 into ten parts, each of length .01. The drawing has been enlarged, as if by a magnifying glass, to enable you to pick out more readily the interval of length .01 containing $\sqrt{2}$. Do you see that $1.41 < \sqrt{2} < 1.42$?

To locate halves of square roots of integers, draw the line whose equation is $x = \frac{y}{2}$. Then, from the graphs of the roots of integers shown on the y-axis draw horizontal lines to meet the line $x = \frac{y}{2}$. The x-coordinates of these points will be half their y-coordinates. Thus, by marking these coordinates on the x-axis you will have filled in the points corresponding to the halves of the square roots of the integers.

To find the point corresponding to $\frac{\sqrt{3}}{2}$, start at the point $(0, \sqrt{3})$ and extend the horizontal line $y = \sqrt{3}$ to the right to meet the line $x = \frac{y}{2}$. By setting $y = \sqrt{3}$ in this latter equation, you see that the x-coordinate of this point is $\frac{\sqrt{3}}{2}$.
The geometric process of finding \( \sqrt{2} \) corresponds to the algebraic process of finding successive pairs of rational numbers between which \( \sqrt{2} \) lies:

You find the first pair by knowing that \( 1 < 2 < 2^2 \). You find the second pair by computing \( (1.4)^2 = 1.96 \) and \( (1.5)^2 = 2.25 \). Hence, \( 1.4 < \sqrt{2} < 1.5 \).

Next, by squaring the numbers that correspond to each pair, you approximate \( \sqrt{2} \) by \( \frac{1.41}{} \). You can continue this process, repeating the first pair by knowing that \( 1 < 2 < 2^2 \). You can find the second pair by computing \( (1.41)^2 = 1.9881 \), and so on. By repeating this process, you approximate \( \sqrt{2} \) by \( \frac{1.41}{}, \frac{1.41}{}, \frac{1.41}{}, \), and so on.

A succession of intervals of this type is called a sequence of nested intervals. This sequence represents a repeating, nonterminating, nonrepeating decimal in the real number system. It is the property of completeness of the real number system that there is one, and only one, number between each pair of nested intervals. This means that a number always closes on exactly one number. Every interval is split into two equal half-intervals, and each half-interval is successively split into two equal half-intervals, until the intervals represent nonterminating, nonrepeating decimal numbers. The sequence of nested intervals represents a repeating, nonterminating, nonrepeating decimal (the irrational number system). This means that a number always closes on exactly one number.

A sequence of nested intervals: \([1.4, 1.5] \), \([1.49, 1.51] \), \([1.499, 1.501] \), \([1.4999, 1.5001] \), etc. This means that \( \frac{3}{5} \) has a repeating decimal representation of \( 0.666 \ldots \). The terminating decimals are based on the rules of sums, for example, to add \( 0.333 \ldots \) and \( 0.666 \ldots \), you obtain as the sum of the digits in the addends, you obtain as the sum with \( \frac{1}{3} + \frac{2}{3} = 1. \)
To close in further on \( \sqrt{2} \), subdivide the interval from 1.41 to 1.42 into 10 equal parts, each of length .001. From the enlarged drawing you can see that 1.414 < \( \sqrt{2} \) < 1.415. By repeating this process you can approximate \( \sqrt{2} \) to more and more decimal places.

The preceding construction is repeated by using the lines \( x = \frac{y}{5}, x = \frac{y}{3}, \) and \( x = \frac{2y}{3} \) to locate these numbers on the x-axis:

\[
\begin{align*}
\sqrt{2} & : 1.41421356237 \ldots \\
\sqrt{3} & : 1.73205080757 \ldots \\
\sqrt{5} & : 2.23606797749 \ldots \\
\sqrt{6} & : 2.44948974278 \ldots \\
\sqrt{7} & : 2.64575131106 \ldots \\
\end{align*}
\]

Since \( \sqrt{a} = \sqrt[ab]{b} \) for any positive integers a and b, you can locate the positive square roots of any rational number on the x-axis. This process does not locate all of the irrational numbers. (It does locate all the rational numbers.) Numbers which are not square roots of rational numbers, such as \( \sqrt{2}, \sqrt{6}, \) and multiples of \( \pi \), cannot be located by this method.
The geometric process of squeezing in on $\sqrt{2}$ corresponds to the algebraic process of finding successive pairs of rational numbers between which $\sqrt{2}$ lies:

The numbers of the first pair differ by 1, [1, 2].
The numbers of the second pair differ by .1, [1.4, 1.5]
The numbers of the third pair differ by .01, [1.41, 1.42]
The numbers of the fourth pair differ by .001, [1.414, 1.415]

You find the first pair by knowing that $1^2 < 2 < 2^2$. You find the second pair by computing squares of successive tenths: $(1.1)^2$, $(1.2)^2$, until you find that $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$; as $1.96 < 2 < 2.25$, you know that $1.4 < \sqrt{2} < 1.5$.

Next, by squaring successive hundredths you discover that $(1.41)^2 = 1.9881$, and that $(1.42)^2 = 2.0164$, so $1.41 < \sqrt{2} < 1.42$. As you repeat this process, you approximate $\sqrt{2}$ more closely by terminating decimals (rational numbers).

Notice that each interval $[1, 2]$, $[1.4, 1.5]$, $[1.41, 1.42]$, ... lies within the preceding interval. Furthermore, their lengths become smaller, so that by continuing the process you can find in the succession an interval that is as short as you wish.

A succession of intervals of this type is called a sequence of nested intervals. The property of completeness states that there is one, and only one, number belonging to every interval in a sequence of nested intervals. This means that a sequence of nested intervals always closes on exactly one number.

Many sequences of nested intervals represent nonterminating, nonrepeating decimals. Since the decimals for rational numbers always terminate or repeat, the rational number system is not complete. By extending the number system to include numbers represented by nonterminating, nonrepeating decimals (the irrational numbers), you obtain the real number system. It is the property of completeness of this set that permits you to pair every real number with exactly one point on the number line and to assign to each point on the line exactly one real number.

Consider this set of nested intervals: $[1.4, 1.5]$, $[1.49, 1.5]$, $[1.499, 1.5]$, ...; do you see that it closes in on 1.5 or $\frac{3}{2}$? This means that $\frac{3}{2}$ has two decimal representations: a terminating one, 1.5 and a nonterminating, repeating one, 1.4999 ... .

In general, every terminating decimal can be expressed as a repeating decimal whose repeating block is $\overline{9}$; for example, $1 = 0.999\ldots$ and $1.32 = 1.31\overline{9}$.

The rules for operating with nonterminating decimals are based on the rules of operation for terminating decimals, for example, to add $0.333\ldots$ and $0.666\ldots$:

<p>| | | | | | |</p>
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<tbody>
<tr>
<td>.3</td>
<td>.33</td>
<td>.333</td>
<td>.3333</td>
<td>.33333</td>
<td>.333333</td>
</tr>
<tr>
<td>.6</td>
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<td>.666666</td>
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<td>.99</td>
<td>.999</td>
<td>.9999</td>
<td>.99999</td>
<td>.999999</td>
</tr>
</tbody>
</table>

You can see that by taking more digits in the addends, you obtain as the sum $0.999999\ldots$, or 1. This checks with $\frac{1}{3} + \frac{2}{3} = 1$. 


To multiply these same numbers, proceed as follows:

<table>
<thead>
<tr>
<th>.3</th>
<th>.33</th>
<th>.333</th>
<th>.3333</th>
<th>.33333</th>
</tr>
</thead>
<tbody>
<tr>
<td>.6</td>
<td>.66</td>
<td>.666</td>
<td>.6666</td>
<td>.66666</td>
</tr>
<tr>
<td>.18</td>
<td>198</td>
<td>1998</td>
<td>19998</td>
<td>199998</td>
</tr>
<tr>
<td>198</td>
<td>1998</td>
<td>19998</td>
<td>199998</td>
<td></td>
</tr>
<tr>
<td>.2178</td>
<td>1998</td>
<td>19998</td>
<td>199998</td>
<td></td>
</tr>
<tr>
<td>.221778</td>
<td>19998</td>
<td>199998</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.22217778</td>
<td>199998</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.2222177778</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Although it was not apparent in the first few products, you can see that the product begins .222. By taking more digits in the factors you can determine additional places in the product. As a matter of fact, \( \frac{1}{3} \div \frac{2}{3} = \frac{2}{9} = .2222 \).

You can tell which of two real numbers is the greater by comparing their decimals; for example, \(.97423 < .97425\). It is also easy to insert between any two real numbers another real number; for instance, between the real numbers just named is the number \(.974233\). Because there is a real number between any two real numbers, the set of real numbers is said to have the property of density.

There are more irrational numbers than there are rational ones. The sequence of drawings on the lower halves of the preceding pages shows you how to locate on the number line the square root of any positive rational number. Unless a rational number is the square of a rational number, its square root is an irrational number, so that many of the points located correspond to irrational numbers. Moreover, there are irrational numbers which cannot be expressed as square roots, cube roots, or as roots at all. The measure of the circumference of a circle whose diameter is 1 unit in length is given by the irrational number \(\pi = 3.14159265\ldots\), which is an unending, nonrepeating decimal which cannot be found by taking roots of rational numbers. However, it can be found by the method of nested intervals, as was first demonstrated by Archimedes.

Archimedes used a circle of radius 1 whose area is measured by \(\pi\). The area of this circle is between the areas of two squares, one with its vertices on the circle and the other with its sides just touching the circle, which are called inscribed and circumscribed squares. The areas of these two squares are given by \((\sqrt{2})^2 \) and \(2^2\), respectively, and, since the area of the circle is between these two numbers, \(2 < \pi < 4\). By increasing the number of sides of the inscribed and circumscribed regular polygon, you obtain smaller and smaller intervals containing \(\pi\). Using regular octagons, for example, you find that \(2.8 < \pi < 3.3\). Archimedes used polygons of 96 sides and found that \(3\frac{10}{71} < \pi < 3\frac{1}{7}\). By taking polygons with enough sides, you can approximate \(\pi\) as closely as you wish.

The system of real numbers can be built up step by step from the natural numbers used for counting, and it can be shown to satisfy all the familiar rules of arithmetic while also possessing the properties of density and completeness.
11-9 Radical Equations

An equation having a variable in a radicand is a radical equation. The simplest kind of radical equation is one like $\sqrt{x} = 3$, which you solve by squaring each of its members.

$$\sqrt{x} = 3 \quad (\sqrt{x})^2 = 3^2 \quad x = 9$$

Check: $\sqrt{9} = 3 \quad \checkmark \quad \therefore \text{The solution set is } \{9\}, \text{Answer.}$

To solve radical equations which have several terms in each member but only one radical term, you first isolate the radical term in one member. Then you can square each member and solve the resulting equation.

**EXAMPLE** Solve $3 = 2\sqrt{x} + x$.

**Solution:**

1. Isolate the radical term in one member of the equation.
   $$3 = 2\sqrt{x} + x$$
   $$3 - x = 2\sqrt{x}$$

2. Square both members.
   $$(3 - x)^2 = (2\sqrt{x})^2$$
   $$9 - 6x + x^2 = 4x$$

3. Solve the resulting equation.
   $$x^2 - 10x + 9 = 0$$
   $$(x - 1)(x - 9) = 0$$
   $$x - 1 = 0 \quad x - 9 = 0$$
   $$x = 1 \quad x = 9$$

4. Check.
   Substitute, and then take the principal root of the number in the radicand.
   $$3 \overset{?}{=} 2\sqrt{1} + 1 \quad 3 \overset{?}{=} 2\sqrt{9} + 9$$
   $$3 \overset{?}{=} 2 \cdot 1 + 1 \quad 3 \overset{?}{=} 2 \cdot 3 + 9$$
   $$3 = 2 + 1 \quad \checkmark \quad 3 \overset{?}{=} 6 + 9 \quad \text{No}$$
   The solution set is $\{1\}$, Answer.

Can you explain why the “squared” equation in Step 2 may not be equivalent to the given equation? Notice that

if $a = b$, then $a^2 = b^2$; but if $a^2 = b^2$, it need not be true that $a = b$.

$5^2 = (-5)^2$, but $5 \neq -5$. 
Solve each radical equation.

<table>
<thead>
<tr>
<th>A</th>
<th></th>
<th>B</th>
<th></th>
<th>C</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\sqrt{2y} = 2$</td>
<td>6.</td>
<td>$\sqrt{r} - \frac{1}{2} = 2$</td>
<td></td>
<td>11. $\sqrt{2y - 3} = 3$</td>
</tr>
<tr>
<td>2.</td>
<td>$\sqrt{3x} = \frac{3}{5}$</td>
<td>7.</td>
<td>$\sqrt{\frac{r}{2}} = 1$</td>
<td></td>
<td>12. $\sqrt{3y + 4} = 1$</td>
</tr>
<tr>
<td>3.</td>
<td>$\sqrt{5x} = \frac{5}{2}$</td>
<td>8.</td>
<td>$\sqrt{\frac{y}{3}} = 2$</td>
<td></td>
<td>13. $4\sqrt{5m} = 20$</td>
</tr>
<tr>
<td>4.</td>
<td>$\sqrt{7z} = \frac{7}{4}$</td>
<td>9.</td>
<td>$\sqrt{x + 10} = 3$</td>
<td></td>
<td>14. $\frac{1}{2}\sqrt{10m} = \frac{5}{2}$</td>
</tr>
<tr>
<td>5.</td>
<td>$\sqrt{s} + 5 = 7$</td>
<td>10.</td>
<td>$\sqrt{x - 4} = 9$</td>
<td></td>
<td>15. $\sqrt{p + 5} = 3$</td>
</tr>
<tr>
<td>16.</td>
<td>$\sqrt{p} + 2 = 1$</td>
<td>19.</td>
<td>$\sqrt{\frac{9x}{2}} - 2 = 7$</td>
<td></td>
<td>20.</td>
</tr>
<tr>
<td>17.</td>
<td>$\sqrt{4y - 3} + 7 = 10$</td>
<td>21.</td>
<td>$\sqrt{\frac{3x - 5}{4}} = 2$</td>
<td></td>
<td>27. Solve for $E$: $v = \sqrt{\frac{2E}{m}}$</td>
</tr>
<tr>
<td>18.</td>
<td>$\sqrt{5y - 1} - 8 = -1$</td>
<td>22.</td>
<td>$\sqrt{\frac{5y + 1}{6}} = 1$</td>
<td></td>
<td>28. Solve for $h$: $r = \sqrt{\frac{7v}{22h}}$</td>
</tr>
<tr>
<td>23.</td>
<td>$\sqrt{x} = 2\sqrt{5}$</td>
<td>24.</td>
<td>$3\sqrt{k} = 4\sqrt{3}$</td>
<td></td>
<td>29. $\sqrt{a^2 + 9} = a + 3$</td>
</tr>
<tr>
<td>25.</td>
<td>$4\sqrt{5r^2 + 5} = 20$</td>
<td>26.</td>
<td>$2\sqrt{3s^2 - 12} = 12$</td>
<td></td>
<td>30. $\sqrt{b^2 - 16} = b - 4$</td>
</tr>
<tr>
<td>27.</td>
<td>$\sqrt{19 - y} = y - 7$</td>
<td>37.</td>
<td>$\sqrt{19 - y} = y - 7$</td>
<td></td>
<td>32. $\sqrt{n} = -\frac{n}{2}$</td>
</tr>
<tr>
<td>28.</td>
<td>$\sqrt{y + 3} = y - 9$</td>
<td>38.</td>
<td>$\sqrt{y + 3} = y - 9$</td>
<td></td>
<td>33. $\sqrt{x^2 + 4} - 1 = x$</td>
</tr>
<tr>
<td>39.</td>
<td>$\sqrt{x + 2} + 3 = 0$</td>
<td>40.</td>
<td>$1 + \sqrt{x + 4} = 0$</td>
<td></td>
<td>34. $\sqrt{x^2 - 4} - 2 = x$</td>
</tr>
</tbody>
</table>

Problems

1. Three times the square root of a number is 30. Find the number.
2. One-third the square root of a number is 5. What is the number?
3. The square root of 3 less than twice a number equals 3. Find the number.
4. When 9 is added to 5 times a certain number, the square root of the result is 8. Find the number.

5. The perimeter of a square in terms of its area is \( p = 4\sqrt{A} \). Solve for \( A \) in terms of \( p \), and find the area of a square whose perimeter is 32 inches.

6. The distance, in miles, from a turret \( h \) feet above ground to the horizon is \( d = \frac{3h}{\sqrt{2}} \). Solve this rule for \( h \) in terms of \( d \), and determine the turret’s height if the lookout sees 9 miles to the horizon.

7. Solve this form of the Pythagorean theorem for the positive value of \( a \):
\[
c = \sqrt{a^2 + b^2}
\]
Then find \( a \) when \( c = 17 \) and \( b = 15 \).

8. Solve the rule \( s = \frac{1}{4}\sqrt{f} \) (the number of seconds \( s \) a body falls a distance of \( f \) feet from a position of rest) for \( f \) to find the approximate height of the Washington Monument, given that an object reaches the ground almost 6 seconds after being dropped from its top.

9. The rule \( d = \frac{H}{\sqrt{1.6n}} \) gives the diameter \( d \) of each of \( n \) cylinders of an automobile with \( H \) horsepower. What horsepower is developed in a six-cylinder engine whose cylinders have a three-inch diameter?

10. A moving car hits a wall with the force with which it would strike the ground in falling from a certain height. The relationship of the car’s rate \( r \), in miles per hour, to the height \( h \), in feet, is:
\[
r = \sqrt{\frac{h}{.0336}}
\]
How far (to the nearest foot) would a car fall and strike the ground with the force with which it would hit a wall at 60 miles per hour?

11. The velocity of sound, in meters, at \( t \) degrees is \( V = 333\sqrt{14.0037t} \). At what temperature will sound travel 11,988 meters per second?

12. The amount \( a \) to which $1 grows when invested for 2 years at \( r\% \) per year is
\[
a = \left(1 + \frac{r}{100}\right)^2
\]
At what rate will $1 become $1.21?

13. The geometric average of two positive numbers is the positive square root of their product. Find a pair of consecutive positive integers whose geometric average is \( 6\sqrt{2} \).

14. The illumination \( i \) in foot-candles on a surface \( f \) feet from a light with \( c \) candle power is:
\[
i = \frac{c}{f^2}
\]
How far away can a lamp of 54 candle power light a screen as well as a lamp of 6 candle power at 10 feet can?
One of the earliest planned cities, a housing project for men working on the Pyramids, was built in Egypt around 3000 B.C. Since that time, city planning has been carried on for a variety of reasons, and in varying degrees of detail. Some cities, like Pompeii, Washington, D.C., and Brazilia, were planned from their inception; others, like Paris and Rome, were re-planned after poverty and congestion had made them almost uninhabitable.

The photograph shows two modern-day civic planners marking suggested improvements on an acetate overlay. These planners must bring into harmony the large populations, intricate transportation networks, and vast industries of today’s urban areas. With the aid of modern techniques of statistical and mathematical analysis, however, the modern civic planner can effectively cope with many of these dynamic problems.

One type of problem a civic planner solves is shown on the pad. Surveys were made of the traffic flow at an intersection where many accidents occurred. The data obtained were analyzed mathematically to show: (1) the traffic in each direction at the intersection; (2) the total traffic through the intersection; (3a) north traffic minus south traffic; (3b) east minus west traffic; and (4) the direction and amount of the heaviest traffic flow.

The civic planner found the greatest flow of traffic to be north and east. He advised that the intersection be modified to favor traffic in these directions, either by having longer traffic-light intervals, or by providing more lanes. These changes would reduce accidents and allow faster, less congested movement of vehicles through the intersection.
Chapter Summary

Inventory of Structure and Method

1. A rational number can be expressed as a fraction in an unlimited number of ways, and as a terminating decimal or a repeating decimal. Between every pair of rational numbers is an infinite set of rational numbers. To approximate a number in decimal notation, you retain the digits unchanged if the first digit dropped is less than 5; you increase the last digit by one if the first digit dropped is 5 or more.

2. If \( a \geq 0 \) and \( b > 0 \), then \( \sqrt{a^2} = \sqrt{a} \cdot \sqrt{b} \); and \( \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \).

3. In any right triangle, if \( c \) is the hypotenuse, and \( a \) and \( b \) are the other sides, \( a^2 + b^2 = c^2 \).

4. Roots of rational numbers are not all rational numbers. Irrational numbers are represented by unending, nonrepeating decimals. The real number system can be put into one-to-one correspondence with the points of a line. If you divide a number by a divisor which is smaller in absolute value than the square root of the number, the quotient is larger in absolute value than the square root.

5. Square roots having the same radicand can be added or subtracted by applying the distributive property.

6. Squaring both members of an equation produces a new equation which includes the roots of the given equation as a subset of its own roots.

Vocabulary and Spelling

number system (p. 397) extracting a root (evolution) (p. 403)

rational number (p. 397) root (p. 403)
rational operations (p. 397) radical (p. 403)
property of density (p. 398) root index (p. 403)
terminating (p. 401) radicand (p. 403)
nonterminating (p. 401) principal root (p. 404)
repeating, periodic (p. 401) perfect square (p. 407)
equals approximately (p. 402) irrational numbers (p. 407)
power of a number (p. 403) real number system (p. 407)
raising to a power (involution) property of completeness (p. 407)
(p. 403) property of pairs of divisors (p. 408)
Chapter Test

11-1 Arrange in order of size, with the largest first.
1. $\frac{6}{15}, \frac{7}{16}, \frac{9}{17}$
2. $\frac{147}{72}, \frac{79}{48}, \frac{479}{288}$
Find a number halfway between these numbers.
3. $\frac{15}{44}, 1$
4. $-\frac{33}{7}, \frac{10}{9}$
Tell whether or not each statement is true.
5. There is no smallest rational number greater than 0.
6. Between two integers is an infinite number of integers.

11-2 Write as terminating or repeating decimals.
7. $\frac{5}{7}$
8. $-\frac{13}{15}$
9. $12\frac{3}{8}$
Write as common fractions.
10. $.7272...$
11. $.8$
12. $2.345$

11-3 Evaluate each expression, using the product and quotient properties.
13. $\sqrt{900}$
14. $-\sqrt{\frac{1}{54}}$
Find each solution set.
15. $x^2 = 400$
16. $3y^2 - 363 = 0$

11-4 Compute the indicated square root to the nearest hundredth.
17. $\sqrt{4.72}$
18. $-\sqrt{8.2314}$

11-5 Find the third side of right triangle $ABC$ if its hypotenuse is $AB$.
19. $BC = 20, AB = 25$
20. $AC = 12, BC = 16$

11-6 Find these products and quotients in simplest form.
21. $4\sqrt{13} \cdot 2\sqrt{13}$
22. $\frac{2\sqrt{49}}{3\sqrt{40}}$
23. $3\sqrt{7} \cdot 2\sqrt{12} \cdot \sqrt{21}$
24. $\frac{2\sqrt{a^3b^7}}{\sqrt{8a^5b^5}}$

11-7 Combine these radicals.
25. $3\sqrt{5} + 2\sqrt{5} - \sqrt{5}$
26. $10\sqrt{\frac{1}{50}} - 6\sqrt{\frac{1}{2}}$
27. $3\sqrt{27} - \sqrt{\frac{1}{3}}$
28. $\frac{1}{2}\sqrt{\frac{3}{5}} + \frac{3}{4}\sqrt{\frac{7}{6}}$
11–8 Express in simplest form.
29. \( (3 - \sqrt{2})(3 + \sqrt{2}) \)  
30. \( \frac{1}{3\sqrt{3} + 2} \)  
31. \( (4 + \sqrt{5})^2 \)  
32. \( \frac{\sqrt{6} + 1}{3 - \sqrt{6}} \)

11–9 Find the solution set.
33. \( \sqrt{3}y + 12y = 0 \)  
34. \( \sqrt{3}z - z = 2 \)

\[ \text{Chapter Review} \]

11–1 The Nature of Rational Numbers  
Pages 397–400
1. A number system is a set of numbers with the operations \( \_? \) and \( \_? \) defined.
2. Of the fractions \( \frac{1}{2} \) and \( \frac{3}{4} \), the greater is \( \_? \).
3. The number halfway between \( \frac{7}{12} \) and \( \frac{5}{8} \) is \( \_? \).
4. The property of density implies that between any two different rational numbers is an \( \_? \) number of others.

Tell whether or not each statement is true.
5. If a rational operation is performed on a rational number, the result is always a rational number.
6. The set of rational numbers between 1 and 2 is finite.

11–2 Decimal Forms of Rational Numbers  
Pages 400–403
Write as decimals.
7. \( 2.5 \)  
8. \( -1\frac{1}{2} \)  
9. \( \frac{3}{2} \)
Write as common fractions.
10. \( 0.636363 \ldots \)  
11. \( 1.46 \)  
12. \( 0.297 \)

Round each number to the nearest tenth.
13. \( 8.446 \)  
14. \( 1.0\bar{6} \)  
15. \( \frac{13}{20} \)  
16. \( 4.97 \)

11–3 Roots of Numbers  
Pages 403–407
17. In \( \sqrt[n]{n+1} \), the root index is \( \_? \), and \( n + 1 \) is the \( \_? \).
18. Every positive number has \( \_? \) square roots.
19. The expression \( \sqrt{9} = \_? \); but \( -\sqrt{9} = \_? \).

Evaluate each expression.
20. \( \sqrt{1024} \)  
21. \( \pm \sqrt{\frac{87}{25}} \)  
22. \( -\sqrt{\frac{1}{361}} \)
Find each solution set.
23. \( n^2 - 64 = 0 \)  
24. \( 32s^2 = 8 \)  
25. \( (r - 5)^2 = 25 \)

### 11-4 Properties of Irrational Numbers

26. In the sentences, \( x^2 = 5, \ y^2 = 9, \ z^2 = .4, \ u^2 = 4\frac{1}{3}, \ v^3 = 8 \), only \( \_?\) and \( \_?\) can represent rational numbers.
27. Every point on the number line corresponds to a \( \_?\) number which may be either \( \_?\) or \( \_?\).
28. The decimal form of an irrational number is neither \( \_?\) nor \( \_?\).

Compute to the nearest hundredth.
29. \( \sqrt{35} \)  
30. \( \sqrt{9.641} \)  
31. \( -\sqrt{.6803} \)

32. The area of a square floor is 250 square feet. How long is it, to the nearest hundredth of a foot?

### 11-5 Geometric Interpretation of Square Roots

33. The \( \_?\) of a right triangle is longer than either of the other sides.

The hypotenuse of right triangle \( ABC \) is \( AB \). How long is the side not given?
34. \( AC = 8 \text{ cm}.; \ BC = 15 \text{ cm}. \)  
35. \( AC = 40''; \ AB = 41'' \)

Determine whether the following are the sides of a right triangle.
36. 39, 36, 15  
37. 18, 30, 24

### 11-6 Multiplication, Division, and Simplification of Radicals

Perform the indicated operations.
38. \( 2\sqrt{7} \cdot 3\sqrt{7} \)  
39. \( \frac{6\sqrt{81}}{2\sqrt{27}} \)  
40. \( 5\sqrt{12} \div 2\sqrt{3} \)  
41. \( \frac{\sqrt{4r^2s^3t^4}}{2\sqrt{9r^6s^6}} \)  
42. \( \sqrt{3}(2\sqrt{2} - 3\sqrt{3}) \)
43. \( \frac{\sqrt{8} - 2\sqrt{28}}{2\sqrt{2}} \)

44. The expression \( \frac{\sqrt{3}}{\sqrt{2} - 1} \) is not in simplest form because its \( \_?\) contains a radical.
45. You \( \_?\) the denominator to transform a fraction having an irrational denominator into an equivalent fraction having a \( \_?\) denominator.

Simplify these radicals.
46. \( \frac{2}{3}\sqrt{162} \)  
47. \( -2\sqrt{\frac{4}{5}} \)
11-7 Addition and Subtraction of Radicals

Simplify.

50. \(6\sqrt{18} - 2\sqrt{98} + 2\sqrt{9}\)

51. \(3\sqrt{6} + 6\sqrt{6} - 8\sqrt{1.5}\)

52. To combine similar radicals by addition or subtraction, apply the \(\square\) property.

53. A rectangle's sides are in the ratio 1 to 3, and its diagonal is 39 inches. Find the width, to the nearest tenth of an inch.

11-8 Multiplication of Binomials Containing Radicals

Express in simplest form.

54. \((2\sqrt{5} + 5)^2\)

55. \((3\sqrt{2} - 1)(3\sqrt{2} + 1)\)

56. \(\frac{2}{\sqrt{6} + 2}\)

57. \(\frac{10}{2\sqrt{3} - \sqrt{2}}\)

11-9 Radical Equations

58. The result of squaring each member of an equation is not always \(\square\) to the original equation.

Find the roots of the following equations.

59. \(\sqrt{2x} = 6\)

60. \(2\sqrt{3n} = 9\)

61. \(2 + \sqrt{r} = 5\)

62. \(\sqrt{2s - 1} + 3 = 2\)

63. \(\sqrt{2x - 2} - x = 1\)

64. \(\sqrt{3y + 13} = y + 3\)

Cumulative Review: Chapters I-II

1. Simplify: If \(x \neq 0\), \(x - \frac{x^2 - y^2}{x} = \square\).

2. Simplify: If \(x \neq 3\) and \(x \neq -3\),

\[
\frac{2x + 6}{x^2 - 9} \cdot \frac{x^2 - 6x + 9}{2} \div (x - 3) = \square.
\]

3. Expand \((a - b)^2\).

4. Give the roots of \(x^2 - 2x = 3\).

5. The lowest common denominator of \(\frac{1}{x^2} + \frac{z}{x} + \frac{x}{z^2}\) is \(\square\).
Complete each statement.

6. A man invests $10,000, part (x) at 4% and the rest at 5%. The amount (in terms of x) invested at 5% is $\square$.

7. A canoeist who paddles r miles per hour in still water paddles $\square$ miles in 1 hour against a current of c m.p.h.

8. The set of rational numbers is a $\square$ of the set of real numbers.

9. The product of a number and its $\square$ is always 1.

Match an answer in the second column to each condition in the first ($m > 1$).

10. The average of three consecutive odd integers beginning with $m - 2$.
   (a) $m$

11. The value of $x$ in $\frac{15}{20m} = \frac{3}{x}$.
   (b) $2m$

12. The radius of a circle whose area is $4\pi m^2$.
   (c) $3m$

13. The root $x$ of $mx - 3m^2 = x - 3m$.
   (d) $4m$

14. The value of $5|m|$.
   (e) $5m$
   (f) $6m$

Solve Exercises 15–18 for each variable.

15. $3 = m^2 + 2m$

16. $\frac{x + 1}{5} + \frac{x - 1}{2} = 9 - \frac{x}{3}$

17. $y^2 - 3y = 0$

18. $9x + 4y = -21$

19. Solve and graph the solution set of $4 - 2(3r - 1) \leq 12$.

20. Solve graphically: $3x - 2y = 12$, $x + y = -1$.

21. Graph the solution set of $3x - 2y > 12$, $x + y < -1$ as a doubly shaded area.

22. If $y = 1 - \frac{2}{t}$, and $t$ is positive and increases, how does $y$ change?

Perform the required operations.

23. Find the rational number midway between $\frac{1}{3}$ and $\frac{5}{8}$.

24. Write $\frac{2}{25}$ as a terminating decimal.

25. Find the rational number represented by $\sqrt[3]{\frac{8}{27}}$.

26. Simplify $\sqrt{5n} \cdot \sqrt{10n^3}$.

27. Simplify $\frac{\sqrt{216}}{2\sqrt{2}}$.

28. Express $(\sqrt{2} + 1)(\sqrt{2} - 1)$ as a rational number.

29. Solve $x + 3\sqrt{x} = 10$. 
30. The total area of the six faces of a cube is 540 square inches. How long, to the nearest tenth, is one edge?

31. Find the third side of a right triangle whose hypotenuse is 17 and whose other side is 8.

32. If 3 tons of hard coal and 2 tons of soft coal cost $112, and 2 tons of hard and 6 tons of soft coal cost $168, what does a ton of each cost?

33. In 2 hours John overtakes scouts walking at 3 m.p.h., who started 1 hour ahead of him. How fast does John walk?

34. Had \( x \) students voted for Alan instead of for Ruth, he would have had four times as many votes as Ruth. However, if \( 2x \) students had voted for Ruth instead of for Alan, the vote would have been a tie. How many of all 500 votes went to each?

**Extra for Experts**

**Algebraic Fallacies**

Seemingly valid methods of working with algebraic statements may lead to illogical results because they conceal violations of the properties of numbers. Test your ability to detect improper procedures by trying to find the errors in the following arguments.

**Questions**

Justify the valid steps in each series of transformations, and explain the error in each invalid step.

1. To "prove" that \( 1 = 0 \), let \( a \) and \( b \) be two integers such that

\[
 a = b + 1.
\]

Then,

\[
(a - b)a = (a - b)(b + 1)
\]

\[
a^2 - ab = ab + a - b^2 - b
\]

\[
a^2 - ab - a = ab + a - a - b^2 - b
\]

\[
a(a - b - 1) = b(a - b - 1)
\]

\[
a = b
\]

\[
b + 1 = b
\]

\[
\therefore 1 = 0
\]

(*Hint: Substitute \( a = b + 1 \) in \( a - b - 1 \).)
2. To “prove” that $1 = 8$, let $y = 3$; then,

$$\frac{y + 9}{y - 1} - 6 = \frac{15 - 5y}{y - 8}$$

$$\frac{15 - 5y}{y - 1} = \frac{15 - 5y}{y - 8}$$

$$\frac{1}{y - 1} = \frac{1}{y - 8}$$

$$y - 1 = y - 8$$

$$-1 = -8$$

$$\therefore 1 = 8$$

(Hint: What is the value of $15 - 5y$ if $y = 3$?)

3. To “prove” that $0 > 3$, let $a$ be any number such that $a > 3$.

Then,

$$3a > 3(3)$$

$$3a - a^2 > 9 - a^2$$

$$a(3 - a) > (3 - a)(3 + a)$$

$$a > 3 + a$$

$$\therefore 0 > 3$$

4. To “prove” that $1 > 1$, let

$$c > 1$$

and

$$c = d.$$  

Then,

$$c - 1 = d - 1.$$  

$$c - 1 = -1(1 - d)$$

$$(c - 1)^2 = (-1)^2(1 - d)^2$$

$$(c - 1)^2 = (1 - d)^2$$

$$c - 1 = 1 - d$$

$$\therefore c - 1 = 1 - c$$

$$2c = 2$$

$$c = 1$$

$$\therefore 1 > 1$$

(Hint: $5^2 = 25$ and $(-5)^2 = 25$, but $5 \neq -5$.)

Having examined these fallacies, you should be aware of the disastrous consequences of dividing by zero, of multiplying an inequality by a negative number without changing the order of the inequality, and of assuming that numbers whose squares are equal must equal each other.
Some words have more than one meaning, foreign words as well as English words. In medieval Italian, *bigollo* meant both "traveler" and "blockhead." When, about 1200, Leonardo of Pisa signed himself Leonardo Bigollo, he may have meant that he was a great traveler, for so he was. Some historians, however, think that by using this name he was saying to those who disdained him: "You called me a blockhead; look what a blockhead can do! Can you smart men do as much?" For Leonardo of Pisa, self-educated son of a merchant, was not popular with the professors of his day. But time has shown that he was far and away the greatest mathematician of the thirteenth century.

One of the discoveries for which he is remembered is called *Fibonacci's series*. Fibonacci is another name Leonardo used; it is short for *filius Bonacci*, "son of Bonaccio." His series is a sequence of numbers, obtained like this:

\[
\begin{align*}
1 + 1 &= 2 \\
1 + 2 &= 3 \\
2 + 3 &= 5 \\
3 + 5 &= 8 \\
5 + 8 &= 13 \\
8 + 13 &= 21 \\
13 + 21 &= 34 \\
21 + 34 &= 55
\end{align*}
\]

Of course, there is no end to the Fibonacci series.

There seems to be no end to the places you find it. Examine some green plant — a weed will do. Start at the bottom of the stalk, and count the leaves on it. When you come to a leaf that is directly over the one you started with, you will find that you have reached some number in the Fibonacci series. Make a similar count of the leaves of head lettuce, or of an artichoke, or of an onion! You will find numbers in Fibonacci's series every time.

Leonardo of Pisa probably did not know most of the implications of his series. Mathematicians today probably do not know all of them. They are still studying the series of numbers discovered by Leonardo the traveler — or blockhead!

*Leonardo of Pisa who discovered a number series in the thirteenth century which is not fully explained even today!*
Functions and Variation

Have you considered the importance of knowing that two things have been paired? The weather bureau (upper and lower photos) can tell you the amount of rainfall, barometric pressure, temperature, and the direction and velocity of the wind during any period for which it has records. A less complex example can be found in a prom in which dancers are grouped into pairs of a boy and a girl, each. The replacement set for one member of each pair is the set of boys; the replacement set for the other member of each pair is the set of girls. By selecting a specified member from one set, you can identify the corresponding partner from the other set, just as the weather bureau can tell you the rainfall for a specified day.

Mathematicians form similar pairs of numbers; when one is designated, the other can be identified. Although this may seem like a simple process, it is one of the unifying concepts of mathematics.

SELECTING PAIRS OF NUMBERS

12-1 Relations

In graphing equations you formed ordered pairs of numbers. Any pairing of the members of two sets of numbers is a relation. Figure 12-1 shows the graph of the relation specified by the roster \{(0, 0), (1, 1), (−1,1), (2,2), (−2, 2), (3,3), (−3, 3), . . . \}, or by the rule: With each integer associate its absolute value. Thus, a relation may be described by a graph, a roster, or a rule.

The two sets of numbers which are paired in a relation are called the domain of definition (domain) and the range of values (range) of the relation. In the graph of a relation, points of the horizontal axis represent members of the domain, and points of the vertical axis represent members of the range. In the roster of a relation, the first coordinate of each ordered pair is a member of the domain, and
the second is a member of the range. In stating the rule for a relation, you must specify the range and domain. In Figure 12–1, the domain is the set of integers, and the range is the set of nonnegative integers.

Frequently you can state a rule as an open sentence: \( y = |x| \), where \( x \in \{\text{integers}\} \). Such an open sentence is called a formula, and the relation it defines consists of the ordered pairs \((x, y)\) in its solution set. In a formula the domain of the relation is the replacement set of one variable, in this case \( x \), and the range is the replacement set of the other, \( y \).

When a formula is used for a relation and the domain and range are not specified, you agree to include in the domain and range those real numbers for which the formula is meaningful. As division by 0 is meaningless, the domain and range of the relation defined by \( y = \frac{1}{x} \) are understood to be \{real numbers except 0\}.

**ORAL EXERCISES**

Give each relation and its domain and range, and a formula, if possible.

**SAMPLE.** The price \( p \) in cents for \( n \) five-cent pencils is five times \( n \).

What you say: The relation is the set of ordered pairs \((n, p)\) for which \( p = 5n \).

The domain is \{nonnegative integers\}.
The range is \{nonnegative integral multiples of 5\}.

1. The diameter \( d \) of a circle is twice the radius \( r \).
2. The perimeter \( p \) of a square is four times the length \( s \) of one side.
3. A batting average is the number of hits \( h \) divided by times at bat \( b \).
4. Divide a length in centimeters \( c \) by 2.54 for the length in inches \( i \).
5. Multiply a length in inches \( i \) by 2.54 for the length in centimeters \( c \).
6. The charge \( c \) for a power mower for \( d \) days is $10, plus $2 a day.
7. The area \( A \) of a square is the square of one side \( s \).
8. Associate with each number \( n \) its absolute value \( |n| \).
9. A stadium has 500 $1 seats. The receipts \( r \) are determined by the number \( n \) of seats sold for any game.
10. A builder makes $1000 profit on each of 75 houses. His daily profit \( p \) is determined by the number of houses \( h \) sold each day.
11. A theater charges teen-agers 50¢ and preteens 30¢. The cost of admission \( c \) is determined by the patron’s age \( a \).
12. A man receives 15¢ for each object he makes. His daily wage $d$ is determined by the number of objects $o$ he makes, which varies from 100 to 200.

Give a formula for each relation, and state the elements of its domain.

13. $t \begin{array}{c}1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$
\[ d \begin{array}{c}3 \\ 6 \\ 9 \\ 12 \\ 15 \end{array} \]

16. $v \begin{array}{c}4 \\ 8 \\ 12 \\ 16 \end{array}$
\[ u \begin{array}{c}2 \\ 4 \\ 6 \\ 8 \end{array} \]

14. $h \begin{array}{c}1 \\ 2 \\ 3 \\ 4 \end{array}$
\[ s \begin{array}{c}35 \\ 70 \\ 105 \\ 140 \end{array} \]

17. $s \begin{array}{c}4 \\ 6 \\ 8 \end{array}$
\[ r \begin{array}{c}3 \\ 5 \\ 7 \end{array} \]

15. $x \begin{array}{c}2 \\ 3 \\ 4 \\ 5 \end{array}$
\[ y \begin{array}{c}6 \\ 7 \\ 8 \\ 9 \end{array} \]

18. $x \begin{array}{c}3 \\ 4 \\ 5 \end{array}$
\[ y \begin{array}{c}8 \\ 10 \\ 12 \end{array} \]

**WRITTEN EXERCISES**

Graph the relation defined by each formula for the domain \{-2, -1, 0, 1, 2\}.

1. $y = x$
2. $y = x^2 - x$
3. $y = |x|$
4. $y = x^2 - |x|$

State each domain and range as a finite set.

5. \{(0, 2), (1, 3), (2, 4)\}
6. \{(1, -1), (2, 0), (3, 1)\}
7. \{(-1, 4), (3, -2), (0, 0)\}
8. \{(5, 0), (-2, 3), (5, 1), (3, 6)\}
9. \{(-5, 4), (4, -4), (6, 4), (-7, -4)\}
10. \{(-6, -1), (-8, -2), (-9, -1), (-10, -2)\}

For each roster write a formula; then copy and complete the chart.

11. Cost of gasoline:

<table>
<thead>
<tr>
<th>Gallons</th>
<th>2</th>
<th>?</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>?</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cents</td>
<td>64</td>
<td>96</td>
<td>?</td>
<td>160</td>
<td>320</td>
<td>384</td>
<td>?</td>
</tr>
</tbody>
</table>

12. Parallelogram:

<table>
<thead>
<tr>
<th>Height</th>
<th>4</th>
<th>5</th>
<th>?</th>
<th>9</th>
<th>10</th>
<th>?</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>28</td>
<td>35</td>
<td>49</td>
<td>?</td>
<td>70</td>
<td>105</td>
<td>?</td>
</tr>
</tbody>
</table>
In Exercises 19–22 diagonals join one vertex to the other vertices of each polygon.

19. Chart the relation between the number of sides \( n \) and the number of triangles \( t \) in each figure. Write a formula for this relation.

20. Chart the relation between the number of sides \( n \) and the number of diagonals \( d \). Write a formula for this relation.

21. Write a formula for \( t \) in terms of \( d \).

22. Write a formula which pairs the sum \( s \) of the angles (in degrees) of each figure with the number of sides \( n \).

### 12–2 Functions

If you graph the two relations \{(2, 1), (1, 0), (2, 3), (−1, −1)\}, and \{(1, 2), (0, 1), (3, 2), (−1, −1)\}, (Figure 12–2), you find that two points of the first relation lie on the same vertical line. Its roster has one element of the domain appearing in two ordered pairs. Such is not true of the second relation. Relations of the second kind are called functions. Thus, a function is a relation which assigns to each element...
of the domain one and only one element of the range. The latter is called the value of the function for the given element of the domain. Relations which are functions are quite common. If a person regularly records his height at different ages, he will never repeat the same age, although eventually he will repeat the same height.

**EXAMPLE 1.** Graph the relation defined by this formula, give its domain and range, and tell whether it is a function:

\[ y = 3x - 5, \]

\[ \text{if } 1 < x < 3. \]

**Solution:**

Domain: \( x \in \{\text{real numbers between 1 and 3}\} \)

Range: \( y \in \{\text{real numbers between } -2 \text{ and } 4\} \)

This relation is a function.

**EXAMPLE 2.** Graph the relation defined by this formula, give its domain and range, and tell whether it is a function:

\[ y = 4 \text{ if } x \geq 0 \text{ and } y = 3 \text{ if } x \leq 0. \]

**Solution:**

Domain: \( x \in \{\text{real numbers}\} \)

Range: \( y \in \{3, 4\} \)

This relation is not a function. Two points lie on the vertical line \( x = 0. \)
Give the domain and range of each relation, and tell if it is a function.

1. \[ y \] domain: \((-1, 0) \) range: \((2, 0)\)\[ y \] function.

2. \[ (0, 2), (-2, 1), (0, 1) \] domain: \((-2, 1)\) range: \((0, 1)\)\[ y \] function.

3. \[ (0, 0), (3, 1), (3, -2) \] domain: \((0, 3)\) range: \((0, -2)\)\[ y \] function.

4. \[ (-1, 1), (0, -1), (2, -1) \] domain: \((-1, 2)\) range: \((-1, -1)\)\[ y \] function.

5. \[ \] domain: \((-2, 2)\) range: \((-2, 2)\)\[ y \] function.

6. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

7. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

8. \[ \] domain: \((-\infty, \infty)\) range: \((0, 0)\)\[ y \] function.

9. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

10. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

11. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

12. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

13. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

14. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.

15. \[ \] domain: \((0, 0)\) range: \((0, 0)\)\[ y \] function.
1. The perimeter of a rectangle: \( p = 2(l + w) \).
   a. Write a formula for the perimeter when \( w = 3 \), and graph the function defined by the formula.
   b. Draw an arrow showing the point on your graph which indicates the perimeter of a rectangle \( 4\frac{1}{2} \) feet long.
   c. Indicate with an arrow the point showing the length of a rectangle whose perimeter is 20.

2. Distance traveled: \( d = rt \).
   a. Write a formula for finding the distance when \( r = 70 \), and graph the function defined by your formula.
   b. Indicate with an arrow the point on your graph which shows the distance traveled in 5 hours and 20 minutes.
   c. Draw an arrow showing the point on the graph which indicates the number of hours required to travel 245 miles.

Write a formula and then graph the function defined by each rule.

3. The number of centimeters is 100 times the number of meters.
4. The number of feet is 3.3 times the number of meters.
5. The number of kilograms is .45 times the number of pounds.
6. The number of U.S. dollars is 2.8 times the number of British pounds.
7. The number of Italian lire is 625 times the number of U.S. dollars.
8. The average density of 100 cubic centimeters of matter is its weight in grams divided by 100.
9. The pressure of water on the bottom of a tank is 62.4 times the depth of water, in feet.
10. The work of pulling a load \( 3\frac{1}{2} \) feet is \( 3\frac{1}{2} \) times the force exerted, in pounds.
11. The total number of dollars to be repaid on \$100\) borrowed at \( 5\% \) per year is 100 more than 5 times the number of years of the loan.
12. The total number of dollars to be repaid on \$200\) borrowed at \( 3\frac{1}{2}\% \) per year exceeds, by 200, 7 times the number of years of the loan.
13. A day’s hire of a car, in dollars, is 10, plus \( .11 \) times the miles driven.
14. The total cost, in dollars, of printing a magazine is 25, plus \( .12 \) times the number of magazines printed.

Graph the function defined by each indicated domain and formula.

15. \( y = \frac{1}{2}x - 5; -3 \leq x < 8 \)  
16. \( y = 3 - \frac{x}{2}; -2 < x \leq 6 \)
17. \( y = x^3 - 1; x \in \{-2, -1, 0, 1, 2\} \)
18. \( y = -x^2; x \in \{-2, -1, 0, 1, 2\} \)
19. \( C = 2\pi r; 0 \leq r \leq 14 \) (Use \( \frac{22}{7} \) for \( \pi \).)
20. \( C = \pi d; 0 \leq d \leq 7 \) (Use \( \frac{22}{7} \) for \( \pi \).)
21. \( F = \frac{8}{5} C + 32; -100 \leq C \leq 100 \)

22. \( P = 2(l + 3); l \geq 0 \)
23. \( r = \frac{100}{t}; t > 0 \)
24. \( S = 16r^2; t > 0 \)

25. Graph the function which assigns the number 1 to each nonnegative number, and the number 2 to each negative number.
26. Graph the function which assigns the number 3 to each positive number, and the number -2 to each nonpositive number.
27. Graph the function which assigns to each number half its absolute value.
28. Graph the function which assigns to each integer the negative of its absolute value.

VARIATION

12–3 Direct Variation and Proportion

The table shows the velocity \( v \) of a stone falling freely for \( t \) seconds. The ratio \( \frac{v}{t} \) for every pair of numbers is the same, that is, \( \frac{v}{t} = 16 \), or \( v = 16t \). Such a formula describes a direct variation.

A direct variation is a function in which the ratio between a number \( y \) of the range and the corresponding number \( x \) of the domain is the same for all pairs of the function.

\[
\frac{y}{x} = k \quad \text{or} \quad y = kx, \text{where } k \text{ is a constant.}
\]

You can say that \( y \) varies directly as \( x \) or \( y \) is directly proportional to \( x \) or \( y \) varies with \( x \). \( k \) is the constant of proportionality.

Figure 12–3 is the graph of \( y = 2x \). For every value of \( m \) the graph of \( y = mx \) is a straight line with slope \( m \) passing through the origin. Thus, the graph of a direct variation is a straight line passing through the origin and having a slope equal to the constant of proportionality.

Figure 12–3
If one ordered pair of a direct variation is \((x_1, y_1)\) (read \(x_{\text{sub 1}}, y_{\text{sub 1}}\)) and another of the same function is \((x_2, y_2)\), then

\[
\frac{y_1}{x_1} = k \quad \text{and} \quad \frac{y_2}{x_2} = k, \quad \text{or} \quad \frac{y_1}{x_1} = \frac{y_2}{x_2}.
\]

Such an equality of ratios is called a \textit{proportion}, and can be read \(y_1\) is to \(x_1\) as \(y_2\) is to \(x_2\). In this proportion, \(x_1\) and \(y_2\) are called the \textit{means} and \(y_1\) and \(x_2\) are called the \textit{extremes}. Because \(x_1y_2 = x_2y_1\),

\[
\text{in any proportion, the product of the means equals the product of the extremes.}
\]

\textbf{EXAMPLE.} A weight of 15 grams stretches a spring 5 centimeters. What weight stretches it 12 centimeters if the elongation is directly proportional to the weight?

\textbf{Solution 1:} Let \(E = \) elongation in centimeters and \(W = \) weight in grams.

\[
\frac{E_1}{W_1} = \frac{E_2}{W_2}; \quad E_1 = 5, \quad W_1 = 15, \quad E_2 = 12
\]

\[
\frac{5}{15} = \frac{12}{W_2}
\]

\[
5W_2 = 180
\]

\[
W_2 = 36
\]

\textbf{Solution 2:} \(E_1 = kW_1\) \(\quad E_2 = kW_2\)

\[
5 = k \cdot 15 \quad 12 = \frac{1}{3}W_2
\]

\[
\frac{1}{3} = k \quad 36 = W_2
\]

\textbf{Check:} \[
\frac{5}{15} = \frac{1}{3}, \quad \frac{12}{36} = \frac{1}{3}
\]

\[
\frac{1}{3} = \frac{1}{3} \quad \checkmark
\]

\(\therefore\) A weight of 36 grams is needed, Answer.
Read each proportion in two ways.

**SAMPLE.** \( \frac{a}{3} = \frac{4}{7} \)

*What you say:* \( a \) divided by 3 equals four-sevenths, and \( a \) is to 3 as 4 is to 7.

1. \( \frac{2}{3} = \frac{x}{6} \)
2. \( \frac{x}{2} = \frac{15}{6} \)
3. \( \frac{a}{b} = \frac{3a}{3b} \)
4. \( \frac{7d}{7b} = \frac{d}{b} \)
5. \( \frac{ab}{a} = \frac{bc}{b} \)
6. \( \frac{ac}{bc} = \frac{ad}{bd} \)
7. \( \frac{17}{m} = \frac{34}{5} \)
8. \( \frac{n}{21} = \frac{3}{63} \)
9. \( \frac{x + 3}{4} = \frac{x - 3}{3} \)

Translate into formulas expressing direct variation.

**SAMPLE.** The area of a circle varies directly as the square of its radius. The constant of proportionality is \( \pi \).

*What you say:* Let \( A = \) area, and let \( r = \) radius.

Then \( \frac{A}{r^2} = \pi \), or \( A = \pi r^2 \), or \( \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2} \).

*Notice* that this exercise introduces *direct variation as a power.*

10. The heat required to melt a substance varies with its mass.
11. A shadow’s length at any hour varies with the object’s height.
12. The force to push an object along a plane varies with the object’s weight.
13. The annual income from an investment varies with the interest rate.
14. The area of an equilateral triangle varies with the square of its side.
15. The surface area of a sphere varies with the square of its radius. The constant of proportionality is \( 4\pi \).
16. The pressure of water varies with its depth.
17. The weight of any liquid varies with its volume.
18. The power to move a ship varies with the cube of its speed.
19. The distance needed to stop a car varies with the square of its speed.
20. The wind’s pressure on a flat surface varies with the square of its velocity.
21. The power an airplane needs varies as the fourth power of its velocity.

State whether or not each formula or roster expresses direct variation.

22. \( p = 4s \)  
24. \( \frac{y}{x} = 1 \)  
26. \( t = n - 2 \)

23. \( \frac{T_1}{T_2} = \frac{V_1}{V_2} \)  
25. \( xy = k \)  
27. \( \frac{x_1}{x_2} = \frac{y_2}{y_1} \)

WRITTEN EXERCISES

In these direct variations, find the value of the indicated variable.

1. \( x_1 = 12; x_2 = 8; y_2 = 14; y_1 = ? \)
2. \( x_1 = 18; y_1 = 8; y_2 = 6; x_2 = ? \)
3. \( u_1 = 15; z_1 = 35; u_2 = 8; z_2 = ? \)
4. \( m_1 = 13; m_2 = 9; n_2 = 6; n_1 = ? \)
5. \( r_1 = 2.6; s_1 = 6.5; s_2 = 4.5; r_2 = ? \)
6. \( t_1 = 2\frac{1}{2}; t_2 = 6\frac{2}{3}; y_1 = 4\frac{1}{2}; y_2 = ? \)

Find all values of the variable for which each proportion is true.

7. \( \frac{x - 3}{x} = \frac{4}{7} \)  
9. \( \frac{y}{y - 16} = \frac{5}{3} \)  
11. \( \frac{5}{9} = \frac{3x + 2}{12} \)
8. \( \frac{w - 4}{w} = \frac{5}{9} \)  
10. \( \frac{z}{z - 3} = \frac{7}{4} \)  
12. \( \frac{11}{6} = \frac{3x - 2}{27} \)
13. \( \frac{12}{x} = \frac{x}{3} \)  
14. \( \frac{x + 1}{x} = \frac{x + 2}{x - 1} \)  
15. \( \frac{x - 1}{x + 3} = \frac{x + 3}{x} \)

16. If \( x \) varies directly as \( y - 2 \), and \( x = 6 \) when \( y = 11 \), find \( y \) when \( x = 4 \).
17. If \( y \) is proportional to \( 2x + 3 \), and \( y = 12 \) when \( x = 6 \), find \( x \) when \( y = 36 \).
18. If \( a, b, \) and \( c \) vary directly as \( 4, 5, \) and 10 in that order, and \( a + c = 42 \), find \( b \).
19. If \( V \) varies directly as \( R^3 \), and \( V = \frac{9\pi}{2} \) when \( R = 1\frac{1}{2} \), write the formula for \( V \) in terms of \( R \).
Prove each of these properties of proportions if neither \( x_1 \) nor \( x_2 \) is zero.

**SAMPLE.** If \( \frac{y_1}{x_1} = \frac{y_2}{x_2} \), then \( y_1x_2 = x_1y_2 \).

**Solution:**

\[
\begin{align*}
\frac{y_1}{x_1} &= \frac{y_2}{x_2} \\
\left( x_1x_2 \right) \left( \frac{y_1}{x_1} \right) &= \left( x_1x_2 \right) \frac{y_2}{x_2} \\
y_1x_2 \left( \frac{x_1}{x_1} \right) &= x_1y_2 \left( \frac{x_2}{x_2} \right) \\
\therefore y_1x_2 &= x_1y_2
\end{align*}
\]

Given

Multiplication property of equality

Commutative and associative properties

Multiplicative property of 1

20. If \( \frac{y_1}{x_1} = \frac{y_2}{x_2} \) and if \( y_2 \neq 0 \), then \( \frac{y_1}{y_2} = \frac{x_1}{x_2} \).

21. If \( \frac{y_1}{x_1} = \frac{y_1}{x_2} \) and if \( y_1 \neq 0 \), then \( x_1 = x_2 \).

22. If \( \frac{x}{y} = \frac{3}{4} \) and \( \frac{m}{n} = \frac{1}{3} \), find the values of \( \frac{4mx - ny}{2nx - my} \).

23. If \( \frac{r}{s} = \frac{3}{5} \) and \( \frac{r}{t} = \frac{9}{10} \), find the value of \( \frac{(s + t)}{s} \).

24. If \( \frac{x + 1}{y - 1} = \frac{5}{1} \) and \( \frac{x - 1}{y + 1} = \frac{1}{1} \), find the value of \( \frac{x}{y} \).

**PROBLEMS**

1. The ratio of an object’s weight on Earth to its weight on Neptune is 5:7. How much would a man who weighs 150 pounds here weigh on Neptune?

2. The ratio of slaked lime to sand in mortar is 2 to 5. How much sand is mixed with 14 bags of the lime in a supply of this mortar?

3. A charge of 3 tons of iron ore yields 2 tons of pig iron. What must the charge of ore be to obtain 78 tons of pig iron?

4. Sixteen tons of sulfur are needed to make 49 tons of sulfuric acid. How many tons of sulfur are needed to make 35 tons of acid?

5. Eighteen grams of hydrochloric acid neutralize 20 grams of lye. How many grams of lye are neutralized by 1080 grams of hydrochloric acid?

6. When an electric current is 20 amperes, the electromotive force is 100 volts. Find the force when the current is 50 amperes.
7. How long is a fifty-pound roll of wire which weighs .3 pounds per foot?
8. Find the resistance of 300 feet of a wire having .00027 ohms of resistance per foot, if the resistance varies directly as the length.
9. What length represents $5\frac{1}{2}$ feet on a map scaled at $\frac{3}{16}$ inch = 1 foot?
10. What is the scale of a drawing in which a 28-inch statue is drawn $1\frac{3}{4}$ inches high?

11. Pounds vary directly with cubic feet of a given substance, and $k$ is 170 for limestone and 140 for sandstone. Which is heavier and by how much — a slab of limestone 5 feet by 3 feet by 8 inches, or a slab of sandstone $4\frac{1}{2}$ feet by 4 feet by 6 inches?

12. In grams per cubic centimeter, $k$ for Celluloid is 1.4, and for ivory is 1.9. Find the difference in weight between Celluloid and ivory spheres 42 millimeters in diameter. Use $\pi = \frac{22}{7}$.

13. A diamond’s price varies as the square of its weight. If one weighing $\frac{7}{4}$ carat is worth $625, find the cost of a similar diamond of $1\frac{5}{8}$ carats.

14. Rod $A$ has 80 equal divisions, and rod $B$ has 100, although they are the same length. If lengths which equal 60 divisions of rod $B$ are cut from each, how many divisions are cut from rod $A$?

15. In a circuit of 48 volts a voltmeter registers 60 on its scale of 0–100. What is the maximum number of volts the meter can measure?

16. If 1 inch equals 50 miles on a map, and Wyoming is a rectangle $5\frac{1}{8}$ inches by $7\frac{1}{8}$ inches, calculate its area to the nearest square mile.

17. A commander assigns 175 men per square mile to a rectangular area that is $\frac{5}{7}$ inch by $\frac{9}{7}$ inch on a map where the scale is 1 inch = 4 miles. How many men will be in the area?

18. The liner United States displaces about 50,000 tons and is almost 1000 feet long. For a hundred-pound model what should be the scale and the length if the weight varies as the cube of the length?

### 12–4 Inverse Variation

Three rectangles whose lengths and widths are $(12, 2)$, $(8, 3)$, and $(6, 4)$ have the same area: $12 \cdot 2 = 24$, $8 \cdot 3 = 24$, $6 \cdot 4 = 24$, or $lw = 24$. Such an open sentence describes inverse variation.

An inverse variation is a function in which the product of the coordinates of its ordered pairs is a constant. For any ordered pair $(x, y)$ of the function, $xy = k$ or $y = \frac{k}{x}$. You say that $y$ varies inversely as $x$ or $y$ is inversely proportional to $x$, because $y = k \left(\frac{1}{x}\right)$, or $y$ varies directly as the inverse of $x$. 
If \((x_1, y_1)\) and \((x_2, y_2)\) are ordered pairs of an inverse variation,

\[ x_1y_1 = k \quad \text{and} \quad x_2y_2 = k, \quad \text{or} \quad x_1y_1 = x_2y_2. \]

If neither \(y_1\) nor \(x_2\) is 0,

\[ \frac{x_1y_1}{x_2y_1} = \frac{x_2y_2}{x_2y_1} \quad \text{or} \quad \frac{x_1}{x_2} = \frac{y_2}{y_1}. \]

You would not expect the graph of an inverse variation to be a straight line, because its equation, \(xy = k\), is not linear; one term, \(xy\), is of the second degree.

You would not expect the graph of an inverse variation to be a straight line, because its equation, \(xy = k\), is not linear; one term, \(xy\), is of the second degree.

![Figure 12-4](image)

As \(x\) increases, \(y\) decreases so that the product is always 1. Because of this, neither \(x\) nor \(y\) can have the value 0. Therefore, the graph consists of two separate branches which cross neither axis. For every nonzero value of \(k\), the graph of \(xy = k\) has this shape and is called a hyperbola (hy-pur-bo-la). The curve is in the first and third quadrants if \(k\) is positive and in the second and fourth quadrants if \(k\) is negative. If \(k\) were 0, what would be the limitation on the range and domain?

When negative answers are meaningless in practical problems, the range and domain are limited to positive numbers. The graph of such an inverse variation has only one branch.

One instance of inverse variation is the lever (lee-ver), a bar pivoted at a point called the fulcrum (ful-krum) (Figure 12-5). If weights \(w_1\) and \(w_2\) are placed at distances \(d_1\) and \(d_2\) from the fulcrum, and the lever is in balance, then

\[ d_1w_1 = d_2w_2 \quad \text{or} \quad \frac{d_1}{d_2} = \frac{w_2}{w_1}. \]

![Figure 12-5](image)
FUNCTIONS AND VARIATION

EXAMPLE 1. If an eighty-kilogram weight is 150 centimeters from the fulcrum of a lever, how far from the fulcrum is a ninety-kilogram weight which balances it?

Solution: Let \( w_1 = 80 \), \( d_1 = 150 \), \( w_2 = 90 \).

\[
d_1w_1 = d_2w_2
\]

Check:

\[
150(80) = d_290
\]

\[
d_2 = 133\frac{1}{3}
\]

\[
\frac{80}{90} = \frac{8}{9} \checkmark
\]

\[\therefore\] Distance of ninety kg. weight from fulcrum = \(133\frac{1}{3}\) centimeters, Answer.

The brightness of the illumination of an object varies inversely as the square of the distance from the source of light to the object. If \( I \) is the amount of illumination and \( d \) is the distance from the light source to the object, \( Id^2 = k \). If two objects are illuminated from the same source, \( I_1d_1^2 = I_2d_2^2 \) or \( \frac{I_1}{I_2} = \frac{d_2^2}{d_1^2} \).

EXAMPLE 2. The illumination of a book 15 feet from a lamp is 4 foot-candles. Find the illumination of the book 5 feet closer to the lamp.

Solution: Let \( I_1 = 4 \), \( d_1 = 15 \), \( d_2 = 10 \)

\[
I_1d_1^2 = I_2d_2^2
\]

Check:

\[
4(15)^2 = I_210^2
\]

\[
\frac{9}{4} = \frac{15^2}{10^2} \checkmark
\]

\[\therefore\] In the second position the illumination is 9 foot-candles, Answer.

ORAL EXERCISES

Translate each statement into a formula expressing inverse variation.

SAMPLE. The number of days required to complete a job varies inversely as the number of men working on it if they work at the same rate.

What you say: Let \( d = \) number of days; \( m = \) number of men.

\[\therefore\] \( dm = k \), or \( d = \frac{k}{m} \), or \( \frac{d_1}{d_2} = \frac{m_2}{m_1} \), or \( d_1m_1 = d_2m_2 \).
1. The base $b$ of a triangle of constant area varies inversely as its altitude $a$.
2. The altitude $a$ of a parallelogram of constant area varies inversely as its base $b$.
3. The volume $V$ of a gas at constant temperature varies inversely as its pressure $P$.
4. Air pressure $P$ is inversely proportional to altitude $A$.
5. The time $t$ required to move from one point to another is inversely proportional to the rate of motion $r$.
6. The share $s$ of one person in group expenses $e$ varies inversely as the number of persons in the group.
7. The temperature $t$ at which water boils varies inversely as the number of feet $h$ above sea level.
8. The amount of capital $P$ needed to yield a given income varies inversely as the rate of interest $r$.
9. When two gears mesh, their speeds in revolutions per minute, $R_1$ and $R_2$, vary inversely as the numbers of teeth on the gears, $T_1$ and $T_2$.
10. The force $F$ necessary to pry up a rock varies inversely as the length $L$ of the crowbar used.
11. When a generator supplies a fixed voltage to a circuit, the current in amperes $I$ is inversely proportional to the resistance in ohms $R$.

Which exercises indicate direct variation? Which show inverse variation?

12. $uv = k$
13. $u = \frac{k}{v}$
14. $\frac{u}{v} = k$
15. $kuv = 1$
16. $\frac{v}{u} = \frac{1}{k}$

17. | $m$ | 3 | 2 | .5 |
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>10</td>
<td>15</td>
<td>60</td>
</tr>
</tbody>
</table>
18. (1.5, .3), (2, .4), (3, .6)
19. (4, 1.5), (6, 1), (12, .5)

In these inverse variations, find the value of the indicated variable.

A 1. $x_1 = 12, x_2 = 15, y_1 = 20, y_2 = ?$
2. $x_1 = \frac{2}{3}, y_1 = \frac{3}{4}, y_2 = 6, x_2 = ?$
3. $p_1 = 1.6, v_1 = 36, p_2 = 1.8, v_2 = ?$
4. $p_1 = .75, v_1 = .4, p_2 = .5, v_2 = ?$
5. If $d = rt$ and $r$ is doubled while $d$ stays constant, how does $t$ change?
6. If $PV = k$ and $P$ is trebled while $k$ stays constant, how does $V$ change?
7. If \( x \) varies inversely as \( t + 3 \), and \( x = 6 \) when \( t = 7 \), find \( t \) when \( x = 15 \).

8. If \( y \) varies inversely as \( 2t - 1 \), and \( y = 8 \) when \( t = 7.5 \), find \( y \) when \( t = 8.5 \).

9. How far from a lamp does a book receive 4 times as much illumination as a book 3 feet from the lamp?

10. If \( F \) varies inversely as \( s^2 \), and \( F = 80 \) when \( s = 100 \), find \( F \) when \( s = 40 \).

11. If \( T \) is inversely proportional to \( d^2 \), what change in \( T \) doubles \( d \)?

12. If \( H \) varies inversely as \( R^2 \), what value of \( R \) causes \( H \) to become one-ninth as much as it is when \( R = 8 \)?

13. If \( S \) varies inversely as the cube of \( L \), and \( S = 20 \) when \( L = 3 \), write a formula for \( S \) in terms of \( L \).

14. If \( R \) is inversely proportional to \( T^2 \), and \( R = 5.5 \) when \( T = 2 \), write a formula for \( R \) in terms of \( T \).

**PROBLEMS**

Use an inverse variation to solve each problem.

1. At 40 m.p.h. how fast is a journey which takes 6 hours at 30 m.p.h.?
2. If 8 men do a job in 9 days, how long do 24 men take?
3. How far from the seesaw support must Mary sit to balance John, who sits 4 feet from it, if she weighs 80 and he weighs 100 pounds?
4. Jim, sitting 5 feet from the seesaw support, balances a friend who weighs 110 pounds and sits 6 feet from the support. How heavy is Jim?
5. A weight of 200 grams is 40 centimeters from the center support of a meter stick. Where would a weight of 400 grams balance the stick?
6. How much will each of 6 girls pay to rent a cottage which costs each of 5 girls $36?
7. At what rate does $1400 yield the same annual income as $2100 at 2%?
8. What sum at 3% yields the same yearly income as $1500 at 4%?
9. The altitude of a triangle is 15 inches, and the base, 6 inches. Find the altitude of a triangle of equal area whose base is 2 inches.
10. How many revolutions will a five-inch wheel make in going a distance which takes 42 revolutions of a thirty-inch wheel?
11. The volume of a gas is 40 cubic feet under 3 pounds pressure. What is its volume at the same temperature when the pressure is 8 pounds?
12. If the current is 18 amperes when the resistance is 5 ohms, what is the current when the resistance is 3 ohms?

13. A ten-inch pulley runs at 150 revolutions per minute (r.p.m.). How fast does the five-inch pulley it drives revolve, if the number of r.p.m. varies inversely as the diameter?

14. A gear with 18 teeth makes 100 r.p.m. and meshes with a gear having 12 teeth. What is the speed of the second gear if the number of r.p.m. varies inversely with the number of teeth?

15. Jack weighs 148 pounds and Jill weighs 116. How far from Jack, on a seesaw 13.2 feet long, is the support balancing them?

16. If the gas pressure in a cylinder varies inversely as the distance of the piston from one end, how far was the piston moved to double the pressure?

17. The weight of a body at or above the earth’s surface varies inversely as the square of the body’s distance from the earth’s center. What does a 540-pound projectile weigh 800 miles out from the earth’s surface? (Use 4000 miles as the earth’s radius.)

18. A three-quarter-inch wire has 12 ohms resistance; how much has the same length of half-inch wire, if resistance varies inversely as the square of the diameter?

12–5 Joint Variation and Combined Variation (Optional)

The area $A$ of a triangle depends upon its altitude $a$ and its base $b$. If $a$ and $b$ are measured in the same units,

$$A = \frac{1}{2} ab \quad \text{or} \quad \frac{A}{ab} = \frac{1}{2}.$$

The area is directly proportional to the altitude and to the base, and you say that the area of a triangle varies jointly as its base and altitude.

**Joint variation** occurs when a variable $z$ varies jointly as the product of variables $x$ and $y$. We say $z$ varies jointly as $x$ and $y$, and for a constant $k$, write:

$$\frac{z}{xy} = k \quad \text{or} \quad z = kxy$$

and

$$\frac{z_1}{x_1y_1} = \frac{z_2}{x_2y_2} \quad \text{or} \quad \frac{z_1}{z_2} = \frac{x_1y_1}{x_2y_2}.$$

**Example 1.** The volume of a circular cylinder varies jointly as its height and the square of its radius. If a right circular cylinder of height 10 inches and radius 3 inches has a volume of $90\pi$ cubic inches, find the volume of one with a height of 9 inches and a radius of 5 inches.
FUNCTIONS AND VARIATION

Solution: Let \( h_1 = 10, r_1 = 3, h_2 = 9, r_2 = 5, V_1 = 90\pi \).

\[
\frac{V_1}{h_1r_1^2} = \frac{V_2}{h_2r_2^2}
\]

Check:

\[
\frac{90\pi}{10(3)^2} = \frac{V_2}{9(5)^2}
\]

\[
\frac{90\pi}{10(3)^2} \Rightarrow \frac{90}{225} = \frac{90}{225} \checkmark
\]

The volume of the second cylinder is \( 225\pi \) cubic inches, Answer.

A second variation involving three variables is combined variation. Combined variation is indicated when a variable \( z \) varies directly as one variable \( x \) and inversely as another \( y \). For a constant \( k \),

\[
z = \frac{kx}{y} \text{ or } \frac{zy}{x} = k \text{ or } zy = kx \text{ and } \frac{z_1y_1}{x_1} = \frac{z_2y_2}{x_2} \text{ or } \frac{z_1}{x_1} = \frac{z_2}{x_2}.
\]

EXAMPLE 2. The force between two small electrical charges varies jointly as the charges on the bodies and inversely as the square of the distance between them. When the charge on one body is 9 units and on the other 8 units, and they are 6 centimeters apart, the force between them is 2 dynes. Determine the force on the bodies when they are 4 centimeters apart.

Solution: Let \( q_1 \) the charge on the first body = 9,

\( q_2 \) the charge on the second body = 8,

\( d \) distance between the two bodies = 6,

\( D \) distance between the two bodies = 4,

\( f \) force between the two bodies at 6 cm. = 2, and

\( F \) force between the two bodies at 4 cm. = \( x \).

\[
f = \frac{kq_1q_2}{d^2}, \quad F = \frac{kq_1q_2}{D^2},
\]

\[
\frac{F}{f} = \frac{kq_1q_2}{D^2} \cdot \frac{d^2}{kq_1q_2} \text{ or } \frac{F}{f} = \frac{d^2}{D^2}
\]

\[
\frac{x}{2} = \frac{6^2}{4^2} \quad \text{Check:} \quad 2 = \frac{k(9)(8)}{36}, \quad \frac{9}{2} = \frac{k(9)(8)}{16}
\]

\[
x = 2\left(\frac{36}{16}\right) = \frac{72}{16} = k, \quad \frac{72}{16} = k, \quad \frac{144}{16} = k
\]

\[
x = \frac{9}{2}
\]

\( x = \frac{9}{2} \)

.: The force when the bodies are 4 cm. apart is \( 4\frac{1}{2} \) dynes, Answer.
Express in words. Assume that $k$ is a constant.

**SAMPLE 1.** \[ \frac{A}{bh} = k \] \text{Solution: } A \text{ varies jointly as } b \text{ and } h.

**SAMPLE 2.** \[ PV = kt \] \text{Solution: } P \text{ varies directly as } t \text{ and inversely as } V.

1. \[ E = kIR \]
2. \[ A = kbh \]
3. \[ \frac{V}{T} = P = k \]
4. \[ \frac{t}{v} = k \]
5. \[ \frac{A_1}{A_2} = \frac{b_1h_1}{b_2h_2} \]
6. \[ \frac{E_1}{E_2} = \frac{I_1R_1}{I_2R_2} \]
7. \[ P = k \frac{T}{V} \]
8. \[ V = k \frac{T}{P} \]
9. \[ I = kPrt \]
10. \[ V = klwh \]
11. \[ \frac{V_1}{V_2} = \frac{B_1h_1}{B_2h_2} \]
12. \[ \frac{A_1}{A_2} = \frac{l_1w_1}{l_2w_2} \]
13. \[ \frac{t_1}{t_2} = \frac{n_1v_2}{n_2v_1} \]
14. \[ \frac{P_1}{P_2} = \frac{T_1V_2}{T_2V_1} \]
15. \[ R = \frac{12S}{\pi D} \]
16. \[ S = \frac{\pi RD}{12} \]

**WRITTEN EXERCISES**

Translate into formulas.

A
1. The area of a trapezoid varies jointly as its altitude and the sum of its bases.
2. The area of a rectangle varies jointly as the length and the width.
3. The volume of a pyramid varies jointly as the altitude and the area of the base.
4. The volume of a circular cone varies jointly as the altitude and the square of the radius.
5. The number of persons needed to do a job varies directly as the amount of work to be done and inversely as the time in which it must be done.
6. The time required for a journey varies directly as the distance traveled and inversely as the speed.
7. The number of gallons stored in a cylindrical tank varies jointly as the tank’s height and the square of the radius of its circular base.
8. The cost of a job varies jointly as the number of men working and the number of days they work.
9. Centrifugal force varies directly as the square of the speed of a moving object and inversely as the radius of its circular path.

10. The pressure necessary to force water through a pipe varies directly as the square of the water’s speed and inversely as the pipe’s diameter.

PROBLEMS

1. In the formula \( H = \frac{I^2Rt}{4} \), \( R \) remains constant. If \( I \) is doubled, and \( t \) is made 3 times as large, how is \( H \) changed?

2. In the formula \( F = \frac{mv^2}{r} \), \( m \) remains constant, \( v \) is halved, and \( r \) is doubled. How does \( F \) change?

3. \( W \) varies jointly as \( x \) and \( y \) and inversely as the square of \( z \). If \( w = 150 \), \( x = 15 \), \( y = 18 \), and \( z = 9 \), find (a) the constant \( k \) of variation, (b) the equation of relation, and (c) \( w \), when \( x = 21 \), \( y = 12 \), and \( z = 6 \).

4. \( r \) varies directly as the cube of \( s \) and inversely as \( t \) and the square of \( u \). If \( r = 4 \), \( s = 6 \), \( t = 2 \), and \( u = 3 \), find (a) the constant \( k \) of variation, (b) the equation of relation, and (c) \( t \) when \( r = 9 \), \( s = 15 \), and \( u = 5 \).

5. If 15 boys pick 360 boxes of berries in 4 hours, how many boys pick 540 boxes in 3 hours?

6. A rod’s weight varies jointly as its length and the area of its cross section. If a rod 4\( \frac{1}{2} \) feet long with a half-inch square cross section is 4.8 pounds, what weight has a similar rod 7\( \frac{1}{2} \) feet long whose cross section is \( \frac{1}{4} \) inch square?

7. When a mass moves at 30 feet per second in a circle whose radius is 6 feet, the centrifugal force is 1260 pounds. Find the force when that mass moves at 40 feet per second in a circle whose radius is 8 feet.

8. The load on a horizontal beam supported at its ends varies directly as the square of the beam’s depth and inversely as its length between supports. A beam 10 feet long and 6 inches deep bears 1350 pounds. What load can one 16 feet long and 8 inches deep bear?

9. The cost of operating an appliance varies jointly as the number of watts drawn, the hours of operation, and the cost per kilowatt-hour. A thousand-watt iron operates for 30 minutes for 2\( \frac{1}{2} \) at 4\( \frac{1}{2} \) per kilowatt-hour. What is the cost of cooking 20 waffles 3 minutes each, if the waffle iron uses 720 watts?
10. The heat developed in an electric wire varies jointly as the wire's resistance, the time the current flows, and the square of the current. In 2 minutes a current of 5 amperes develops 1200 heat units in a wire of 8 ohms resistance. What resistance has a similar wire which develops 6000 heat units with a current of 10 amperes in 5 minutes?

11. The wind pressure on a plane varies jointly as the surface area and the square of the wind's velocity. Under a velocity of 8 miles per hour, the pressure on 1 square foot is \( \frac{1}{2} \) pound. What is the velocity when the pressure on a rectangle 2 feet by 3 feet is 27 pounds?

12. The heat lost through a windowpane varies jointly as the difference of the inside and outside temperatures and the window area, and inversely as the thickness of the pane. If 180 heat calories are lost through a pane 50 by 24 centimeters, \( \frac{1}{2} \) centimeter thick, in one hour when the temperature difference is 30°C, how many are lost in one hour through a pane .4 centimeter thick having half the area, when the temperature difference is 32°C?

Chapter Summary

Inventory of Structure and Method

1. The ordered pairing of the members of two sets is a relation that can be shown by graph, roster, or rule. A formula is a rule stated as an open sentence. The domain of definition (domain) and the range of values (range) of the relation must be specified in each case. A function is a relation which assigns only one element of the range to an element of the domain.

2. Direct variation and inverse variation are special types of functions. If \( k \) is a constant, equations like \( y = kx \) and \( y = \frac{k}{x} \) are associated with, respectively, a direct and an inverse variation. In either case, you find the constant of proportionality, \( k \), by substituting in the equation a pair of values for the variable.

Vocabulary and Spelling

<table>
<thead>
<tr>
<th>Relation</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>relation</td>
<td>435</td>
</tr>
<tr>
<td>domain of definition (domain)</td>
<td>435</td>
</tr>
<tr>
<td>range of values (range)</td>
<td>435</td>
</tr>
<tr>
<td>formula</td>
<td>436</td>
</tr>
<tr>
<td>function</td>
<td>438</td>
</tr>
<tr>
<td>value of a function</td>
<td>439</td>
</tr>
<tr>
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<td>442</td>
</tr>
<tr>
<td>constant</td>
<td>442</td>
</tr>
<tr>
<td>constant of proportionality</td>
<td>442</td>
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<td>proportion</td>
<td>443</td>
</tr>
</tbody>
</table>
Chapter Test

12-1 Give the domain and range, with a formula, if possible.

1. The weight \( p \), in pounds, of \( n \) cubic feet of water is 62.4 times \( n \).

2. \[
\begin{array}{c|ccccc}
  x & 0 & 1 & 2 & 3 & 4 & 5 \\
  y & -1 & 1 & 3 & 5 & 7 & 9 \\
\end{array}
\]

3. Give the roster of \( s = 2x - x^2 \), when \(-4 < x \leq 2\).

12-2 4. Is the following relation a function? Justify your answer. 
\{(3, -1), (1, -1), (0, 0), (1, 1), (3, 1)\}

5. Graph \( y = 3 - 2x \), if \(|x| \leq 3\). Give its domain and range, and tell whether it is a function.

12-3 6. If \( t = 5.6 \) when \( r = 3.2 \), and \( t \) varies directly as \( r \), find \( r \) when \( t = 4.2 \).

7. A freely falling body falls a distance directly proportional to the time it is falling. If a body falls 576 feet in 6 seconds, find the constant of proportionality and how far it falls in 10 seconds.

12-4 8. If \( P \) is inversely proportional to \( V \), and \( P = 12 \) when \( V = 8 \), find \( V \) when \( P = 16 \).

9. The intensity \( H \) of heat radiation on a surface varies inversely as the square of the distance \( d \) from the heat source to the surface. If \( H = 2.7 \) when \( d = 4 \), find \( H \) when \( d = 3 \).

10. If \( m \) men do a job in \( h \) hours, how many can do it in 2 hours less? \((h > 2)\).

12-5 (Optional)

11. The volume of a pyramid varies jointly as its altitude and the area of its base. A pyramid whose base is a four-inch square and whose altitude is 6 inches has a volume equal to 32 cubic inches. Find the volume of one 8 inches high with a three-inch square base.

12. If \( y \) varies directly as \( n \) and inversely as \( t \), and \( y = 42 \) when \( n = 14 \) and \( t = 18 \), find \( n \) when \( y = 36 \) and \( t = 21 \).
**Chapter Review**

**12-1 Relations**

Pages 435–438

1. Any pairing of the elements of two sets of numbers is a ___.
2. A relation may be shown by ___, ___, or ___.
3. The first coordinate of the roster of a relation is a member of the ___; the second is a member of the ___.
4. A formula is a rule stated as an ___ sentence.
5. In stating the rule for a relation, always specify the ___.
6. Using \{3, 1, 0, −1, −3\} as the domain, give the roster of the relation \(m = t^2 - 3t\).
7. Give the domain and range of relation shown by the roster below. Find a formula for the relation.

<table>
<thead>
<tr>
<th>(x)</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>(y)</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
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</table>

**12-2 Functions**

Pages 438–442

8. A function is a kind of ___.
9. In a function, an element of the domain may appear ___ in the roster; an element of the range may appear ___ times.
10. Two or more points in the graph of a ___ may lie on the same vertical line.
11. A vertical line may not cut the graph of a ___ more than once.

Is the given relation a function? Justify your answer.

12. \{(−1, −2), (0, −3), (1, −2), (2, 1)\}

13. | \(x\) | −2 | 2 | 2 | 7 | 7 |
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</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>2</td>
<td>−2</td>
<td>3</td>
<td>−3</td>
</tr>
</tbody>
</table>

**12-3 Direct Variation and Proportion**

Pages 442–447

14. A direct variation is a kind of ___.
15. When the ___ of each pair of numbers of a function is the same throughout, the function is a ___ of ___.
16. A direct variation may be shown by \(\frac{y}{x} = k\) or \(y = ___\).
17. The constant \(k\) is called the ___ of ___.
18. If \(r\) varies directly as \(s\), then \(r_1:r_2 = ___:___\).
19. An equality of two ratios is called a ___.
20. In $a:b = c:d$, the means are $?$ and $?$. The extremes are $?$ and $?$. 

21. In a proportion, the $?$ of the means equals the product of the $?$. 

22. The graph of a direct variation is a $?$, whose slope is the $?$ of $?$, and whose $y$-intercept is $?$. 

23. If $c$ is directly proportional to $n$, and $c = 30$ when $n = 12$, find $n$ when $c = 40$. 

24. The lift on an airfoil $L$ is directly proportional to the square of the air speed $s$. If $L = 756$ when $s = 450$, find $L$ when $s = 300$. 

25. On a map, $1\frac{1}{2}$ inches represents 30 miles. Find the actual distance between two points $6\frac{1}{4}$ inches apart on the map. 

12-4 Inverse Variation

26. An inverse variation is a kind of $?$. 

27. An inverse variation is shown by $xy = k$ or $y = ?$, $x \neq ?$. 

28. If $r$ varies inversely as $s$, then $r_1:r_2 = ?:?$. 

29. The graph of an inverse variation is called a $?$. 

30. If $n$ varies inversely as $z$, then $n$ varies directly as the $?$ of $z$. 

31. Joan, weighing 108 pounds, sits 12 feet from the pivot of a seesaw. How many feet from the pivot should Bob sit in order to balance the seesaw if he weighs 144 pounds? 

32. The force $F$ of gravity on an earth satellite varies inversely as the square of its distance $s$ from the center of the earth. If $F = 120$ when $s = 60,000$, find $F$ when $s = 40,000$. 

33. If $x$ is inversely proportional to $y$ and $x = 1.5$ when $y = 6$, find the rule specifying the function. 

12-5 Joint Variation and Combined Variation (Optional)

34. When a variable varies jointly as other variables, it varies $?$ as the $?$ of the other variables. 

35. The centripetal force $f$ on a body varies jointly as its mass $m$ and the square of its speed $v$. If $f = 1,350,000$ when $m = 60$ and $v = 300$, find $m$ when $f = 1,920,000$ and $v = 400$. 

36. In a combined variation, a variable varies $?$ as one variable and $?$ as another. 

37. If 12 men can dig a trench 150 feet long in 3 hours, how fast can 8 men dig a similar trench 200 feet long?
Nylon, paraffin, acrylic fibers, plastics, and high-power fuel oils are so common a part of modern life that we rarely stop to consider their origin. Yet, if we do reflect, we realize that each of these invaluable products was the result of research in petroleum chemistry.

Petroleum in its natural state cannot be used as a fuel. Called crude oil, it is a complex mixture of gases and liquids, known as hydrocarbons. The common fuel oils are most easily obtained by distillation from raw petroleum. Chemists like the one in the photograph work continually to improve the quality and increase the yield of fuel from petroleum.

Polymerization, the process which gives us many plastics and synthetic fibers, is probably the major development in petroleum research to date. It is a process whereby two or more molecules of the same substance are combined to form one large molecule. Although the new substance retains the same proportion of elements as the original substance, it has a higher molecular weight and a different set of physical properties.

The work pad shows the mathematical relationships between two polymer groups and the molecules from which they were formed. The original substance, isoprene, has five carbon atoms to eight hydrogen atoms and a molecular weight of $5 \times 12.001 + 8 \times 1.008$. The first polymer, dipentene, also has a ratio of five carbon : eight hydrogen, but since each dipentene molecule is formed from two isoprene molecules, the molecular weight of dipentene is twice that of isoprene. The polyterpenes, the third of these polymers, are important because of their rubber-like characteristics. They are composed of $n$ number of isoprene units ($n$ is an element of a set of whole numbers), and their molecular weight is $n$ times that of isoprene.
**Linear Programming**

Graphing sets of linear inequalities can help you to make decisions such as this:

Each day Mr. Kay needs at least 40 milligrams of niacin, 12 milligrams of riboflavin, and 24 milligrams of thiamine. He can buy \( A \) pills for \( 2\theta \) each, containing 6 milligrams of niacin, 1 milligram of riboflavin, and 8 milligrams of thiamine, or \( B \) pills at \( 3\theta \) each, containing 8 milligrams of niacin, 4 milligrams of riboflavin, and 3 milligrams of thiamine. What combination of pills will satisfy his minimum needs at the smallest cost daily?

Let \( x = \) number of \( A \) pills used daily, and 
\( y = \) number of \( B \) pills used daily.

If \( C = \) total daily cost, in cents,
\[
C = 2x + 3y, \quad \text{or} \quad y = -\frac{2}{3}x + \frac{C}{3}
\]

You are trying to minimize \( C \) (find its smallest value) within the limits of these inequalities (constraints):

1. \( 6x + 8y \geq 40 \)
2. \( x + 4y \geq 12 \)
3. \( 8x + 3y \geq 24 \)
4. \( x \geq 0 \)
5. \( y \geq 0 \)

The total amount of each vitamin must equal at least the daily need. Mr. Kay cannot use a negative amount of either pill.
The graphs of these inequalities indicate the points in their common solution set (see figure, shaded area). Because the points satisfying all the constraints lie within or on its boundary line, this region is called the feasible region.

For any value of $C$, such as $C = 21$, the graph of the solution set of $C = 2x + 3y = 21$ is a straight line. For a smaller value of $C$, like $C = 4$, the graph is a line parallel to the graph of $C = 21$, with a smaller $y$-intercept. Different values of $C$ give a family of parallel lines, each having $y$-intercept, $C \over 3$. As the figure suggests, the line of this family, having the smallest $y$-intercept and, therefore, smallest $C$, and containing a point of the feasible region, must intersect this region at a corner point. Substituting the coordinates of the corner points in $C = 2x + 3y$ shows that point $D(4, 2)$ gives the least value, $C = 14$, since at the other corners $C$ has the values 24, 14.6, 24. Thus, the cheapest combination is four $A$ pills and two $B$ pills, with a total daily cost of 14¢.

Because the constraints, as well as the variable $C$, are linear in $x$ and $y$, this is called a linear programming problem. Under such conditions a linear expression takes on its maximum and minimum values at corner points.

Questions

1. Graph the set of points defined by $y \geq \frac{x}{2} + 1, y \leq 4$, and $y \geq -3x + 8$.
2. Over the polygon obtained in Question 1, determine the points where $2.5x + 0.8y$ has its maximum and minimum values, which are ___ and ___.
3. Find $x$ and $y$, maximizing $R = 4x + y$ subject to the constraints, $x \geq 0, y \geq 0, x + 2y \leq 6$, and $x + y \leq 3$.
4. A druggist wishes to display Toanup and Freshall bath salts on 18 inches of a shelf 10 inches deep with 10 inches of space above it. Toanup is at least four times as popular as Freshall, and a box of Toanup takes 35 cubic inches of space, while one of Freshall takes 40 cubic inches. If his profit on a box of Toanup is 28¢ and on a box of Freshall, 35¢, how many of each should he display to realize the maximum profit when all are sold? Find the maximum profit.
5. Machine $A$ runs for an hour for $1$, producing 120 bolts and 50 screws in that time. Machine $B$ runs for an hour at $1.20$ and produces 80 bolts and 80 screws. With a combined running time of no more than 15 hours, how long should each of the machines run to produce an order of 1000 bolts and 750 screws at the minimum operating cost?
A Friendship Pact and What Came of It

It was a great honor to study with the Imam Mowaffek. Did not all Persia know that everyone who did so attained fame and fortune, honor and happiness? Any boy would be proud to be accepted as his pupil. Surely Nizam and Hasan and Omar were proud.

But the three boys couldn’t quite believe that they were all destined for success. So one day they made a friendship pact, pledging each other, as Nizam later reported in his autobiography, “that to whomsoever this fortune falls, he shall share it equally with the rest.” Fortune favored Nizam. About the year 1050, he became Vizier to the Sultan; that is, the chief administrative officer in all Persia. True to his boyhood pact, he shared his good fortune with Hasan and Omar.

Hasan asked for and was given a position in the government. But he was jealous of Nizam, and tried to get his job for himself. As a result, the Sultan banished Hasan from his court. Hasan became the leader of a gang of outlaws. When he died, he was a fugitive from justice.

Omar’s request was also granted — a place to live and freedom to study. He used his freedom well. He made thousands of observations of the heavens, recorded the results in tables that other astronomers could use, and reformed the calendar. And he wrote a really excellent book on algebra. When Omar died, in 1123, he was famous as a mathematician and a scientist.

Omar is still famous today — primarily as a poet. For he was the Omar Khayyam who wrote the well-known Rubaiyat.

Do you recognize the problem in the multiplication of polynomials on this page of Omar Khayyam’s algebra? Here, also, is some of his poetry:

Ah, but my computations, people say,
Reduced the year to better reckoning?
Nay
’Twas only striking from the calendar
Unborn tomorrow and dead yesterday.
 Quadratic Equations and Inequalities

How long a cable should an engineer plan for a suspension bridge? What are the orbits of planets, asteroids, and satellites as they flash through the sky? What is the trajectory a missile follows to its destination (radar screen, bottom)? Mathematically, all of these problems have one common characteristic — their solutions involve quadratic equations.

Just as the natural world is not made up entirely of figures with straight lines, the world of algebra is not confined to linear equations. The study of quadratic equations will extend your insight into the properties and relationships of numbers; to the types of problems you can solve will be added those involving relationships of the second degree.

GENERAL METHODS OF SOLVING QUADRATIC EQUATIONS

13–1 The Square-Root Property

A quadratic equation can be put into the standard form \(ax^2 + bx + c = 0\), where \(a\), \(b\), and \(c\) are real numbers and \(a \neq 0\). At present your chief tool in solving such equations is factoring. If the equation is of the form \(ax^2 + c = 0\), it is a pure quadratic and may be solved by using the property of square roots of equal numbers:

\[
\text{If } r \text{ and } s \text{ are real numbers } r^2 = s^2 \text{ if, and only if, } r = s \text{ or } r = -s.
\]

EXAMPLE 1. Solve \(y^2 - 12 = 0\).

Solution:

\[
y^2 - 12 = 0 \\
y^2 = 12
\]

\[
y = \sqrt{12} \text{ or } y = -\sqrt{12}
\]

\[
y = 2\sqrt{3} \text{ or } y = -2\sqrt{3}
\]

Check:

\[
(2\sqrt{3})^2 - 12 \geq 0 \quad (-2\sqrt{3})^2 - 12 \geq 0
\]

\[
12 - 12 = 0 \checkmark \quad 12 - 12 = 0 \checkmark
\]

\[
\therefore \text{ The solution set is } \{2\sqrt{3}, -2\sqrt{3}\}, \text{ Answer.}
\]
EXAMPLE 2. Solve $64x^2 + 1 = 0$.

Solution:

$$64x^2 + 1 = 0$$
$$64x^2 = -1$$
$$x^2 = -\frac{1}{64}$$

Since negative numbers have no square roots in the set of real numbers, $64x^2 + 1 = 0$ is not solvable in the real number system.

This method also solves quadratic equations having a trinomial square as one member and a nonnegative constant as the other.

EXAMPLE 3. Solve $t^2 - 6t + 9 = 25$.

Solution:

$$t^2 - 6t + 9 = 25$$
$$(t - 3)^2 = 25$$
$$t - 3 = \pm\sqrt{25}$$
$$t - 3 = 5 \quad t - 3 = -5$$
$$t = 8 \quad t = -2$$

Do the elements of $\{-2, 8\}$ check as the roots of the given equation?

ORAL EXERCISES

Give the roots of these equations.

1. $r^2 = 4$
2. $s^2 = 9$
3. $z^2 = \frac{1}{9}$
4. $k^2 = \frac{1}{16}$
5. $x^2 - 49 = 0$
6. $y^2 - 64 = 0$
7. $g^2 - 5 = 0$
8. $p^2 - 8 = 0$
9. $4y^2 = 1$
10. $25x^2 = 1$
11. $81s^2 = 16$
12. $49t^2 = 4$
13. $r^2 - 2r + 1 = 0$
14. $v^2 + 2v + 1 = 0$
15. $8u^2 - \frac{1}{2} = 0$
16. $9t^2 - \frac{1}{4} = 0$

WRITTEN EXERCISES

Solve each equation.

A

1. $9x^2 - 25 = 0$
2. $25x^2 - 9 = 0$
3. $27 - y^2 = 0$
4. $125 - y^2 = 0$
5. $27m^2 - 3 = 0$
6. $8m^2 - 2 = 0$
7. $4y^2 - \frac{1}{25} = 0$
8. $\frac{1}{9} - 4y^2 = 0$
9. $3w^2 - 16 = 0$
10. $2w^2 - 121 = 0$
13. $(y + 2)^2 = 1$
16. $9(z - 5)^2 = 49$
11. $(x - 1)^2 = 4$
14. $(y - 2)^2 = 1$
17. $(r - \frac{1}{2})^2 = 36$
12. $(x + 1)^2 = 9$
15. $4(z + 3)^2 = 25$
18. $(r + \frac{1}{2})^2 = 25$
19. $(s + \frac{3}{2})^2 = 4$
22. $y^2 - 12y + 36 = 4$
25. $(t - 6)^2 = 5$
28. $(k - 1)^2 = 2$
31. $\frac{1}{3}z^3 - 5z = 0$
20. $(s - \frac{3}{2})^2 = 25$
23. $t^2 - \frac{1}{12}t + \frac{1}{121} = 1$
26. $(t + 6)^2 = 7$
29. $r^3 - 4r = 0$
32. $4z - \frac{1}{3}z^3 = 0$
27. $(k + 1)^2 = 3$
30. $9s - s^3 = 0$
33. $k^3 - 5k = 0$
34. $k^3 - 7k = 0$
35. $m^4 + 4m^3 + 4m^2 = 0$
36. $m^4 - 6m^3 + 9m^2 = 0$
37. $x^2 - 2\sqrt{2}x + 2 = 0$
39. $x^2 - 2\sqrt{2}x + 2 = 9$
40. $x^2 - 2\sqrt{3}x + 3 = 4$
41. Is $x^2 + 7 = 0$ solvable over the set of real numbers? Why?
42. Is $x^2 + \frac{9}{2} = 0$ solvable in the real number system? Why?

13–2 Checking Solution Sets

A method of checking roots of quadratic equations involves the property of factors whose product is zero. If $r$ and $s$ are the roots of a quadratic equation, you know that $x - r = 0$ and $x - s = 0$. Therefore, you can write

$$(x - r)(x - s) = 0 \text{ or } x^2 - (r + s)x + rs = 0.$$  

This latter equation is identical to one in the form $x^2 + bx + c = 0$, if $-(r + s) = b$ or $r + s = -b$, and $rs = c$. This is the property of the sum and product of the roots of a quadratic equation.

If the roots of a quadratic equation of the form $x^2 + bx + c = 0$ are $r$ and $s$, then $r + s = -b$ and $rs = c$.

EXAMPLE 1. Is $\{2, 3\}$ the solution set of $x^2 - 5x + 6 = 0$?

Solution: $x^2 - 5x + 6 = 0; b = -5, c = 6$.

$r + s = 2 + 3 = 5 = -( -5) = -b \checkmark$

$rs = 2 \cdot 3 = 6 = c \checkmark$

$\therefore \{2, 3\}$ is the solution set of $x^2 - 5x + 6 = 0$, Answer.
EXAMPLE 2. Is \( \{-3 + \sqrt{10}, -3 - \sqrt{10}\} \) the solution set of \( x^2 + 6x - 1 = 0 \)?

Solution:
Is the sum of the roots \(-6\)?
\[-3 + \sqrt{10} - 3 - \sqrt{10} \geq -6\]
\[-6 = -6 \checkmark\]

Is the product of the roots \(-1\)?
\[-(3 + \sqrt{10})(-3 - \sqrt{10}) \geq -1\]
\[-9 - 10 = -1 \checkmark\]

\( \therefore \{-3 + \sqrt{10}, -3 - \sqrt{10}\} \) is a solution set of \( x^2 + 6x - 1 = 0 \), Answer.

The numerals \(-3 + \sqrt{10}\) and \(-3 - \sqrt{10}\) may be written jointly as \(-3 \pm \sqrt{10}\) (read \(-3\) plus or minus \(\sqrt{10}\)).

**ORAL EXERCISES**

Give the sum and the product of the roots of each equation.

1. \( x^2 - 20x + 9 = 0 \)
2. \( x^2 + 17x + 10 = 0 \)
3. \( x^2 + x + 3 = 0 \)
4. \( x^2 - x + 2 = 0 \)
5. \( y^2 - 3y - 1 = 0 \)
6. \( y^2 - 19y - 4 = 0 \)
7. \( 2n^2 + 8n - 22 = 0 \)
8. \( 3m^2 + 3m - 15 = 0 \)
9. \( z^2 + 4z = 2 \)
10. \( z^2 + 10z = 10 \)

Is the set given for each equation its solution set?

11. \( x^2 + 2x - 8 = 0; \{-4, 2\} \)
12. \( A^2 - 5A = 0; \{5, 0\} \)
13. \( 2A^2 + 6A = 0; \{3, 0\} \)
14. \( x^2 - 8x - 20 = 0; \{10, -2\} \)
15. \( x^2 - 3x = 4; \{4, -1\} \)
16. \( r^2 + 5r = 14; \{-7, 2\} \)
17. \( 9 - n^2 = 0; \{\pm3\} \)
18. \( 2n^2 = 32; \{\pm4\} \)

**WRITTEN EXERCISES**

Determine whether or not the set given for each equation is its solution set.

A

1. \( 2t^2 - 7t - 4 = 0; \{4, -\frac{1}{2}\} \)
2. \( 2x^2 - x - 6 = 0; \{2, -1\frac{1}{2}\} \)
3. \( 4x^2 - 8x - 21 = 0; \{\frac{7}{2}, -\frac{3}{2}\} \)
4. \( 4x^2 - 4x - 15 = 0; \{\frac{3}{2}, -\frac{5}{2}\} \)
5. \( u^2 - 18u = 7; \{9 \pm 2\sqrt{11}\} \)
6. \( v^2 - 20v = 25; \{10 \pm 5\sqrt{5}\} \)
7. \( w^2 + 14w = 49; \{\pm7\} \)
8. \( x^2 = 2 - 10x; \{5 \pm 3\sqrt{3}\} \)
9. \( 3x^2 + 18x = 33; \{-3 \pm 2\sqrt{5}\} \)
10. \( x^2 = 9 - 12x; \{6 \pm 9\sqrt{15}\} \)  
11. \( n^2 - n - 3 = 0; \left\{ \frac{1 \pm \sqrt{13}}{2} \right\} \)  
12. \( n^2 + n - 9 = 0; \left\{ \frac{-1 \pm \sqrt{35}}{2} \right\} \)

Find a quadratic equation having the given solution set.

13. \( \{5, 2\} \)  
14. \( \{-2, 3\} \)  
15. \( \{-4\} \)  
16. \( \{0, -3\} \)  
17. \( \{\sqrt{7}, -\sqrt{7}\} \)  
18. \( \{-\sqrt{3}, \sqrt{3}\} \)  
19. \( \{1 + \sqrt{2}, 1 - \sqrt{2}\} \)  
20. \( \{3 - \sqrt{5}, 3 + \sqrt{5}\} \)  
21. \( \{3\sqrt{5}\} \)  
22. \( \{-2, -\frac{3}{2}\} \)

### 13–3 Completing a Trinomial Square

If you can transform a quadratic equation into one having a trinomial square as a member, you can find its solution set.

**EXAMPLE 1.** Solve \( x^2 + 2x - 8 = 0 \).

**Solution:**

1. Write an equivalent equation \( x^2 + 2x - 8 = 0 \) with the constant term as right member. 
2. Add to both members the number making the left member a trinomial square. 
3. Use the property of square roots. 
4. Form two linear equations. 
5. Solve. 
6. Check. 

\[ \begin{align*} 
2^2 + 2(2) - 8 & \equiv 0 \quad (-4)^2 + 2(-4) - 8 & \equiv 0 \\
0 & = 0 \checkmark & 0 = 0 \checkmark 
\end{align*} \]

\( \therefore \) The solution set is \( \{-4, 2\} \), Answer.

The only unfamiliar step in this solution is the second, called completing a trinomial square. To apply the method of this example to any quadratic equation in the form \( x^2 + bx = k \), you must be able to determine what to add to \( x^2 + bx \) to produce a trinomial square.

Analyze the following trinomials which are squares of binomials.

\[
\begin{align*}
(x + 5)^2 &= x^2 + 2 \cdot (5)x + (5)^2 \\
(x - n)^2 &= x^2 + 2(-n)x + (-n)^2 \\
(x - 3)^2 &= x^2 + 2(-3)x + (-3)^2 \\
\left(x + \frac{b}{2}\right)^2 &= x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2
\end{align*}
\]
Do you see that in each case the constant term is the square of half the coefficient of the linear term?

**EXAMPLE 2.** What value of \(c\) makes \(m^2 - \frac{2m}{5} + c\) a trinomial square?

*Solution:* Half the coefficient of the linear term is \(\frac{1}{2} \left(-\frac{2}{5}\right) = -\frac{1}{5}\).

\[
c = \left(-\frac{1}{5}\right)^2 = \frac{1}{25}
\]

*Check:* Is \(m^2 - \frac{2m}{5} + \frac{1}{25}\) a trinomial square?

\[
m^2 - \frac{2m}{5} + \frac{1}{25} = \left(m - \frac{1}{5}\right)\left(m - \frac{1}{5}\right)
\]

\[
= \left(m - \frac{1}{5}\right)^2 \checkmark
\]

\[
\therefore c = \frac{1}{25}, \text{ Answer.}
\]

To solve an equation whose quadratic term has a coefficient other than 1, you first may use the division property of equality and divide each term by the coefficient of the quadratic term.

**EXAMPLE 3.** Solve \(3x^2 - 2x - 9 = 0\).

*Solution:*

\[
3x^2 - 2x - 9 = 0
\]

\[
x^2 - \frac{2}{3}x - 3 = 0
\]

\[
x^2 - \frac{2}{3}x = 3
\]

\[
x^2 - \frac{2}{3}x + \frac{1}{9} = 3 + \frac{1}{9}
\]

\[
(x - \frac{1}{3})^2 = \frac{28}{9}
\]

\[
x - \frac{1}{3} = \pm\sqrt{\frac{28}{9}}
\]

\[
x - \frac{1}{3} = \sqrt{\frac{28}{9}} \quad x - \frac{1}{3} = -\sqrt{\frac{28}{9}}
\]

\[
x = \frac{1}{3} + \frac{2\sqrt{7}}{3} \quad x = \frac{1}{3} - \frac{2\sqrt{7}}{3}
\]

\[
x = \frac{1 + 2\sqrt{7}}{3} \quad x = \frac{1 - 2\sqrt{7}}{3}
\]
Check: Is the sum of the roots \(-(-\frac{2}{3})\)?

\[
\frac{1 + 2\sqrt{7}}{3} + \frac{1 - 2\sqrt{7}}{3} = -\left(\frac{-2}{3}\right)
\]

\[
\left(\frac{1 + 2\sqrt{7}}{3}\right)\left(\frac{1 - 2\sqrt{7}}{3}\right) = -3
\]

\[
\frac{2}{3} = \frac{2}{3}\sqrt{7}
\]

\[\therefore\text{ The solution set is } \left\{\frac{1 + 2\sqrt{7}}{3}, \frac{1 - 2\sqrt{7}}{3}\right\}, \text{ Answer.}\]

For computational purposes, you sometimes need rational approximations of such roots. To approximate them to the nearest tenth, use a two-decimal-place approximation of \(\sqrt{7}\), and perform the indicated operations.

\[
\frac{1 + 2\sqrt{7}}{3} \approx \frac{1 + 2(2.64)}{3} = \frac{1 + 5.28}{3} = \frac{6.28}{3} = 2.09 \approx 2.1
\]

\[
\frac{1 - 2\sqrt{7}}{3} \approx \frac{1 - 2(2.64)}{3} = \frac{1 - 5.28}{3} = \frac{-4.28}{3} = -1.42 \approx -1.4
\]

\[\therefore\text{ To the nearest tenth, the roots are 2.1 and } -1.4, \text{ Answer.}\]

**ORAL EXERCISES**

What value of \(c\) will make each trinomial a square?

1. \(x^2 + 4x + c\)
2. \(x^2 + 8x + c\)
3. \(n^2 - 14n + c\)
4. \(n^2 - 12n + c\)
5. \(y^2 + y + c\)
6. \(y^2 - y + c\)
7. \(z^2 - 3z + c\)
8. \(u^2 + 5u + c\)
9. \(r^2 - .2r + c\)
10. \(s^2 - .6s + c\)
11. \(z^2 + 1.2z + c\)
12. \(b^2 + 1.8b + c\)
13. \(t^2 + \frac{1}{6}t + c\)
14. \(v^2 + \frac{v}{5} + c\)
15. \(h^2 - \frac{5}{4}h + c\)
16. \(w^2 + \frac{4w}{3} + c\)

**WRITTEN EXERCISES**

Solve by completing the square. Give irrational roots in radical form, and to the nearest tenth.

1. \(x^2 + 2x = 7\)
2. \(x^2 + 4x = 14\)
3. \(y^2 + y - 6 = 0\)
4. \(y^2 - 4y + 3 = 0\)
5. \( m^2 - 8m + 2 = 0 \)  
6. \( n^2 + 6n + 4 = 0 \)  
7. \( a^2 + 7a + 5 = 0 \)  
8. \( b^2 - 5b + 3 = 0 \)  
9. \( m^2 - 3m = 0 \)  
10. \( m^2 + 2m = 0 \)  
11. \( v^2 = 20v - 19 \)  
12. \( x^2 = 24x - 23 \)  
13. \( 2r^2 - 10r - 10 = 0 \)  
14. \( 3p^2 + 9p - 81 = 0 \)  
15. \( 4y^2 + 12y + 9 = 0 \)  
16. \( 9y^2 - 6y + 1 = 0 \)  
17. \( 3n^2 + 7n = 1 \)  
18. \( 5n^2 - 8n = 1 \)  
19. \( s^2 + \frac{2}{3}s = 0 \)  
20. \( t^2 - \frac{3}{2}t = 0 \)  
21. \( z^2 + \frac{5z}{2} = 25 \)  
22. \( w^2 + \frac{2w}{5} = 3 \)  
23. \( \frac{1}{y} + \frac{1}{y - 2} = 2 \)  
24. \( \frac{1}{x + 1} + \frac{1}{x} = 1 \)  
25. \( \frac{3}{s + 1} + \frac{1}{s - 1} = 2 \)  
26. \( \frac{2}{t + 1} - \frac{1}{t - 1} = 3 \)  
27. \( \frac{2m}{m + 3} - \frac{m}{m - 3} + \frac{9}{m^2 - 9} = 0 \)  
28. \( \frac{n}{n + 2} - \frac{2n}{2 - n} = \frac{3}{n^2 - 4} \)  

### PROBLEMS

Give irrational answers to the nearest tenth. Reject inappropriate roots.

A 1. The dimensions of a rectangle can be represented by consecutive even integers. Its area is 224 square inches. Find its dimensions.
2. A rectangular foyer is 10 feet longer than it is wide. Its floor area is 119 square feet. Find its length and width.
3. To cover two floors completely takes 690 square feet of carpet. One floor is 3 feet longer than it is wide. The other floor is 2 feet wider than the first is long, and its length is twice the length of the first. Find the dimensions of the floors.
4. Two tin squares together have an area of 325 square inches. One square is 5 inches longer than the other. Find the side of each.
5. Two buses travel at right angles from one spot. In one hour they are 50 miles apart. If one goes 10 m.p.h. faster than the other, what is the rate of each?
6. Some pupils buy a $3 gift and share the cost equally. With 5 more pupils, each would give 2¢ less. How large is each share?
7. Two boys join a team, thus lowering by $2 each member's share of $120 spent on equipment. What is each share now?
8. A plane goes 600 miles with a wind of 30 m.p.h. in \( \frac{1}{3} \) hour less than it returns against the wind. What is its rate in still air?

9. A dealer bought a number of old trucks for $4000. Although one truck was useless, the dealer made a $200 profit on each of the rest and so broke even on the transaction. How many trucks did he buy?

10. A dealer bought some items for $10.40. He kept 4 and sold the rest at a gain of 60¢ each, with a total profit of $2.20. How many did he sell?

13–4 The Quadratic Formula

For a given set of coefficients \( a, b, \) and \( c, \) the quadratic equation \( ax^2 + bx + c = 0 \) can be transformed to express the variable \( x \) directly in terms of \( a, b, \) and \( c \) by completing the square. Study carefully the following parallel treatment of the standard quadratic equation and of a special quadratic equation.

\[
ax^2 + bx + c = 0 \quad 5x^2 + 8x + 1 = 0
\]

\[
x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad x^2 + \frac{8}{5}x + \frac{1}{5} = 0
\]

\[
x^2 + \frac{b}{a}x = -\frac{c}{a} \quad x^2 + \frac{8}{5}x = -\frac{1}{5}
\]

\[
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad x^2 + \frac{8}{5}x + \left(\frac{4}{5}\right)^2 = -\frac{1}{5} + \left(\frac{4}{5}\right)^2
\]

\[
\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \left(x + \frac{8}{5}\right)^2 = -\frac{1}{5} + \frac{16}{25}
\]

\[
\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad 
\left(x + \frac{4}{5}\right)^2 = \frac{11}{25}
\]

\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad x + \frac{4}{5} = \pm \sqrt{\frac{11}{25}}
\]

\[
x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad x = -\frac{4}{5} \pm \sqrt{\frac{11}{25}}
\]

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad x = -\frac{4}{5} \pm \frac{\sqrt{11}}{5}
\]

\[
x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}} \quad x = -\frac{4}{5} \pm \frac{\sqrt{11}}{5}
\]
The last step is actually two sentences written as one and is called the \textit{quadratic formula}. If either expression is taken as the value of \( x \) and substituted in the general quadratic equation, the resulting sentence is \( 0 = 0 \). In developing the quadratic formula, notice the assumptions that \( a 
eq 0 \), and that \( \sqrt{b^2 - 4ac} \) is a real number (\( b^2 - 4ac \geq 0 \)).

To solve any quadratic equation of the form \( ax^2 + bx + c = 0 \), substitute the coefficients in the quadratic formula, and evaluate.

**EXAMPLE.** Solve \( 5x^2 - 8x + 1 = 0 \) by using the quadratic formula.

\[
5x^2 - 8x + 1 = 0 \\
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; \ a = 5, \ b = -8, \ c = 1 \\
x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(1)}}{2(5)} \\
x = \frac{8 \pm \sqrt{64 - 20}}{10} = \frac{8 \pm \sqrt{44}}{10} \\
x = \frac{8 \pm 2\sqrt{11}}{10} = \frac{4 \pm \sqrt{11}}{5}
\]

\textbf{Check:} Is the sum of the roots \(- \left( \frac{-8}{5} \right) \)? Is the product of the roots \( \frac{1}{5} \)?

\[ \therefore \text{The solution set is } \left\{ \frac{4 + \sqrt{11}}{5}, \frac{4 - \sqrt{11}}{5} \right\}, \text{ Answer.} \]

\textbf{ORAL EXERCISES}

State the values of \( a, b, \) and \( c \) for these quadratic equations.

1. \( 2x^2 + 4x + 1 = 0 \) 
2. \( 3y^2 - 7y - 3 = 0 \) 
3. \( z^2 + 12z - 9 = 0 \) 
4. \( u^2 - 14u + 1 = 0 \) 
5. \( 17w - 8w^2 = -1 \) 
6. \( 19x - 4x^2 = 3 \) 
7. \( 7y^2 = 12y - 3 \) 
8. \( 5z^2 = 8z + 2 \) 
9. \( u^2 - 14 = 0 \) 
10. \( 3v^2 - 17 = 0 \) 
11. \( 2w^2 - 5 = 0 \) 
12. \( x^2 + 9x = 0 \) 
13. \( y^2 - 7y = 0 \) 
14. \( 5z^2 = 0 \) 
15. \( 12v^2 = 0 \) 
16. \( 0 = 4t^2 - t \)
Use the quadratic formula to solve each equation. Give irrational roots in simplest radical form, and correct to tenths.

1. $3x^2 + 5x + 1 = 0$
2. $4x^2 + 7x + 2 = 0$
3. $2x^2 - 8x + 3 = 0$
4. $4x^2 - 6x + 1 = 0$
5. $2x^2 - 5x - 12 = 0$
6. $6x^2 + x - 35 = 0$
7. $x^2 + 4x = 3$
8. $x^2 + 6x = 4$
9. $x^2 = 2x + 1$
10. $x^2 = 11 - x$
11. $20x^2 - 17x = -3$
12. $10x^2 - 17x = -3$
13. $3x^2 - x = 0$
14. $5x^2 - 17 = 0$

Factor these polynomials over the set of real numbers.

15. $x^2 - 2x - 2$
16. $x^2 - 2x - 4$
17. $y^2 + 6y + 3$
18. $y^2 + 8y + 13$

Show that these equations have no real roots.

19. $x^2 - 2x + 2 = 0$
20. $x^2 - 2x + 4 = 0$
21. $\sqrt{y} = y + 1$
22. $2\sqrt{y} = y + 3$

Give irrational answers to the nearest tenth. Reject inappropriate roots.

A. A rectangular floor of 147 square feet is three times as long as it is wide. It is divided into a rectangle twice as long as it is wide and a square (see diagram). Find the dimensions of each division.

B. A section of a wood floor is 72 square inches in area. This section has six rectangular pieces of equal area, three laid vertically and three, horizontally (see diagram). The length of each piece is three times its width; find these dimensions.
3. A square table top has a two-inch border. If two-thirds of its area is within the border, what are the dimensions of the table top?

4. One hundred square tiles would cover a floor now covered by 150 square tiles, 1 inch shorter on a side. How long is each small tile?

5. The perimeter of a triangle is 2 feet. Two sides form a right angle and are in the ratio of 3 to 4. Find the lengths of all three sides.

6. A group hikes east along a road. Another group starts cycling north at the same time from that point. The cyclers travel 7 miles an hour faster than the hikers. The groups are 13 miles apart at the end of 1 hour. Find the rate of each group.

7. Motor trouble reduced a bus's usual speed by 10 miles per hour, lengthening the time of its journey of 400 miles by 2 hours. What is the bus's usual speed?

8. A rectangular paper is 11 inches longer than it is wide. Eight-inch squares are cut from each corner, and the ends folded to form an open box whose volume is 2800 cubic inches. Find the paper's dimensions.

9. In a town with 1000 daily bus riders, the fare is 25¢. If the company raises the fare, for each increase of 1¢, it loses 4 riders.
   a. What fare increase would yield the company $44 more daily?
   b. How many answers are possible and how many are practical?

10. A telephone company has a net profit of $16 a year for each of its 1000 subscribers. This profit is decreased 1¢ for each additional subscriber.
   a. How many new subscribers would increase the annual profit by $900?
   b. How many answers are possible?

13–5 The Nature of the Roots of a Quadratic Equation (Optional)

If to each real number \( x \) you assign the number \( y \) given by \( y = x^2 + 24x + 140 \), the resulting quadratic function has as its graph the parabola farthest left in Figure 13–1 (page 477). In that figure also appear the graphs of other quadratic functions:

\[
\begin{align*}
  y &= -x^2 + 26x - 169 \\
  y &= \frac{1}{2}x^2 - 16x + 134 \\
  y &= x^2 - x - 6 \\
  y &= -x^2 - 7 \\
  y &= x^2
\end{align*}
\]

Since the \( x \)-axis is the line \( y = 0 \), by replacing \( y \) with 0 in the formula for the function and solving the resulting quadratic equation, you can determine the \( x \)-intercepts of the graph of the function.
Case 1  
\[ y = x^2 + 24x + 140 \]
\[ 0 = x^2 + 24x + 140 \]
\[ x = \frac{-24 \pm \sqrt{576 - 560}}{2} \]
\[ x = -12 \pm 2 \]
\[ x = -10 \text{ or } x = -14 \]

Case 2  
\[ y = -x^2 + 26x - 169 \]
\[ 0 = -x^2 + 26x - 169 \]
\[ x = \frac{-26 \pm \sqrt{676 - 676}}{-2} \]
\[ x = 13 \pm 0 \]
\[ x = 13 \text{ or } x = 13 \]

Case 3  
\[ y = \frac{1}{2}x^2 - 16x \]
\[ + 134 \]
\[ 0 = \frac{1}{2}x^2 - 16x \]
\[ + 134 \]
\[ x = 16 \]
\[ \pm \sqrt{256 - 268} \]
\[ x = 16 \pm \sqrt{-12} \]

But \( \sqrt{-12} \) does not exist in the set of real numbers.

The three cases are analyzed in the following chart:

<table>
<thead>
<tr>
<th></th>
<th>Number of points in common with the x-axis</th>
<th>Number of different real roots of the equation</th>
<th>Value of ( b^2 - 4ac )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>2</td>
<td>2</td>
<td>a positive number</td>
</tr>
<tr>
<td>Case 2</td>
<td>1</td>
<td>1, a double root</td>
<td>zero</td>
</tr>
<tr>
<td>Case 3</td>
<td>0</td>
<td>0</td>
<td>a negative number</td>
</tr>
</tbody>
</table>
Do you see that the value of \( b^2 - 4ac \) is the key to these cases? If \( b^2 - 4ac > 0 \), then \( \sqrt{b^2 - 4ac} \) is positive. Thus, \( ax^2 + bx + c = 0 \) has two different roots for \( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \neq \frac{-b - \sqrt{b^2 - 4ac}}{2a} \).

But if \( b^2 - 4ac = 0 \), then \( \sqrt{b^2 - 4ac} = 0 \) and you find that \( \frac{-b + 0}{2a} = \frac{-b - 0}{2a} = \frac{-b}{2a} \), so that the roots are equal. But, for \( b^2 - 4ac < 0 \), no real root exists, because square roots of negative numbers do not exist in the real number system.

A quadratic equation with real coefficients can have

1. two different real roots,
2. a double real root, or
3. no real roots.

Because the value of \( b^2 - 4ac \) distinguishes the three cases it is called the discriminant of the quadratic equation.

**WRITTEN EXERCISES**

Determine the nature of the roots of these equations graphically and by use of the discriminant.

**A**

1. \( x^2 + 2x - 3 = 0 \)
2. \( x^2 + 4x - 5 = 0 \)
3. \( 2x^2 + 8x + 15 = 0 \)
4. \( -2x^2 + 6x + 8 = 0 \)
5. \( -\frac{3}{2}x^2 - 5x + 6 = 0 \)
6. \( \frac{1}{2}x^2 - 7x + 6 = 0 \)
7. \( -x^2 - x - 1 = 0 \)
8. \( -x^2 + x - 1 = 0 \)
9. \( 3x^2 - 4x + 2 = 0 \)
10. \( 4x^2 + 6x + 1 = 0 \)

Determine whether these polynomials can be factored over the set of real numbers. If they can, find the factors.

**B**

11. \( x^2 + 1 - 2x \)
12. \( x^2 + 4x - 4 \)
13. \( -2x^2 + 4x - 1 \)
14. \( -3x^2 - 5 - 3x \)
15. \( u^2 + \frac{1}{4}u - \frac{1}{8} \)
16. \( u^2 - \frac{1}{6}u + \frac{1}{6} \)
17. \( \frac{1}{3}y - \frac{1}{6}y^2 - \frac{3}{2}y^3 \)
18. \( \frac{1}{15}z + \frac{1}{3}z^2 - \frac{3}{2}z^3 \)
PROBLEMS

Find the roots in the most efficient way; reject inappropriate roots. Find irrational roots to the nearest tenth. Use the indicated formulas.

1. A ball is thrown down from the top of the Statue of Liberty, \( h \) (288) feet above the ground. In how many seconds \( t \) does it strike the ground if it starts falling with a velocity \( v \) of (a) 48 feet per second? (b) 112 feet per second? (c) 30 feet per second? \( h = vt + 16t^2 \)

2. How high \( h \) is a circular arch with a radius \( r \) of 30 feet and a span \( s \) of 36 feet? \( s^2 = 8rh - 4h^2 \)

3. The greatest number of lines \( l \) that can join a number of points \( p \) is 10. Find the number of points, if \( l = \frac{p(p - 1)}{2} \).

4. How many vertices are in a figure, if only 6 lines connect them?

5. What is the escape velocity \( v \) of a rocket fired from the earth’s surface, if the radius \( r \) of the earth is 4000 miles? \( v^2 = \frac{64.4}{5280} \)

6. Find the circular orbital speed \( v \) of (a) Sputnik I, if \( h = 500 \) miles; (b) the moon, if \( h = 236,000 \) miles; and (c) Discoverer XIII, if \( h = 425 \) miles. \( v^2 = \frac{gR^2}{R + h} \); \( g = 32.2 \); \( R = 4000 \)

7. At what speed \( v \) should you be driving 50 feet from an intersection in order to stop at the intersection? \( d = \frac{0.45v^2 + 1.1v}{d^2} \)

The distance \( d \) between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[ d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \]

8. Find the ordinates of the two points on the line \( x = 7 \) which are 13 units from the point \((2, 1)\). \( d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \)

9. Find the distance between the points \((-2, -1)\) and \((-2, 3)\).

10. Show that points \((-2, -2), (4, 0), \) and \((7, 1)\) are on one straight line.

11. How far \( d \) can a man in an airplane at an altitude of 4 miles see to the horizon? \( d^2 = h(h + 8000) \)

12. At what altitude can a pilot see 446 miles to the horizon?

THE SOLUTION OF QUADRATIC INEQUALITIES

13–6 Solving Quadratic Inequalities (Optional)

To solve quadratic equations by factoring, you use the property of factors whose product is zero. To solve a quadratic inequality,
you use the property of the nonzero product of two real numbers:

A product is greater than zero if, and only if, both factors are greater than zero or both are less than zero; and a product is less than zero if, and only if, one factor is greater than zero and the other, less than zero.

**EXAMPLE 1.** Graph the solution set of \( x^2 - 2x > 3 \).

*Solution:*

\[
x^2 - 2x > 3 \\
x^2 - 2x - 3 > 0 \\
(x + 1)(x - 3) > 0
\]

Both factors are less than zero.

\[
x + 1 < 0 \quad \text{and} \quad x - 3 < 0 \\
x < -1 \quad \text{and} \quad x < 3
\]

The only numbers which satisfy both conditions satisfy \( x < -1 \).

Both factors are greater than zero.

\[
x + 1 > 0 \quad \text{and} \quad x - 3 > 0 \\
x > -1 \quad \text{and} \quad x > 3
\]

The only numbers which satisfy both conditions satisfy \( x > 3 \).

\[
\therefore \text{The solution set is} \quad x < -1 \quad \text{or} \quad x > 3 \quad , \quad \text{Answer.}
\]

**EXAMPLE 2.** Graph the solution set of \( x^2 - 2x \leq 3 \).

*Solution:*

\[
x^2 - 2x - 3 \leq 0 \\
(x + 1)(x - 3) \leq 0
\]

or

\[
x + 1 \geq 0 \quad \text{and} \quad x - 3 \leq 0 \\
x \geq -1 \quad \text{and} \quad x \leq 3
\]

The numbers satisfying both conditions satisfy \(-1 \leq x \leq 3\).

\[
x + 1 \leq 0 \quad \text{and} \quad x - 3 \geq 0 \\
x \leq -1 \quad \text{and} \quad x \geq 3
\]

\[
x \in \emptyset
\]

\[
-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4
\]
The solution set is \[-1 \leq x \leq 3\], Answer.

**ORAL EXERCISES**

1. \((x + 2)(x - 1) > 0\)
   a. If \(x + 2 > 0\), then \(x - 1 > 0\).
   b. If \(x + 2 < 0\), then \(x - 1 < 0\).
   c. If \(x + 2 > 0\) and \(x - 1 > 0\), \(x > ?\).
   d. If \(x + 2 < 0\) and \(x - 1 < 0\), \(x < ?\).

2. \((x - 1)(x - 2) > 0\)
   a. If \(x - 1 > 0\), then \(x - 2 > 0\).
   b. If \(x - 1 < 0\), then \(x - 2 < 0\).
   c. If \(x - 1 > 0\) and \(x - 2 > 0\), \(x > ?\).
   d. If \(x - 1 < 0\) and \(x - 2 < 0\), \(x < ?\).

3. Consider \(x^2 - x - 2 < 0\) or \((x + 1)(x - 2) < 0\).
   a. If \(x + 1 > 0\), then \(x - 2 < 0\).
   b. If \(x + 1 > 0\) and \(x - 2 < 0\), then \(? < x < ?\).
   c. If \(x + 1 < 0\), then \(x - 2 > 0\).
   d. If \(x + 1 < 0\) and \(x - 2 > 0\), then \(x < ?\) and \(x > ?\).
   e. Since \(x\) cannot satisfy the two conditions in (d) simultaneously, the solution set is ?.

4. Consider \(x^2 - 3x \geq 0\) or \(x(x - 3) \geq 0\).
   a. If \(x \geq 0\), then \(x - 3 \geq 0\).
   b. If \(x \geq 0\) and \(x \geq 3\), then \(x \geq ?\).
   c. If \(x \leq 0\), then \(x - 3 \leq 0\).
   d. If \(x \leq 0\) and \(x \leq 3\), then \(x \leq ?\).

5. Consider \(x^2 - 9 \leq 0\) or \((x + 3)(x - 3) \leq 0\).
   a. If \(x + 3 \geq 0\), then \(x - 3 \leq 0\).
   b. If \(x + 3 \geq 0\) and \(x - 3 \leq 0\), then \(? \leq x \leq ?\).
   c. If \(x + 3 \leq 0\), then \(x - 3 \geq 0\).
   d. If \(x + 3 \leq 0\) and \(x - 3 \geq 0\), then \(? \leq x \leq ?\).
CHAPTER THIRTEEN

WRITTEN EXERCISES

Graph the solution set of each inequality.

A 1. \( x^2 - 5x + 4 > 0 \)  
2. \( x^2 - 5x + 4 < 0 \)  
3. \( x^2 + 4x + 3 < 0 \)  
4. \( x^2 + 4x + 3 > 0 \)  
5. \( y^2 + y > 2 \)  
6. \( y^2 + y < 2 \)  
7. \( 2n^2 < n + 15 \)  
8. \( 5m^2 > 12 - 4m \)  
9. \( r^2 + 6 \geq 7r \)  
10. \( s^2 - 5s \leq 6 \)  
11. \( x^2 \geq 9 \)  
12. \( x^2 \leq 25 \)  
13. \( x^2 + 2x + 1 > 0 \)  
14. \( x^2 - 6x + 9 \geq 0 \)  
15. \( x^2 - 4x + 4 < 0 \)  
16. \( x^2 + 10x + 25 \leq 0 \)  
17. \( x^2 > 0 \)  
18. \( x^2 < 0 \)  

Find the values of \( x \) for which each expression is a real number.

B 19. \( \sqrt{x^2 - 2x - 35} \)  
20. \( \sqrt{x^2 - 12x + 35} \)  
21. \( \sqrt{x^2 + 5x} \)  
22. \( \sqrt{x^2 - x} \)  

13–7 Using Graphs of Equations to Solve Inequalities (Optional)

Figure 13–2 shows the graph of \( y = x^2 - 4 \). The abscissas of points at which \( y = 0 \) form the solution set of \( x^2 - 4 = 0 \). The values of \( x \) for which \( y > 0 \) give the solution set of \( x^2 - 4 > 0 \). And the values of \( x \) for which \( y < 0 \) give the solution set of \( x^2 - 4 < 0 \). Do you see that you can find the solution sets of these open sentences by determining the values of \( x \) for which the graph of the equation is on, above, or below the \( x \)-axis?
Figure 13–3 shows the graph of \( y = x^2 + 2x + 1 \). How many \( x \)-intercepts does it have? Does the point \((-1, 0)\) satisfy \( y = 0 \), \( y > 0 \), or \( y < 0 \)? Which sentence is satisfied by values of \( x \) to the left of \((-1, 0)\)? to the right of \((-1, 0)\)? Do you see that no value of \( x \) will make \( y < 0 \), since the graph does not go below the \( x \)-axis?

Notice that the values of \( x \) for which the graph of \( y = ax^2 + bx + c \) lies above the \( x \)-axis form the solution set of \( ax^2 + bx + c > 0 \), while the \( x \)-coordinates of the points on the graph below the \( x \)-axis comprise the solution set of \( ax^2 + bx + c < 0 \).

**ORAL EXERCISES**

1. If \( y = x + 1 \),
   a. What is the value of \( x + 1 \) when \( x = -1 \)?
   b. is \( x + 1 > 0 \) or is it \( < 0 \) when \( x > -1 \)?

2. If \( y = x - 2 \),
   a. What is the value of \( x - 2 \) when \( x = 2 \)?
   b. is \( x - 2 > 0 \) or is it \( < 0 \) when \( x < 2 \)?

3. Consider the equation \( y = (x - 1)(x - 4) \).
   a. What are the \( x \)-intercepts of the graph of the equation?
   Is \( (x - 1)(x - 4) > 0 \) or is it \( < 0 \) when
   b. \( x < 1 \)?
   c. \( 4 > x > 1 \)?
   d. \( x > 4 \)?

4. If \( y = (x - 1)^2 \),
   a. What are the \( x \)-intercepts of the graph of the equation?
   Is \( (x - 1)^2 > 0 \) or is it \( < 0 \) when
   b. \( x < 1 \)?
   c. \( x > 1 \)?
   d. \( x \neq 1 \)?

5. Examine \( y = x^2 - 1 \) in the form \( y = (x + 1)(x - 1) \).
   a. What are the \( x \)-intercepts of the graph of the equation?
   Is \( x^2 - 1 > 0 \) or is it \( < 0 \) when
   b. \( x < -1 \)?
   c. \( -1 < x < 1 \)?
   d. \( x > 1 \)?

6. Examine \( y = x^2 - 4 \) in the form \( y = (x + 2)(x - 2) \).
   a. What are the \( x \)-intercepts of the graph of the equation?
   Is \( x^2 - 4 > 0 \) or is it \( < 0 \) when
   b. \( x < -2 \)?
   c. \( -2 < x < 2 \)?
   d. \( x > 2 \)?
7. If \( y = x(1 - x) \),
   a. What are the x-intercepts of the graph of the equation?
   Is \( x(1 - x) > 0 \) or is it \( < 0 \) when
   b. \( x < 0? \)  
   c. \( 0 < x < 1? \) 
   d. \( x > 1? \)
8. If \( y = 2x(2 - x) \),
   a. What are the x-intercepts of the graph of the equation?
   Is \( 2x(2 - x) > 0 \) or is it \( < 0 \) when
   b. \( x < 0? \)  
   c. \( 0 < x < 2? \) 
   d. \( x > 2? \)

**WRITTEN EXERCISES**

Graph these equations, and mark the sections of the x-axis each determines
as solution sets for \( y = 0, y > 0, y < 0 \).

**A**
1. \( x^2 + 3x + 2 = y \) 
2. \( x^2 - 4x + 3 = y \) 
3. \( x^2 - 8 - 2x = y \) 
4. \( x^2 + 3x + 10 = y \) 
5. \( 3x^2 - 5x + 2 = y \) 
6. \( 4x^2 + 13x - 3 = y \) 
7. \( x^2 - 1 = y \) 
8. \( x^2 - 4 = y \) 
9. \( x^2 - 2x = y \) 
10. \( x - 2x^2 = y \) 
11. \( x^2 - 3x = y \) 
12. \( x - 3x^2 = y \) 

**B**
13. \( x^2 - 2x + 1 = y \) 
14. \( x^2 + 8x + 16 = y \) 
15. \( 4x^2 + 1 - 4x = y \) 
16. \( 9x^2 + 6x + 1 = y \) 

Find the values of x for which each expression is a real number.

**G**
17. \( \sqrt{x^2 - 3x - 10} \) 
18. \( \sqrt{x^2 - 3x} \)

**Chapter Summary**

**Inventory of Structure and Method**

1. A quadratic equation containing no linear term can be solved by using the property of square roots of equal numbers: If \( r \) and \( s \) are real numbers, \( r^2 = s^2 \) if, and only if, \( r = s \) or \( r = -s \).

2. In a quadratic equation of the form \( x^2 + bx + c = 0 \), the sum of the roots is equal to \( -b \), the opposite of the coefficient of the linear term;
the product of the roots equals the constant term \( c \). These facts may be used to check the solution set of a quadratic equation.

3. To solve a quadratic equation in one variable by the method of completing the square: transform it into an equivalent equation whose quadratic term has coefficient 1; write it in the form \( x^2 + bx = -c \); add to each member \( \left( \frac{b}{2} \right)^2 \), thus making the left member a square; apply the property of square roots of equal numbers; and solve the resulting linear equations. A rational approximation of an irrational root can be calculated by substituting a rational approximation for each radical which is accurate to one more place than desired in the root and performing the operations. Expressions such as \( \frac{a \pm b}{c} \) represent two expressions \( \frac{a + b}{c} \) and \( \frac{a - b}{c} \).

4. The roots of the standard quadratic equation \( ax^2 + bx + c = 0 \) are given by the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

5. To solve a quadratic equation in one variable graphically, write it in the standard form \( ax^2 + bx + c = 0 \), and find ordered pairs \((x, y)\) satisfying \( ax^2 + bx + c = y \). Then draw a parabola through the points represented by those ordered pairs. Determine the abscissas of the points of intersection of the parabola \((y = ax^2 + bx + c)\) with the x-axis \((y = 0)\).

6. To solve a quadratic inequality in one variable, first transform it into an inequality whose right member \( R \) is zero, and then factor the left member \( L \). (If \( L > 0 \), the factors must be both positive or both negative; if \( L < 0 \), the factors must be opposite in sign.) Solve the resulting linear inequalities.

7. The solution set of \( ax^2 + bx + c > 0 \) is the set of values of \( x \) for which the graph of \( y = ax^2 + bx + c \) lies above the x-axis. The solution set of \( ax^2 + bx + x < 0 \) is the set of values of \( x \) for which the graph of \( y = ax^2 + bx + c \) lies below the x-axis.

**Vocabulary and Spelling**

- completing a trinomial square (p. 469)
- general quadratic equation (p. 473)
- quadratic formula (p. 474)
- nature of roots (p. 476)
- discriminant (p. 478)
- quadratic inequality (p. 479)
**Chapter Test**

13-1 Solve, using the property of square roots of equal numbers.

1. \( \frac{1}{8} - 2y^2 = 0 \)
2. \( 4(x - \frac{1}{2})^2 = 9 \)
3. \( \frac{1}{4} - n + n^2 = 0 \)

13-2 Are the roots given for each equation correct?

4. \( x^2 - 22x = 11; x \in \{11 \pm 2\sqrt{33}\} \)
5. \( x^2 + 30x = -29; x \in \{1.1, -31.1\} \)

13-3 Find the number \( c \) which completes the square in \( x^2 - 1.4x + c \), and write the trinomial as the square of a binomial.

7. Transform \( x^2 - 3x = 9 \) into an equation whose left member is shown as the square of a binomial.

8. Solve \( 32x - 2x^2 - 16 = 0 \) by completing the square. Express the roots in simplest radical form and to the nearest tenth.

13-4 Using the quadratic formula, find the roots of \( 4x^2 + 12x + 7 = 0 \) in simplest radical form and to the nearest tenth.

10. A rectangular picture, 18 inches by 24 inches, has a frame of uniform width. The area of both picture and frame is 720 square inches. How wide is the frame?

13-5 How many real roots do these equations have?

11. \( x^2 - 3x - 5 = 0 \)
12. \( 4x^2 + 4x + 1 = 0 \)
13. \( x^2 + x + 3 = 0 \)
14. \( 5x^2 - 3x + 7 = 0 \)
15. (Optional) Graph the solution set of \( x^2 - 3x > 4 \).
16. (Optional) Graph \( y = x^2 + 4x + 3 \), and from the graph determine the solution set of \( x^2 + 4x \leq -3 \).

**Chapter Review**

13-1 The Square-Root Property

1. If \( u^2 = v^2 \), you know that \( ? = ? \) or \( ? = ? \).

Solve by using the square-root property.

2. \( a^2 = 4 \)
3. \( b^2 - 64 = 0 \)
4. \( (c + 1)^2 = 25 \)
13-2 Checking Solution Sets

5. When one member of the equation is zero and \( x^2 \) has a coefficient of 1, the sum of the roots of a quadratic equation equals the \( \frac{?}{?} \) of the \( \frac{?}{?} \) of the \( \frac{?}{?} \) term, and the product of the roots equals the \( \frac{?}{?} \) term.

Use the sum and product method to check the solution sets.

6. \( x^2 - 4x = 4; \{2, -2\} \)
7. \( x^2 + 26x + 19 = 0; \{-13 \pm 5\sqrt{6}\} \)

13-3 Completing a Trinomial Square

State the value of \( k \) which makes each trinomial a square, and express the trinomial as the square of a binomial.

8. \( s^2 - 20s + k \)
9. \( y^2 + y + k \)
10. \( z^2 + .5z + k \)

Solve by completing the square. Express irrational roots in simplest radical form and to the nearest tenth.

11. \( x^2 + 2x = 23 \)
12. \( 2x^2 - 12x = 24 \)
13. \( y^2 + 5y - 3 = 0 \)
14. \( y^2 - 6y + 9 = 20 \)
15. A man rows 18 miles downstream and back in 8 hours, in a current of 3 miles per hour. Find his rate of rowing in still water.
16. A car and a train each travel 400 miles. The train goes 10 m.p.h. faster than the car and arrives 2 hours sooner. What is the rate of each vehicle?

13-4 The Quadratic Formula

17. In a quadratic equation, the coefficient of the quadratic term may be any real number except \( \frac{?}{?} \).
18. The linear term in a quadratic equation may have any \( \frac{?}{?} \) number as its coefficient.
19. The roots of \( ax^2 + bx + c = 0 \) are \( \frac{?}{?} \) and \( \frac{?}{?} \).

Solve by the quadratic formula. Express the irrational roots in simplest radical form and correct to tenths.

20. \( 2x^2 + 11x + 3 = 0 \)
21. \( x^2 - 18x + 4 = 0 \)
22. \( 3x^2 - 14x - 5 = 0 \)
23. \( 5x^2 + 8x = 3 \)
24. A dealer sold 200 radios a month at $35 each. For each price increase of $1, he sells two fewer radios each month. What price increase will yield $1100 more monthly?
25. A tray with a volume of 16 cubic inches was made from a square of tin by cutting one-inch squares from the corners and turning up the sides (see sketches). What size was the square of tin?

13-5 The Nature of the Roots of a Quadratic Equation (Optional)  

26. An equation in the form $ax^2 + bx + c = 0$ can be solved by graphing the equation _?_.

27. The graph of $x^2 + 3x + 1 = y$ crosses the x-axis at _?_ points.

28. How many real roots does each equation have?
   a. $2x^2 - 7x + 2 = 0$
   b. $2x^2 = 8x - 8$
   c. $x^2 + 3x + 2 = 0$
   d. $x^2 + 3x = -3$

29. If $h = vt - 16t^2$, in how many seconds $t$ will a ball thrown with a velocity $v$ of 48 feet per second reach a height $h$ of 32 feet?

30. If $s^2 = 8rh - 4h^2$, how high $h$ to the nearest tenth of a foot, is an arch with a radius $r$ of 20 feet and a span $s$ of 22 feet?

31. If $c = \frac{3}{8}(t - 1)$, how many telephones $t$ are served by a switchboard with 630 connections $c$?

13-6 Solving Quadratic Inequalities (Optional)  

32. If $(x + 1)(x - 2) > 0$, then $x + 1 > 0$ and $x - 2 \ ? \ 0$, or $x + 1 < 0$ and $x - 2 \ ? \ 0$.

33. If $(x + 3)(x - 1) \leq 0$, then $x + 3 \geq 0$ and $x - 1 \ ? \ 0$, or $x + 3 \leq 0$ and $x - 1 \ ? \ 0$.

34. There are _?_ values of $x$ that satisfy both $x + 3 \leq 0$ and $x - 1 \geq 0$.

Solve each quadratic inequality.

35. $x^2 + 3x + 2 > 0$
36. $x^2 \leq x + 12$
37. $x^2 - 25 \geq 0$
38. $x^2 - 1 < 0$
13-7 Using Graphs of Equations to Solve Inequalities (Optional)

Pages 482-484

Items 39-40 refer to the solution of $x^2 - x > 2$.

39. The number line segments $x < -1$ and $x > 2$ represent abscessas of points for which $y > 0$.

40. If $-1 < x < 2$, $(x + 1)(x - 2) > 0$, and the values of $x$ are members of the solution set of $x^2 - x > 2$.

Solve each of the following inequalities by graphing a quadratic equation in two variables.

41. $x^2 + 4x + 3 \geq 0$

42. $x^2 < x + 6$

43. $x^2 - 9 > 0$

44. $x^2 - 4 \leq 0$

Extra for Experts

The Factor Theorem

Knowing the factors of the left member of a polynomial equation like $2x^3 - 7x^2 - 7x + 30 = 0$, you can determine its roots. But if you know the roots, can you determine the factors of the left member? The answer is yes, and it usually is explained in terms of the factor theorem.

If a polynomial equation in standard form is satisfied by a specific value $a$ of $x$, then $x - a$ is a factor of the polynomial.

**EXAMPLE 1.** Factor $2x^3 - 7x^2 - 7x + 30$.

**Solution:** Set $2x^3 - 7x^2 - 7x + 30 = 0$, and test values of $x$ in it.

When $x = 1$:

$x = 2$:

$x = 3$:

$\therefore x - 3$ is a factor of $2x^3 - 7x^2 - 7x + 30$.

$\frac{2x^3 - 7x^2 - 7x + 30}{x - 3} = 2x^2 - x - 10 = (x + 2)(2x - 5)$

$\therefore 2x^3 - 7x^2 - 7x + 30 = (x - 3)(x + 2)(2x - 5)$, Answer.

The quadratic formula can help you factor a quadratic polynomial. In equations with integral coefficients like $x^3 + 2x^2 - 14x + 5 = 0$, where the coefficient of $x^3$ is 1, all integral roots of the equation are factors of the
constant term 5. This is so because if the equation is factorable the left member will be \((x - a)(x - b)(x - c)\), where \(a, b,\) and \(c\) are real numbers, and the constant term will equal \((-a)(-b)(-c)\). If there is no constant term, 0 is a root of the equation, and \(x - 0\) is a factor of the left member.

**EXAMPLE 2.** Factor \(y^3 + 2y^2 - 14y + 5\) into linear factors.

**Solution:** Check \(\{1, -1, 5, -5\}\) in \(y^3 + 2y^2 - 14y + 5 = 0\).

\[
(1)^3 + 2(1)^2 - 14(1) + 5 \neq 0; \quad (-1)^3 + 2(-1)^2 - 14(-1) + 5 \neq 0 \\
(5)^3 + 2(5)^2 - 14(5) + 5 \neq 0; \quad (-5)^3 + 2(-5)^2 - 14(-5) + 5 = 0 \checkmark
\]

\[
y^3 + 2y^2 - 14y + 5 = \frac{y^3 + 2y^2 - 14y + 5}{y - 5} = y^2 + 3y + 1
\]

Apply the factor theorem to \(y^2 - 3y + 1\):

\[
y^2 - 3y + 1 = 0; \text{ by the quadratic formula, } y = \frac{3 \pm \sqrt{5}}{2}
\]

\[
y^3 + 2y^2 - 14y + 5 = (y + 5) \left( y - \frac{3 + \sqrt{5}}{2} \right) \left( y - \frac{3 - \sqrt{5}}{2} \right)
\]

**Questions**

1. Show by substitution that each polynomial has the indicated factor.
   - a. \(n^{156} - 2n^{41} + 1; n - 1\)
   - b. \(x^7 - 3x^2 + 4; x + 1\)
   - c. \(x^3 - x^2 - x + 10; x + 2\)
   - d. \(3t^4 - 16t^3 + 19t^2 + 5t + 3; t - 3\)

2. Factor these polynomials.
   - a. \(x^3 - 7x + 6\)
   - b. \(2y^3 + 3y^2 - 2y - 3\)
   - c. \(n^4 + 16n^3 - 14n^2 - 24n - 9\)
   - d. \(2m^4 + 2m^3 - 17m^2 - 23m + 6\)

3. Solve these equations. Leave irrational roots in radical form.
   - a. \(x^3 - 4x^2 - x - 4 = 10\)
   - b. \(3z^3 - 5z^2 = 3z - 2\)
   - c. \(2z^3 - 14z = 3z^2 - 15\)
   - d. \(5y^2 + 40y = 7y^3 + 48 - y^4\)

4. Find \(k\) so that:
   - a. \(x + 1\) is a factor of \(2x^3 + kx - 4\).
   - b. \(x - 2\) is a factor of \(x^3 + kx^2 - 14x + 2k\).

5. Show these statements to be true.
   - a. \(x - y\) is a factor of \(x^n - y^n\).
   - b. \(x + y\) is a factor of \(x^n + y^n\) when \(n\) is an odd integer, but not when \(n\) is an even integer.
If insurance plans are to be financially sound, premiums must be charged in proportion to the risk, the likelihood that the insurance company will have to pay out money to the policyholder. Determining the degree of risk and fixing the rates accordingly is the main job of an actuary.

Actuarial science relies heavily on the branch of mathematics known as the theory of probability and statistics. Shown in the photograph is a class of actuarial trainees who are learning to compute the value of annuities. In classes such as these, actuaries learn to interpret data gathered on millions of people, covering birth, death, and accident rates, illness, disability, and unemployment.

Analyzing statistics on more than 10 million people, actuaries determined that babies born in 1941 had an average life expectancy of approximately 62 years, while babies born in 1958 had an average expectancy of nearly 70 years. From statistics like these, the actuary can determine the probability that a boy now 15 years old will live to be sixty-eight. This kind of problem, important in adjusting the premium for pension, old-age, or life insurance, is illustrated on the work pad. The actuary uses two facts: (1) 6,144,088 of ten million people born 68 years ago are still living, and (2) 9,743,175 of the ten million people born 15 years ago are still living. He finds that there is a sixty per cent probability that the 15-year-old boy will live to be 68.
Geometry and Trigonometry

Looking ahead in mathematics may be looking ahead into your future. Through the screen of Cartesian coordinates you see mathematical symbols extending into infinity, representing the boundlessness of mathematics (upper photo). Man’s knowledge of this field is limited only by his imagination; its challenges are as broad as the constellations of the heavens.

As you look into your own future, can you see the role mathematics may play in it? The astronomers who work in the observatory (lower left) require a knowledge of mathematics greater than that you now possess. They did not learn their mathematics as part of their jobs. They learned it in order to get their jobs. Since many occupations which are challenging require a knowledge of mathematics, you should plan to include it in your schooling.

GEOMETRY

14–1 Geometric Assumptions

Like numbers, geometric points, lines, and planes are abstract concepts, not concrete objects. However, representing points by dots, lines by strokes, and planes by flat surfaces, as in Figure 14–1, helps you in your thinking.

Your study of geometry begins with certain relationships, stated as axioms, which are assumed to hold for geometric figures and for numbers. From these, other relationships, stated as theorems, can be proved.

A few geometric axioms are these:

I. Every line is a set of points containing at least two different points.

II. Any two different points lie on one, and only one, line.

By the axioms just stated you can identify a line by giving any two of its points; you can call line $PQ$ in Figure 14–1 $QT$ or $PT$, also. Points like $P$, $Q$, and $T$ which lie on the same line are said to be collinear.
(ko-lin-e-ar). From these axioms you know that no two different lines can have more than one common point.

**III.** Every plane is a set of points containing at least three points not on the same line (noncollinear).

**IV.** Any three noncollinear points belong to one, and only one, plane.

Using these two axioms, you identify a plane by naming three of its points, for example, plane \(UWV\) in Figure 14–1.

In putting coordinates on a number line, you assume:

**V.** There is a one-to-one correspondence between real numbers and points of a line such that if \(A\) and \(B\) are points with coordinates \(a\) and \(b\), the distance between \(A\) and \(B\) \((AB)\) is the absolute value of the difference of their coordinates: \(AB = |a - b|\).

This assumption implies that a line extends indefinitely in both directions without holes or gaps. It also suggests this definition:

If point \(C\) lies on line \(AB\), then \(C\) is between \(A\) and \(B\) if \(a < c < b\) or \(b < c < a\), where \(a, b,\) and \(c\) are coordinates of points \(A, B,\) and \(C\).

The line segment determined by points \(A\) and \(B\) is the set of points consisting of \(A\) and \(B\) and the points on the line between them. \(A\) and \(B\) are the end points of the line segment \(AB\). The length of a line segment is the distance between its end points. That is, the length of segment \(AB\) is \(AB\).

A plane, also, extends indefinitely, as this assumption implies:

**VI.** If two points of a line are in a plane, every point of the line is in that plane.

**WRITTEN EXERCISES**

Tell which axiom justifies each statement.

1. If \(P\) is a point on line \(m\), there is another point on \(m\).
2. If \(P\) and \(Q\) are different points on line \(m\) and if line \(n\) also contains \(P\) and \(Q\), then \(m\) and \(n\) are the same line.
3. Any three points lie in a plane.
4. Any two points are collinear.
5. If $A$, $B$, and $C$ are noncollinear points, then $AB$, $AC$, and $BC$ are different lines.
6. If $A$, $B$, $C$, and $D$ are points not in the same plane, then $ABC$, $BCD$, $ACD$, and $ABD$ designate different planes.
7. If $L$ and $T$ are two different points of line $m$ and if $P$ is a point not on $m$, there is a plane containing $P$ and $m$.

Which point makes each statement about the adjoining figure true?

8. Points $S$, $W$, and ____ are collinear.
9. Points $Y$, $T$, and ____ are collinear.
10. Points $R$, $V$, and ____ are noncollinear.

Given collinear points with coordinates as indicated in the figure, find each distance.

$\begin{align*}
A &= -1 \\
B &= 0 \\
C &= 3 \\
D &= 5 \\
\end{align*}$


Diagram the points $K$, $L$, $M$, and $P$ on a line in the given relationships. In each case, at least two orders are possible.

17. $KM + LM = KL$, $P \in KM$
18. $PL + LP = MP$, $L \in KP$
19. $K \in LP$, $LP < LM$
20. $KL > KP$ and $PM < PL$

14–2 Rays and Angles

Any point $P$ with coordinate $p$ on a line is the end point of two rays on that line, one consisting of all the points on the line whose coordinates equal or exceed $p$, the other consisting of all the points whose coordinates equal or are less than $p$. If $P$ is between $S$ and $T$ on the line, then ray $PT$ and ray $PS$ designate the two rays (opposite rays) determined by $P$. Describe ray $TP$. What is its end point?
An angle is the set of points on two rays (its sides) with a common end point (its vertex). To name an angle, use the sign $\angle$ (read angle) followed by a letter naming its vertex, as $\angle G$ or $\angle K$ in Figure 14-2. When two or more angles have the same vertex, use $\angle$ followed by three letters, naming first a point on one side, then the vertex and, finally, a point on the other side, as $\angle MHP$ or $\angle RHP$. Why would the designation $\angle H$ be ambiguous?

Any line in a plane separates the plane into two half-planes, each having the line as an edge. If $Q$ is a point on the edge of a half-plane, a protractor which measures angles in units called degrees is made by putting the set of all rays lying in the half-plane, and having end point $Q$, into one-to-one correspondence with the real numbers $r$, where $0^\circ < r \leq 180^\circ$.

In Figure 14–3 $\angle NQL$ measures fifty degrees, or $m\angle NQL = 50^\circ$. Similarly, $m\angle NQM = 140^\circ$. An angle whose measure is $90^\circ$ is a right angle, $\angle NQK$; an angle whose measure is $180^\circ$ is a straight angle, $\angle NQP$; an angle whose measure is such that $0^\circ < m < 90^\circ$
is an **acute angle**, \( \angle NQL \); and an angle whose measure is such that \( 90^\circ < m < 180^\circ \) is an **obtuse angle**, \( \angle NQM \).

If \( A, B, \) and \( C \) are three noncollinear points, the set of points of the segments \( \overline{AB}, \overline{BC}, \) and \( \overline{AC} \) is a **triangle** having \( A, B, \) and \( C \) as vertices, segments \( \overline{AB}, \overline{BC}, \) and \( \overline{AC} \) as sides, and \( \angle A, \angle B, \) and \( \angle C \) as **angles**.

To name a triangle, use the symbol \( \triangle \) (read triangle) followed by the letters for its vertices: \( \triangle ABC \).

You already have used this property of triangles:

**THEOREM:** The sum of the measures of the angles of a triangle is \( 180^\circ \).

This property implies another whose proof is outlined in the exercises:

**THEOREM:** If the measures of two angles of one triangle are equal to the measures of two angles of another triangle, the measures of the remaining angles are equal.

**WRITTEN EXERCISES**

Supply the reason for each step (1 to 5) in this proof:

**Given:** Triangles \( ABC \) and \( DEF \) with \( m\angle A = m\angle D \) and with \( m\angle B = m\angle E \).

**Prove:** \( m\angle C = m\angle F \)

**Proof:**

1. \( m\angle A + m\angle B + m\angle C = 180^\circ \)
2. \( m\angle A + m\angle B + m\angle C \)
3. \( m\angle D + m\angle E + m\angle F = 180^\circ \)
4. \( m\angle D + m\angle E + m\angle F \)
5. \( m\angle C = m\angle F \)
6. Why can a triangle have only one right angle or one obtuse angle?
7. Explain why all points of a triangle lie in the same plane.
8. What may be true of angles \( A \) and \( K \), if triangles \( ABC \) and \( KLM \) are both right triangles and \( m\angle B = m\angle L < 90^\circ \)?
On a scaled line indicate the graph of each open sentence, and identify the graph as a ray, a point, a line segment, a line, or none of these.

9. \( x \geq 2 \)  
11. \( x < 5 \)  
13. \( |x| \leq 1 \)  
15. \( |x| \leq 2 \)
10. \( x = 1 \)  
12. \( |x| \geq 0 \)  
14. \( -2 \leq x < 1 \)  
16. \( |x| \geq 2 \)

The measures of angles \( A \) and \( B \) of \( \triangle ABC \) are given. Find \( m \angle C \).

17. \( 40^\circ, 60^\circ \)  
19. \( 45^\circ, 45^\circ \)  
21. \( 90^\circ, 35^\circ \)  
23. \( 110^\circ, 50^\circ \)
18. \( 70^\circ, 50^\circ \)  
20. \( 60^\circ, 30^\circ \)  
22. \( 24^\circ, 90^\circ \)  
24. \( 130^\circ, 10^\circ \)

### 14-3 Similar Triangles

A correspondence between two triangles is a pairing of each vertex of one triangle with a particular vertex in the other. For example, in two triangles of Figure 14-4, you can pair \( A \) and \( P \), \( B \) and \( Q \), \( C \) and \( R \). Under this pairing, you can also speak of corresponding sides and corresponding angles: \( AB \) corresponds to \( PQ \), \( \angle C \) corresponds to \( \angle R \), and so forth.

A correspondence of great practical importance is called similarity. Two triangles are similar if their vertices can be paired so that the measures of corresponding angles are equal. If \( m \angle A = m \angle A' \) (read \( A \)-prime), \( m \angle B = m \angle B' \), and \( m \angle C = m \angle C' \), then \( \triangle ABC \) is similar to \( \triangle A'B'C' \), written as \( \triangle ABC \sim \triangle A'B'C' \) (\( \sim \) means similar). You can prove the following theorem from those in the previous section.

**THEOREM:** If the measures of two pairs of corresponding angles of two triangles are equal, the triangles are similar.

Another property of similar triangles can be observed in Figure 14-5, where in \( \triangle ABC \), \( a \), \( b \), and \( c \) denote the lengths of the sides opposite vertices \( A \), \( B \), and \( C \), respectively; \( a = BC \), \( b = AC \), and \( c = AB \). Compare the lengths of corresponding sides in \( \triangle ABC \) and \( A'B'C' \). Do
you see that \( \frac{a}{c} = \frac{3}{5} = \frac{a'}{c'} \), \( \frac{a}{b} = \frac{3}{4} = \frac{a'}{b'} \), and \( \frac{b}{c} = \frac{4}{5} = \frac{b'}{c'} \) ? In general:

**THEOREM:** If triangles \( ABC \) and \( A'B'C' \) are similar, then

\[
\frac{a}{b} = \frac{a'}{b'}, \quad \frac{b}{c} = \frac{b'}{c'}, \quad \text{and} \quad \frac{a}{c} = \frac{a'}{c'}.
\]

You can use this theorem to determine inaccessible distances.

**EXAMPLE.** A boy 5'6" tall casts a shadow 4' long at the same time that a tree casts a shadow 12' long. Find the height of the tree.

**Solution:**

Let \( a = \) boy's height (ft.) = \( 5\frac{1}{2} \),

\( a' = \) tree's height (ft.) = \( a' \),

\( b = \) length of boy's shadow (ft.) = 4,

\( b' = \) length of tree's shadow (ft.) = 12.

\( \triangle ABC \sim \triangle AB'C' \)

\[
\frac{a'}{b'} = \frac{a}{b}
\]

\[
\implies \frac{a'}{b'} = \frac{a}{b}
\]

\[
\frac{a'}{12} = \frac{5\frac{1}{2}}{4}
\]

\[
a' = 16\frac{1}{2}
\]

Check this solution. \( \therefore \) The height of the tree is 16'6", Answer.
1. Two angles of a triangle measure 20° and 65°. What is the measure of the largest angle in a similar triangle?

2. In a triangle two angles measure 40° and 55°. Find the largest angle of a similar triangle.

Exercises 3–6 refer to the adjoining figure.

3. Find $a'$ when $a = 3$ ft., $b = 5$ ft., and $b' = 15$ ft.

4. Find $b'$ when $a = 4$ ft., $b = 2$ ft., and $a' = 30$ ft.

5. Find $BB'$ when $AB = .39$ meter, $a = .30$ meter, and $a' = .50$ meter.

6. Find $b'$ when $AB = 4.2$ meters, $BB' = 6.0$ meters, and $b = 3.5$ meters.

7. A triangle has sides of 12, 16, and 20 inches. The longest side of a similar triangle is 15 inches. How long are its other sides?

8. A triangle has sides of 8, 10, and 15 meters. The shortest side of a similar triangle is 12 meters. How long are its other sides?

Exercises 9–14 refer to the figure, which has right angles at $B$, $C$, and $D$.

9. Find the ratio of $CF$ to $AC$.

10. Find the ratio of $DE$ to $AD$.

11. If at a point $H$ on $AE$ a line were drawn to a point $J$ on $AD$, so that $m \angle AJH = 90°$, what would be the ratio of $JH$ to $AJ$?

12. Name two ratios of the sides of triangles in the figure, each of which equals the ratio of $CF$ to $AF$.

13. Name two ratios of the sides of triangles in the figure, each of which equals the ratio of $AB$ to $AG$.

14. If $BC = 8$ inches and $CD = 8.8$ inches, how long is $DE$?

PROBLEMS

1. A pole six feet high casts an eight-foot shadow, and a tree nearby casts a forty-foot shadow. How high is the tree?
2. How high is a flagpole which casts a fifty-foot shadow when a man six feet tall casts a seven-foot shadow?

3. A tree has a shadow 50 feet long when a vertical yardstick casts a five-foot shadow. How high is the tree?

4. A girl places a mirror on the ground and stands where she can see the top of a tree in it. How tall is the tree as shown in the figure?

5. A method of finding the distance across a stream is illustrated. If two boys make the measurements noted, what is the distance w?

6. Taking the distances as shown, determine the length of the pond.

NUMERICAL TRIGONOMETRY

14–4 The Tangent Function

As the name implies, numerical trigonometry is concerned with the measurement of triangles. It is based on the properties of similar triangles, but, if you had this problem, where could you find an appropriate similar triangle?

An engineer digs a straight tunnel at an angle of 8° with the level surface of the ground. How deep should he drill a vertical shaft to meet this tunnel one-half mile along the surface from the tunnel entrance?

Do you see that the tunnel, the shaft, and the level surface form a right triangle (Figure 14–6)? If you draw a right triangle having one
acute angle of 8° is it similar to the engineer's triangle? By measuring
the sides of your triangle, can you set up an appropriate proportion and
find the required depth?

All right triangles containing an 8° angle are similar to each other,
because each contains an 8° and a 90° angle. Thus, the ratio of the
lengths of two sides of one equals the ratio of the lengths of the cor¬
responding sides of any other such triangle: \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \) (see Figure 14–6).

The ratio \( \frac{a}{b} \) for all right triangles \( ABC \) having an angle of 8° at \( A \) is a
constant, depending only on \( m \angle A \) and not on the lengths of the sides
of each triangle. Similarly, this ratio for all right triangles for which
\( m \angle A = 15° \) is another constant.

Thus, when an acute angle is part of a right triangle, you can pair
its measure with the ratio \( \frac{a}{b} \). This pairing is called the tangent function
of the measure of acute angles. For a specific angle \( A \), you may
designate this function as \( \tan A \) (read tangent of angle \( A \)) and define
it thus: If \( A \) is an acute angle in a right triangle,

\[
\tan A = \frac{\text{length of side opposite angle } A}{\text{length of side adjacent to angle } A} = \frac{a}{b}.
\]

For your convenience a table of values of this trigonometric function
is printed in the Appendix of this book.

To solve the mining engineer's problem you need only determine
from the table of the function that \( \tan 8° \) is approximately .141 and
substitute the known values in the formula (\( \frac{1}{2} \) mi. = 2640 ft.).

\[
\tan A = \frac{a}{b}
\]

\[
\tan 8° = \frac{a}{2640}
\]

\[
.141 = \frac{a}{2640}
\]

\[
.141(2640) = a
\]

\[
a = 372
\]

\( \therefore \) The depth of the shaft is approximately 372 feet, Answer.

In many practical problems, the measured angle is referred to as an
angle of elevation or an angle of depression (see Figure 14–7). \( \angle CAB \)
is an angle of elevation; the point \( B \) is elevated with respect to the
observer at $A$ and the horizontal line $AC$ through $A$. $\angle SRQ$ is called an angle of depression; the point $Q$ is depressed with respect to the observer at $R$ and the horizontal line $SR$ through $R$.

![Figure 14-7](image)

To find four-decimal-place approximations to the tangent of an angle in Table 5, Appendix, look along the row beginning with the measure of the angle until you reach the appropriate column. Thus, for an angle of $25^\circ$, you find: $\tan 25^\circ = .4663$.

**PROBLEMS**

Solve the following problems, drawing a diagram for each one. Express the distances to the nearest foot. Use the tangent ratios in the Appendix.

1. Find the height of a tree casting a forty-foot shadow when the sun’s rays make a thirty-seven-degree angle with the ground.
2. Find the height of a flagpole casting a fifty-foot shadow when the sun’s rays strike the ground at an angle of 40 degrees.
3. At 120 feet from a building, the angle of elevation to the roof is 40 degrees. Find the height of the building.
4. Find the height of a tower whose shadow is 50 feet long when the angle of elevation of its top from the end of the shadow is 60 degrees.
5. A lighthouse built at sea level is 150 feet high. From its top the angle of depression of a buoy is 25 degrees. How far is the buoy from the foot of the lighthouse?
6. From 1200 feet above an airport at sea level a pilot finds the angle of depression of a ship at sea is $14^\circ$. How far is the ship from the airport?
7. Find to the nearest degree the sun’s angle of elevation when a seventy-foot flagpole casts a one-hundred-foot shadow.
8. Find to the nearest degree the sun's angle of elevation when a vertical pole 6 feet high casts a shadow 10 feet long.

9. From the end of its shadow, a tree 40 feet high has a 28-degree angle of elevation. How long is the shadow?

10. From one bank of a river, 10 feet above the level of the river, the angle of elevation of a 210-foot cliff on the opposite shore is 32 degrees. How wide is the river at that point?

14–5 The Sine and Cosine Functions

If you go 300 feet into a tunnel that slopes downward at an angle of 6° from level ground, how far beneath the surface are you?

\[ c = 300 \text{ ft} \]

\[ \text{Figure 14–8} \]

This problem cannot be solved by direct measurement, nor can you solve it by using the tangent ratio. You want to find \( a \), but you do not know \( b \). The only length you know is \( c \). Of course, if you knew \( \frac{a}{c} \), you could find \( a \). The ratio \( \frac{a}{c} \) in right triangle \( ABC \) is called the sine of angle \( A \) and is abbreviated in writing as \( \sin A \)

\[ \sin A = \frac{\text{length of side opposite angle } A}{\text{length of hypotenuse}} = \frac{a}{c} \]

You can use the value \( \sin 6^\circ \approx 0.105 \) to solve the tunnel problem thus:

\[ \sin A = \frac{a}{c} \]

\[ \sin 6^\circ = \frac{a}{300} \]

\[ 0.105 = \frac{a}{300} \]

\[ a \approx 31.5 \]

\[ \therefore \text{ You will be approximately 32 feet beneath the surface, Answer.} \]
Now suppose that the same tunnel has a ventilating shaft whose base is 200 feet from the entrance. How far from the entrance is the upper opening of the shaft, if it is on the same level as the entrance?

\[ \text{Figure 14–9} \]

To solve this problem still another ratio is used, \( \frac{b}{c} \). This is the \textbf{cosine} function (ko-sine). Its value for angle \( A \) is called the \textit{cosine of angle} \( A \), abbreviated \( \cos A \), and is defined for an acute angle as

\[ \cos A = \frac{\text{length of side adjacent to angle } A}{\text{length of hypotenuse}} = \frac{b}{c}. \]

As the \( \cos 6^\circ = .995 \), the solution of the ventilation shaft problem is:

\[
\begin{align*}
\cos A &= \frac{b}{c} \\
\cos 6^\circ &= \frac{b}{200} \\
.995 &= \frac{b}{200} \\
&\quad b \approx 199
\end{align*}
\]

\[ \therefore \text{The upper opening is approximately 199 feet from the entrance, Answer.} \]

**WRITTEN EXERCISES**

Refer to the similar right triangles \( ABC \) and \( A'B'C' \) in giving the value of each of the following ratios.

\[ \begin{align*}
1. \frac{BC}{AB} & \quad 5. \frac{a'}{c'} \\
2. \frac{a}{c} & \quad 6. \sin A' \\
3. \sin A & \quad 7. \frac{A'C'}{A'B'} \\
4. \frac{B'C'}{A'B'} & \quad 8. \frac{b'}{c'}
\end{align*} \]
CHAPTER FOURTEEN

PROBLEMS

Give each answer to the nearest tenth unless otherwise specified. Use the values of the trigonometric functions in the Appendix.

1. A straight road up a hill is 1250 feet long and makes an angle of 18 degrees with the horizontal. Find the height of the hill.

2. A straight cable 800 feet long is run from the top of a tower to the ground. The cable makes an angle of 27 degrees with the ground. How high is the tower?

3. How far from a building must the base of a forty-foot ladder be placed to make a safe angle of 75 degrees with the ground?

4. A thirty-foot ladder makes a 15° angle with the wall it leans against. How high up the wall does the ladder reach?

5. A rectangle is 25 inches wide. Its diagonal makes an angle of 42 degrees with the longer side. How long is the diagonal?

6. How long a ladder is needed to reach a window 25 feet above the ground if the ladder makes an angle of 75° with the ground?

7. A road is inclined 9 degrees to the horizontal. How far must one walk up the road to increase one’s altitude 40 feet?

8. An airplane rises at an angle of 10 degrees with the ground. How far is it from its starting point when it attains a height of 600 feet?

9. How many feet of wire will be needed to brace a pole with a wire from the top of the pole to a stake on the ground 18 feet from the foot of the pole if the taut wire makes an angle of 40° with the ground?

10. When Don’s kite is directly over a landmark 200 feet away, his kite string makes an angle of 42° with the ground. How many feet of string has Don let out? Assume that the string is straight.

14–6 A Table of Trigonometric Function Values

In Table 5 (p. 543) what are the values listed for sin 90° and cos 90°? No value exists for tan 90°. Values for functions of 0° are:

\[ \sin 0° = 0 \quad \cos 0° = 1 \quad \tan 0° = 1 \]

With this information, you see, as you look down the sine column, that as an angle increases from 0° to 90°, the sine increases from...
0 to 1. How do the cosine and tangent change as the angle increases?

If the value of one of its trigonometric functions is given, the measure of an acute angle can be determined. The table indicates that an angle whose cosine is .7660 has a measure of 40°. Suppose \( \tan A = 2.5101 \), a number not listed in your table. To find the measure of \( \angle A \) to the nearest degree, locate in the tangent column the entries between which 2.5101 lies: \( \tan 68° = 2.4751 \) and \( \tan 69° = 2.6051 \). Since \( \tan A \) is closer to \( \tan 68° \) than it is to \( \tan 69° \), the measure of \( \angle A \), to the nearest degree, is 68°.

### WRITTEN EXERCISES

From the table give the sines, cosines, and tangents of these angles.

| A | 1. \( 5° \) | 4. \( 27° \) | 7. \( 48° \) | 10. \( 1° \) |
| 2. \( 2° \) | 5. \( 36° \) | 8. \( 59° \) | 11. \( 57° \) |
| 3. \( 72° \) | 6. \( 60° \) | 9. \( 89° \) | 12. \( 52° \) |

To the nearest degree find the measure of the acute angle named.

| 13. \( \sin A = .5446 \) | 16. \( \sin G = .9272 \) | 19. \( \sin C = \frac{1}{2} \) |
| 14. \( \cos B = .5446 \) | 17. \( \cos K = .2076 \) | 20. \( \cos F = \frac{3}{2} \) |
| 15. \( \tan D = .5543 \) | 18. \( \tan M = 1.1914 \) | 21. \( \tan J = 1\frac{1}{3} \) |

### PROBLEMS

Give answers to tenths unless otherwise specified.

| A | 1. A ship has traveled 180 miles from its starting point (A in the drawing below left). Its bearing is 70 degrees from the north. (a) How far east has the ship sailed? (b) How far north? |
| 2. The pitch angle of the roof in the diagram below right is 32°. The length of the rafters (exclusive of overhang) is 20 feet. (a) Find the rise of the roof (the value of \( x \)). (b) How wide is the house (\( w \))? |
3. An airplane flies 2500 feet at a constant angle of 16° with the horizontal. Find to the nearest foot (a) the ground distance covered, (b) the change in altitude.

4. A tower 90 feet high is situated near another tower 65 feet high. From the top of the taller tower, the angle of depression of the top of the smaller tower is 11°. Find the distance between the towers.

5. A surveyor lays off base line $AC$ 100 feet in length along a river bank. Point $B$ is on the opposite bank directly across from $C$. Angle $A$ is 50 degrees. Find the distance $BC$ (below left).

6. To determine the height of a cloud, a light is projected vertically upward from $C$ to the under portion of the cloud at $B$. At point $A$, 600 feet from the searchlight, the angle of elevation of the spot of light on the cloud is 63°. What is the height of the cloud? (above right)

7. Find the angle of climb of an airplane that travels 12,500 meters through the air to rise 1500 meters.

8. A ship sails from port in a direction 155° from north. When it is 70 miles east of the port, (a) how far has it sailed? (b) How far south of the port is it?

9. Find to the nearest degree the angles of a triangle whose sides are 8', 15', and 17' long.

10. How long is a funicular railway which is inclined 14° and goes up a hill 210 feet high?

11. Two flagpoles are 100 yards apart. The angles of elevation to their tops from the center of the ground between them are 20° and 15°, respectively. How much higher is one pole than the other?

12. From the crow's nest of a ship, an observer finds the angle of elevation of the top of a lighthouse to be 5° and the angle of depression of the foot of the lighthouse to be 8°. If the observer is 80 feet above sea level, find the height of the top of the lighthouse above sea level.

13. In right triangle $ABC$, show that $\tan B = \frac{\sin B}{\cos B}$.

14. In right triangle $ABC$, show that $(\sin A)^2 + (\cos A)^2 = 1$. 
VECTORS

14–7 Working with Vectors

To specify the velocity of an airplane you must give both the direction and magnitude of the velocity. Quantities whose designation requires both magnitude and direction are called vectors and are represented by directed line segments or arrows. Units of length on the arrow indicate the units of magnitude of the vector, and the direction of the arrow indicates the direction of the vector.

In Figure 14–10 vectors represent:

a. A velocity of 30 m.p.h. at 135° (in navigation the angle is called the heading or bearing and is measured clockwise from north).

b. A force of 100 pounds at an angle of 200° (in mathematics the angle is measured counterclockwise from the positive x-axis).

c. Motion in a straight line from \(A(1, 0)\) to \(B(-3, 3)\).

d. Motion from the origin to \(C(-4, 3)\).

Do you see that a vector may be specified by giving its magnitude and direction as a number pair, for example, \((100, 200°)\) or by giving its initial and terminal points, for example, \(A(1, 0)\) to \(B(-3, 3)\)?

A short way to represent a vector is to place an arrow above the names of its end points, thus: \(\overrightarrow{AB}\) represents the vector from \(A\) to \(B\) and \(\overrightarrow{BA}\) represents the vector from \(B\) to \(A\). \(\overrightarrow{AB}\) and \(\overrightarrow{BA}\) are opposite vectors because they have the same magnitude \(AB\) and are collinear but have opposite directions; therefore, you may write \(\overrightarrow{AB} = -\overrightarrow{BA}\).

Vectors like \(\overrightarrow{AB}\) and \(\overrightarrow{OC}\) in Figure 14–10 are equivalent vectors; they have the same magnitude and act in the same direction; you can see
that $AB = OC$, and $\overrightarrow{AB}$ is parallel to $\overrightarrow{OC}$ with the same direction.

The vector representing the sum of two or more vectors is their **resultant**. The resultant of the forces exerted by a man pulling horizontally to the right with a force of 140 pounds and by a boy pushing in the same direction with a force of 100 pounds is a horizontal force of 240 pounds to the right (Figure 14–11, left). If the boy pushes to the left, the resultant is a force of 40 pounds to the right.

**Example.** Determine the resultant velocity $\vec{V}$ of an airplane heading north at 300 m.p.h., $\vec{V}_A$, if it meets westerly winds blowing at 60 m.p.h., $\vec{V}_W$.

**Solution 1:** To add $\vec{V}_A$ and $\vec{V}_W$, construct to scale a right triangle having $\vec{V}_A$ as one side and a vector $\vec{V}'_W$ equivalent to $\vec{V}_W$ as the other side. The hypotenuse $\vec{V}$ is the resultant. From the scale drawing you can estimate that the plane is flying at 310 m.p.h. at a heading of 11°, Answer.

**Solution 2:** You may also solve the problem by trigonometry.

<table>
<thead>
<tr>
<th>$\tan A = \frac{CB}{AC}$</th>
<th>magnitude of $\vec{V}$ is $AB$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{60}{300} = .2000$</td>
<td>$(AB)^2 = (AC)^2 + (CB)^2$</td>
</tr>
<tr>
<td>$A \approx 11^\circ$</td>
<td>$= (300)^2 + (60)^2$</td>
</tr>
<tr>
<td></td>
<td>$= 93600$</td>
</tr>
<tr>
<td></td>
<td>$AB \approx 306$</td>
</tr>
</tbody>
</table>

\[ . \quad \text{The airplane is flying at approximately 306 m.p.h. at a heading of 11°, Answer.} \]
Does each expression describe a vector?

1. Your glove size
2. A batting average of .297
3. A south wind at 6 miles per hour
4. A force of 50 pounds upward
5. A strike thrown by a pitcher
6. An elevator ride to the third floor
7. A three-mile bus ride downtown
8. A speed of 30 miles per hour

Name a vector equivalent to each vector in square $KLMN$.

9. $\overrightarrow{KL}$
10. $\overrightarrow{MN}$
11. $\overrightarrow{LM}$
12. $\overrightarrow{NK}$

What vector in rectangle $RSTV$ makes each statement true?

13. $\overrightarrow{VR} + \overrightarrow{RS} = \overrightarrow{?}$
14. $\overrightarrow{ST} + \overrightarrow{TV} = \overrightarrow{?}$
15. $\overrightarrow{SW} + \overrightarrow{?} = \overrightarrow{SR}$
16. $\overrightarrow{RT} + \overrightarrow{?} = \overrightarrow{RW}$
17. $\overrightarrow{?} + \overrightarrow{TW} = \overrightarrow{SW}$
18. $\overrightarrow{VW} + \overrightarrow{?} = \overrightarrow{VS}$

Find each resultant by using (a) a scale drawing (b) trigonometry.

19. Forces of 100 lb. to the right and 70 lb. up
20. Forces of 520 lb. to the left and 90 lb. up
21. Forces of 60 kg. to the right and 80 kg. down
22. Forces of 70 g. to the left and 150 g. down

Draw a diagram; then find the answer to the nearest integer by trigonometry.

1. Two men stretch a wire fence around a post, making a right-angle corner. One pulls an end north, and the other, east, each with a force of 100 pounds. Describe the force acting on the post at that moment.
2. A man walks north at 5 m.p.h. across a ship sailing east at 12 m.p.h. At what rate and in what direction is he actually moving?
3. An airplane whose speed in still air is 150 m.p.h. heads west in a wind blowing south at 22 m.p.h. In what direction does it move and at what rate?
4. A plane flying at 210 m.p.h. heads west into a wind blowing north at 22 m.p.h. In what direction does the plane move, and at what speed?
5. A balloon pulls up on its anchor rope with a force of 500 pounds, while the east wind blows against it with a force of 200 pounds. Describe the actual pull and the angle the rope makes with the ground.

6. An east wind acts on an anchored ship with a force of 2500 pounds, as a tide flows north with a force of 3500 pounds. Describe the resultant force on the ship.

14–8 Resolving a Vector

Two vectors, such as $\vec{V}_1$ and $\vec{V}_2$ in Figure 14–12, whose sum is $\vec{V}$ are called components of $\vec{V}$. Of particular interest are the horizontal and vertical components of a vector because they refer to a rectangular coordinate system.

**EXAMPLE 1.** A force of 30 pounds is applied to a lawn mower whose handle is 40° from the horizontal. What is the horizontal force moving the lawn mower?

**Solution:**

1. Draw a right triangle and label it to represent the known forces and angles.

2. $\cos 40° = \frac{CB}{30} = .766$

   $CB = 23$

   $\therefore$ The horizontal force is 23 pounds, Answer.

By using $\sin 40°$, you can find the vertical component to be 19 pounds.

Can you find the resultant of two vectors whose horizontal and vertical components are known? The component of the resultant in each direction is the sum of the components of the vectors in that direction.
EXAMPLE 2. Two boys lift a heavy rock by means of rope attached to it. One exerts a force of 30 pounds at 30° to the left of the vertical, and the other exerts a force of 40 pounds at 45° to the right of the vertical. What is the force moving the rock vertically?

Solution:

Let $y_L$ = the vertical component of the force to the left
$y_R$ = the vertical component of the force to the right
$y$ = the total vertical force lifting the rock

In $\triangle RKM$: $\sin 60° = \frac{y_L}{30}$
In $\triangle RTS$: $\sin 45° = \frac{y_R}{40}$

$\therefore y_L = 30 \sin 60° = 30(0.866) = 26$
$y_R = 40 \sin 45° = 40(0.707) = 28$

$y = y_L + y_R = 26 + 28 = 54$

$\therefore$ The total vertical force is approximately 54 pounds, Answer.

Can you explain why the rock is also subject to a net horizontal force of about 13 pounds to the right?

**WRITTEN EXERCISES**

Find the horizontal and vertical components of each vector.

A 1. Force of 150 lb. downward at an angle of 45° from horizontal
2. Force of 210 lb. downward at an angle of 60° from horizontal
3. Velocity of 400 m.p.h. at a heading of 70°
4. Velocity of 650 m.p.h. at a heading of 20°
5. Velocity of a southwest wind blowing at 25 m.p.h.
6. Velocity of a northeast wind blowing at 15 m.p.h.

Find the $x$- and $y$-coordinates of $P$, if $\overrightarrow{OP}$ is the given vector.

B 7. $(17, 30°)$ 8. $(6, 45°)$ 9. $(10, 4°)$ 10. $(8, 86°)$

Each ordered pair gives the horizontal and vertical components of a vector. Find the components of the resultant of the vectors in each exercise.

11. $(4, 3); (5, 6)$ 12. $(7, 8); (9, 4)$
13. $(10, -7); (3, 5)$ 14. $(-3, 1); (4, 8)$
1. Bill pushes a lawn mower with a force of 45 pounds at 37° from the horizontal. How great a force moves the mower?

2. How much of a thirty-pound force applied along the handle of a carpet sweeper at 55° from the floor is effective?

3. Tom pulls the rope of a sled with a force of 15 pounds at 35° from the ground. How many pounds of force are effective?

4. A man operates a pneumatic drill at 60° from the horizontal applying a force of 150 pounds. What is the magnitude of the effective force?

5. A ship's towline makes a 20° angle with the direction of motion when a force of 1200 pounds is applied. Find the pull on the ship.

6. Wind at 52° from the direction of motion of a boat blows against the sail with a 500-pound force. What force pushes the boat forward?

7. A balloon with a lifting force of 3000 pounds is blown so that its anchor rope makes an angle of 35° with the ground. Find the horizontal force of the wind and the pull on the rope.

8. A boy weighing 125 pounds sits in the center of a hammock. Find the pull on each supporting rope if each is 45° from the horizontal.

9. Two forces of 10 pounds each act on an object at an angle of 60° to each other. Find the magnitude of the resultant force.

10. Two forces of 50 pounds each act on an object at an angle of 45° to each other. Find the magnitude of the resultant force.

11. Find the vertical and horizontal components of the resultant of forces of 50 pounds at 20° and 70 pounds at 35° with the horizontal.

12. Find the vertical and horizontal components of the resultant of forces of 70 pounds each, one acting downward at 10° and the other upward at 15° with the horizontal.

Inventory of Structure and Method

1. Theorems stating properties of geometric figures are proved on the basis of axioms stating assumptions for numbers, points, lines, and planes. Once a theorem is proved, you can use it in proving other theorems.

2. Every line segment has a length, and every angle has a measure. In a triangle the sum of the measures of the angles is 180°. The vertices of similar triangles can be paired so that corresponding angles have the same measure and so that the ratio of the lengths of any two sides of one
triangle equals the ratio of the lengths of the corresponding sides in the other triangle. Two triangles are similar if the measures of two pairs of corresponding angles are equal. Two right triangles are similar if an acute angle in one triangle is equal to an acute angle of the other.

3. The **resultant of vectors** acting in the same or opposite directions at a point may be found by adding or subtracting their magnitudes. To find the resultant of two vectors acting at right angles to each other, find the hypotenuse of a right triangle whose sides represent the vectors.

4. To find the **vertical and horizontal components of a vector**, determine the sides of a right triangle whose hypotenuse represents the vector.

**Vocabulary and Spelling**

- collinear \((p. 493)\)
- distance \((p. 494)\)
- line segment \((p. 494)\)
- end points \((p. 494)\)
- length of line segment \((p. 494)\)
- ray \((p. 495)\)
- opposite rays \((p. 495)\)
- angle \((p. 496)\)
- sides of an angle \((p. 496)\)
- vertex of an angle \((p. 496)\)
- edge \((p. 496)\)
- degrees \((p. 496)\)
- right angle \((p. 496)\)
- straight angle \((p. 496)\)
- acute angle \((p. 497)\)
- triangle \((p. 497)\)
- obtuse angle \((p. 497)\)
- correspondence between triangles \((p. 498)\)
- corresponding sides \((p. 498)\)
- corresponding angles \((p. 498)\)
- similar triangles \((p. 498)\)
- tangent function \((p. 502)\)
- angle of elevation \((p. 502)\)
- angle of depression \((p. 502)\)
- sine function \((p. 504)\)
- cosine function \((p. 505)\)
- trigonometric function \((p. 506)\)
- vector \((p. 509)\)
- bearing (heading) \((p. 509)\)
- opposite vectors \((p. 509)\)
- equivalent vectors \((p. 509)\)
- resultant \((p. 510)\)
- components \((p. 512)\)

**Chapter Test**

**14-1** Let point \(Q\) be between points \(P\) and \(R\) on a line. Explain why each statement is true or why it is false.

1. There is a point \(S\) such that \(Q, R,\) and \(S\) are collinear, but \(P, Q,\) and \(S\) are noncollinear.

2. There is a point \(D\) such that \(D \in \overline{PQ}\) and \(D \in \overline{QR}\).
3. In \( \triangle ABC \) find \( m \angle C \) if \( m \angle A = 25^\circ \) and \( m \angle B = 40^\circ \).

State whether the graph of each open sentence is a ray, a point, a line segment, a line, or none of these.

4. \( y \leq -3 \)

5. \(-2 \leq y \leq 6\)

6. \(|y| \leq 0\)

7. \( y > 4\)

8. In the adjacent figure find \( LM \) if \( TP = 8, \ TD = 40, \) and \( TL = 17.\)

9. From a cliff rising 150 feet vertically from the water's edge the angle of depression of a boat is \(32^\circ\). How far is the boat from the base of the cliff? (\( \tan 58^\circ = 1.60\))

10. What is the change in vertical and in horizontal position in going 2 miles up a road with a \(10^\circ\) grade? (\( \sin 10^\circ = .174, \ \cos 10^\circ = .985\))

11. At a point 100 feet from the foot of a building 75 feet high, the angle of elevation of a flagpole on top of the building is \(41^\circ\) degrees. How high, to the nearest foot, is the flagpole?

12. Find the magnitude of the resultant, to the nearest pound.

13. Find, to the nearest degree, the angle that the resultant makes with the smaller of the two forces.

14. A force of 145 pounds acts at an angle of \(15^\circ\) with the vertical. Find the horizontal and vertical components, to the nearest pound.

Chapter Review

14-1 Geometric Assumptions

1. Every line contains at least \(?\) points, and every plane contains at least \(?\) noncollinear points.

2. If \(c\) is the coordinate of \(C\) on a line where the coordinate of \(P\) is 7, and if the coordinate of \(Q\) is 3 and \(3 < c < 7\), then point \(?\) is between points \(?\) and \(?\).
14-2 Rays and Angles

3. If on line \( l \) the coordinate of \( D \) is 5 and of \( K \) is 8, write an open sentence whose graph on \( l \) is the ray \( DK \).
4. The sum of the measures of the angles of a triangle is \( ? \).

14-3 Similar Triangles

5. The measures of the corresponding angles of similar triangles are \( ? \).
6. The sides of a right triangle are 10 inches, 12 inches, and 13 inches. The corresponding sides of a similar triangle are \( ? \) inches, \( ? \) inches, and 52 inches.
7. A right triangle contains a 35° angle. Another right triangle contains a 55° angle. Are the triangles similar?

Items 8–10 refer to the accompanying diagram.

8. If \( a = 4 \), \( b = 5 \), and \( b' = 20 \), then \( a' = \ ? \).
9. If \( a:b = 2:3 \), and \( b' = 21 \), then \( a' = \ ? \).
10. If \( a' = 28 \) and \( b' = 70 \), then \( a:b = \ ? \).

14-4 The Tangent Function

11. The tangent of an angle in a right triangle is the ratio of the length of side \( ? \) that angle to the length of side \( ? \) to it.
12. In right triangle \( ABC \), \( \tan A = \ ? : \ ? \).
13. In right triangle \( ABC \), \( \tan B = \ ? : \ ? \).
14. A boy stands 50 feet from the base of a flagpole and sights its top along the hypotenuse of an isosceles right triangle, holding the triangle at eye level, 5 feet above the ground. (Each acute angle in an isosceles right triangle contains 45°; \( \tan 45^\circ = 1 \).) Find the height of the flagpole.

14-5 The Sine and Cosine Functions

To answer Items 15–17, refer to the figure below. Use these trigonometric functions: \( \sin 28^\circ \approx .47; \sin 35^\circ \approx .57; \cos 28^\circ \approx .88; \cos 35^\circ \approx .82 \). Triangle \( PDQ \) is not a right triangle, but its altitude, \( a \), divides it into two right triangles. \( PD \) is 1.3 inches long. Give answers to tenths.

15. Find the length of the altitude \( a \) of triangle \( PDQ \).
16. Find \( DQ \).
17. Find \( PQ \).
18. In right triangle $ABC$, $\sin A = \ldots : \ldots$.
19. Which has the greater value, $\sin 15^\circ$ or $\sin 55^\circ$?

**14-6 A Table of Trigonometric Function Values**

*Use Table 5, Appendix, to answer Items 20–29.*

- 20. $\sin 55^\circ = \ldots$
- 21. $\cos 80^\circ = \ldots$
- 22. $\tan 13^\circ = \ldots$
- 23. $\sin \theta = 0.2079$
- 24. $\cos \theta = 0.3090$
- 25. $\tan \theta = 1.1106$

26. The measure of an angle whose cosine is 0.5588 is a little larger than ___.

27. A wire used to brace a telephone pole is 25 feet long and makes an angle of 55 degrees with the ground. How high up the pole (to the nearest foot) does the wire reach?

28. How long (to the nearest foot) is a ladder that makes an angle of 50 degrees with the ground at a point 40 feet from the base of the building it leans against?

29. What is the angle of ascent (to the nearest degree) of a flight of stairs in which each step has an eleven-inch tread (horizontal) and a seven-inch rise (vertical)?

**14-7 Working with Vectors**

- 30. A force of 100 pounds acts on an object horizontally to the right. Another force of 100 pounds acts vertically upward on it. Draw to scale the vectors and their resultant.

- 31. The magnitude of the resultant in Item 30 is ___; it acts at an angle of ___ with the horizontal.

- 32. Describe the motion of a canoe when a boy paddles north at 5 m.p.h. and the current is 2 m.p.h. to the east.

**14-8 Resolving a Vector**

- 33. A small boy is pulling a wagon with a force of 10 pounds. The handle makes an angle of 30 degrees with the wagon. Make a scale drawing showing the vector of the force the boy applies, and its horizontal and vertical components.

- 34. A force of 200 pounds is acting at an angle of $55^\circ$ with the horizontal. Find the vertical and horizontal components, each to the nearest pound.
At the University of Cambridge, in the year 1705, a man knelt before Queen Anne of England to be dubbed a Knight of the Realm. Many men before him had been thus honored, but this man was vastly different from his predecessors. Their distinction had been won on the battlefields; his, in the sphere of natural philosophy and mathematics. His name was Isaac Newton.

The abilities which set Newton apart from other men were displayed at an early age. After a brief schooling, he was to learn the business of his father's farm, but he spent most of his time making mechanical models, reading, and solving problems. His mother wisely resolved to encourage his interests and, on the advice of an uncle, sent him to Cambridge in 1661. In his first five years there, Newton not only formulated the theory of gravitation but began investigating the properties of light. He also invented the calculus, a new branch of mathematics which could express the principles of gravitation as well as many discoveries yet to be made in the science of physics. No wonder that in 1669, Newton's former professor resigned his post in favor of this extraordinary student! No wonder that the Queen chose for Newton an honor never before granted a scientist!

Newton once said, "... I ... have been ... like a boy, playing on the seashore ... now and then finding a smoother pebble, or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.... If I have seen a little farther than others, it is because I have stood on the shoulders of giants." Three hundred years later we can say that Newton, for all his modesty, was one of the greatest giants and, by standing on his shoulders, scientists have begun perhaps to discover some of the ocean of truth.

Isaac Newton as a young man. This engraving was made from a portrait of Newton painted when he was at Trinity College, Cambridge.
Complex Numbers

In a plane a point like $P(3, 2)$ determines a vector $(\overrightarrow{OP}$ in Figure 14–13) from the origin to the point. To pair a number with each such vector, begin with the real numbers as the partners of the horizontal vectors from $O$ (Figure 14–14).

Since you use all the real numbers as partners of the horizontal vectors, you need new numbers as partners of the vertical vectors. Use the letter $i$ to designate the partner of $\overrightarrow{OR}$, $R(0, 1)$, and, in general, $ai$ to name the partner of the vector to $(0, a)$ (Figure 14–15). By agreement, $0i = 0$.

The vector $\overrightarrow{OP}$ in Figure 14–16 has $\overrightarrow{ON}$ as its horizontal component and $\overrightarrow{OS}$ as its vertical component. Since $\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{OS}$, pair $\overrightarrow{OP}$ with the sum of the partners of $\overrightarrow{ON}$ and $\overrightarrow{OS}$: $3 + 2i$. In general, pair the vector from $O$ to $(a, b)$ with the sum: $a + bi$. 
Expressions of the form $a + bi$, where $a$ and $b$ are real numbers, represent the elements of the set of complex numbers. You add two complex numbers just as you add vectors whose horizontal and vertical components you know.

$$(2 + 3i) + (4 - 7i) = (2 + 4) + (3 - 7)i = 6 - 4i$$

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$ 

The identity element for addition is $0 + 0i$, or 0. Because

$$(a + bi) + (-a - bi) = 0 + 0i,$$

the additive inverse of $a + bi$ is $-a - bi$.

To multiply complex numbers, first define $i^2 = -1$

Then, multiply (as you multiply binomials) as follows:

$$(a + bi) \cdot (c + di) = ac + i^2bd + adi + bci$$

$$= (ac - bd) + (ad + bc)i$$

Thus, $(2 + 3i)(4 - 7i) = 8 - i^2 \cdot 21 - 14i + 12i$

$$= (8 + 21) + (-14 + 12)i$$

$$= 29 - 2i$$

The identity element for multiplication is $1 + 0i$, or 1. To find the reciprocal of a complex number such as $3 + 2i$, proceed as follows:

$$\frac{1}{3 + 2i} = \frac{1}{3 + 2i} \cdot 1 = \frac{1}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{9 + 4}$$

$$\therefore \frac{1}{3 + 2i} = \frac{3}{13} - \frac{2}{13}i$$

Numbers like $3 + 2i$ and $3 - 2i$ are conjugate complex numbers. The product of a complex number and its conjugate is a real number:

$$(a + bi)(a - bi) = a^2 + b^2.$$
The complex number system is closed under addition, subtraction, multiplication, and division (except by 0).

**EXAMPLE.** If \( r = 3 + i \) and \( s = 4 - 3i \), find \( r + s \), \( r - s \), \( r \cdot s \), \( r \div s \).

**Solution:**

\[
\begin{align*}
    r + s & = (3 + i) + (4 - 3i) = 7 - 2i \\
    r - s & = (3 + i) - (4 - 3i) = (3 + i) + (-4 + 3i) = -1 + 4i \\
    r \cdot s & = (3 + i) \cdot (4 - 3i) = 12 - 3(-1) - 9i + 4i = 15 - 5i \\
    r \div s & = \frac{3 + i}{4 - 3i} = \frac{3 + i}{4 - 3i} \cdot \frac{4 + 3i}{4 + 3i} = \frac{9 + 13i}{25} = \frac{9}{25} + \frac{13}{25}i.
\end{align*}
\]

The complex number system has the commutative and associative properties of addition and multiplication and also the distributive property. Moreover, any polynomial equation of degree \( n \) having this system as the domain of its coefficients and variable has \( n \) roots in this system. For example, the solution set of \( x^2 + 1 = 0 \) is \( \{i, -i\} \). These complex numbers are thus the square roots of \(-1\): \( i = \sqrt{-1}, \) \(-i = -\sqrt{-1} \). Similarly, \( \sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i \) and \( \sqrt{-48} = 4i\sqrt{3} \).

**Questions**

1. Express the following in a form with \( i \) as a factor.
   
   a. \( \sqrt{-64} \)  
   b. \( \sqrt{-27} \)  
   c. \( \sqrt{-\frac{81}{25}} \)  
   d. \( \sqrt{-0.25} \)

2. Name the additive inverse and the conjugate.
   
   a. \( 5 + 2i \)  
   b. \( 6 - 7i \)  
   c. \( -3 - \sqrt{-1} \)  
   d. \( 7 + \sqrt{-9} \)

3. Let \( r \) be the first complex number and \( s \), the second in the following pairs. Compute \( r + s \), \( r - s \), \( r \cdot s \), \( r \div s \).
   
   a. \( 3 + 7i, i \)  
   b. \( 15 + i, 1 - i \)  
   c. \( -i, 1 - 2i \)  
   d. \( i, -3 + 2i \)  
   e. \( 9 - 3i, 7 + 6i \)  
   f. \( -3 - 5i, 6 + i \)

4. Simplify:

   a. \( i^3 \)  
   b. \( i^4 \)  
   c. \( (-3 + 7i)^2 \)  
   d. \( (x - yi)(x + yi) \)

   e. \( (5 + 3i) + (2 - 7i) - (3 - 8i) \)  
   f. \( \frac{3 + 3i}{8 + 6i} - \frac{7 + 4i}{2 - 3i} \)

   g. \( (-i)(1 + 2i)(-3 + i) \)  
   h. \( \frac{3 - 9i}{7 + 2i} + \frac{3 - 4i}{8 + 3i} \)

5. Solve the following equations, if the domain of the variable is the set of complex numbers.

   a. \( x^2 + 25 = 0 \)  
   b. \( y^2 + 75 = 0 \)  
   c. \( x^2 - 2x + 2 = 0 \)  
   d. \( z^2 - 4z + 13 = 0 \)  
   e. \( i^2 - 6t + 34 = 0 \)  
   f. \( i^2 + 6t + 34 = 0 \)
Navigators and Mathematics

On the open sea, when the skies are clear, a navigator takes readings of his position by sighting on the stars with a sextant, an instrument which measures the angle of a star above the horizon. With these sightings and the information contained in various mathematical tables, the navigator is able to determine his ship's position, which he plots on charts in terms of longitude and latitude.

The known and exactly plotted orbits of certain satellites enable marine navigators and aeronauts to plot positions with greater accuracy than from the stars. Some of these satellites, called "lighthouses in the sky," are sighted with a sextant in the same way as stars, while others send radio signals which may be used to compute the location of a ship or plane and keep it on course.

Whether a navigator uses radar or a sextant or plots his theoretical course on a map as the civilian navigator in the photograph is doing, he needs to know algebra, geometry, and trigonometry for plotting the direction and speed of his craft. Applied mathematics may also enable a navigator to maintain a safe course when approaching or avoiding another vessel. It is possible to find the approximate distance to the other ship by measuring with a sextant the angle between its water-line and the top of its mast, as illustrated on the work pad. Knowing the height of the ship from its type, the navigator can calculate the approximate distance to it from his own vessel. And, by taking such readings at regular intervals, he can compute the other ship's speed and plot its course relative to his own.
For any real numbers \( a, b, c \):

\[
\begin{align*}
    a + b &= b + a \\
    (a + b) + c &= a + (b + c) \\
    ab &= ba \\
    (ab)c &= a(bc) \\
    a(b + c) &= ab + ac
\end{align*}
\]

If \( a > b \), \( b > c \), then \( a > c \)

\[
\begin{align*}
    1 \cdot a &= a \\
    -1 \cdot a &= -a \\
    |a| &= |-a| \\
    a + (-a) &= 0
\end{align*}
\]
Comprehensive Review and Tests

When you climb a mountain trail, you pay close attention to each obstruction until at some moment you pause, look back, and realize how far you have come. Sometimes at the end of a school year, you may have this same sense of accomplishment. You have met each difficulty as it has appeared, but not until the close of the year can you evaluate your total progress.

In this chapter you have an opportunity to review your learning for the year. In the light of your total knowledge of algebra, you may now restudy topics you found difficult. Furthermore, you will be able to relearn subjects which have become hazy with time. These next few lessons provide an opportunity for you to clinch your understanding, to sharpen your knowledge, to deepen your insight, and to review the framework needed for further work in mathematics.

REVIEW YOUR ALGEBRA

15–1 Properties of Numbers: Structure

Give a reason (property, principle, definition, etc.) to justify each statement. The set of real numbers is the replacement set for each variable, with exceptions as noted.

1. \( \frac{4a - 6}{2} = 2a - 3 \)
2. \( 729 + (93 + 71) = 729 + (71 + 93) \)
3. \( a - b = a + (-b) \)
4. If \( a = -a' \), then \( a + a' = 0 \).
5. \( \frac{d}{5} = q \), if \( d = 5q \).
6. If \( rs = 0 \) and \( r \neq 0 \), then \( s = 0 \).
7. \( (a - b)(c - d) = a(c - d) - b(c - d) \)
8. \( (a - b)^3 = (a - b)(a - b)(a - b) \)
9. If \( x^2 - 6x + 3 = 0 \), then \( x^2 - 6x = -3 \).
10. \( a + \frac{b}{c} = a \cdot \frac{c}{b} \) when \( b \neq 0 \) and \( c \neq 0 \).

11. If \( m \neq n \), then \( m > n \) or \( m < n \).

12. If \( p + q = r \), and \( p \) and \( q \) are integers, then \( r \) is an integer.

13. If \( S_1 = \{-2, -1, 0\} \) and \( S_2 = \{1, 2\} \), the only subset of both is \( \emptyset \).

14. If \( n = -n \), then \( n = 0 \) only.

15. If \( \frac{m}{3} \) is an integer, then \( m \) is an integral multiple of 3.

16. If \( x > y \) and \( y > z \), then \( x > z \).

17. \( x + 2 = 5 \) and \( y - 2 = 1 \) are equivalent equations.

If a statement is always true, answer true; if not, answer false.

18. If a set is closed under an operation, any subset (except \( \emptyset \)) of that set is closed under that operation.

19. \{points on the equator\} is infinite.

20. The solution set of \(-2 < x < 1\) is shown by \(-2 -1 0\).

21. The associative property for addition in the set of real numbers is proved from other properties.

22. The solution of \(3x + 5y = 7\) is a set of ordered number pairs.

23. If \( ab < 0 \), then \( a < 0 \).

24. In right triangle \(ABC\) if \(a = 3, b = 4, c = 5\), then \(\sin A = \frac{4}{5}\).

Express as an ordered pair the number of each statement and the letter of the reason justifying the conclusion drawn. Each variable represents a real number.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Conclusions</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>25. ((3x - 2)(x + 3))</td>
<td>(= 3x^2 + 9x - 2x - 6)</td>
<td>A. Symmetric property of equality</td>
</tr>
<tr>
<td>26. (3x - 2 = k + 4) (k + 4 = 5)</td>
<td>(3x - 2 = 5)</td>
<td>B. Transitive property of equality</td>
</tr>
<tr>
<td>27. (\frac{3x - 2}{k - 1} = \frac{k - 1}{2})</td>
<td>(k \neq 1)</td>
<td>C. Substitution principle</td>
</tr>
<tr>
<td>28. (3x - 2 = 5)</td>
<td>(5 = 3x - 2)</td>
<td>D. Division by 0 is undefined</td>
</tr>
<tr>
<td>29. (3x - 2 &gt; x - 1)</td>
<td>(2 - 3x &lt; 1 - x)</td>
<td>E. Distributive property</td>
</tr>
<tr>
<td>30. (3x - 2 = k + 4) (k = x - 1)</td>
<td>(3x - 2 = x + 3)</td>
<td>F. Multiplicative property of (-1)</td>
</tr>
</tbody>
</table>
15-2 Algebraic Representation

1. What is the average of the numbers \(a, p,\) and \(n\)?
2. The area of a square is \(9n^2\). Represent its perimeter.
3. A rectangle is 5 units long and \(n\) units wide. Represent its diagonal.
4. From a board \(f\) feet long, a carpenter cut a piece \(t\) feet 3 inches long. Represent, in inches, the remaining piece.
5. If \(n\) articles cost \(c\) cents, how many can you buy for a quarter?
6. Charles spent \(c\) cents for \(n\) notebooks, at \(t\) cents each, and \(p\) pencils. Represent the cost of a pencil.
7. An article cost the dealer \(m\) dollars. Represent the selling price with (a) a 20\% profit, and (b) a 10\% loss.
8. Represent a man’s possible annual salary \(n\) on a job which starts at a minimum of \(s\) dollars monthly and can go up to \(t\) dollars annually.
9. Represent the area \(A\) of a circle whose radius is between 7 and 8 centimeters.

Using \(a = 3, b = -1, c = \frac{1}{3}, d = -2,\) and \(e = 0\), find these values.

10. \(3ab^2\)
11. \(\frac{2ad}{c - b}\)
12. \(2a^2 - 4c\)
13. \(5ab - 2c^3d^2 + 1\)
14. \(\sqrt{4(d - e)^2 - 6abc}\)
15. \(a \times d \div 2b \div c \times d\)

Round each number to one place fewer.

16. 58.6
17. 5.84
18. 0.06
19. 0.9
20. 0.95

Find to the nearest tenth.

21. \(\frac{5}{13}\)
22. \(\sqrt{57}\)
23. \(\sqrt{7.1}\)

Give each ratio in its simplest form.

24. 3 feet to 2 yards
25. 3000 pounds to 4\(\frac{1}{2}\) tons
26. 2 weeks to 26 days
27. The volume of a one-inch cube to that of a cube 3 inches on a side.

28. Find the area of the figure at the right, to the nearest tenth of a square inch.
29. Triangles $ABC$ and $DEF$ are similar; $\angle A = \angle D$, $\angle B = \angle E$, $AB = 6$, $BC = 7$, $DE = 12$, $EF = ?$

30. In triangle $ABC$, $\angle C = 90^\circ$, $\angle A = 40^\circ$, and $AB = 12$; find $BC$ and $AC$ if $\sin 40^\circ = .643$, $\cos 40^\circ = .766$, $\tan 40^\circ = .839$.

**15–3 Fundamental Operations and Factoring**

Simplify each expression.

1. $5(a + 2) - 3(a + 4)$
2. $2m^2n(3m^3n - 5n^2)$
3. $(3x - 2)(2x + 1)$
4. $(a^2 + 3ab + 9b^2)(a - 3b)$
5. $(2a - b)^5 \div (2a - b)^4$
6. $(1 - 4a)^2$
7. $(x^3 + 8) \div (x + 2)$
8. $(18ax^3 - 6a^2x^2 + 3a^3x) \div (-3ax)$
9. $(5t^3 + 13t^2 + 5t + 7) \div (t + 3)$
10. $8(1 - a) - 2[(a + 2b) - 3(2b - 1)]$
11. $3[14c - (-2 + 3c)] - 5[(2c - 1) + (c - 5)] - 3$

Find the prime factors of each polynomial over the set of real numbers.

18. $3x^2 - 21x^3$
19. $1 - 9a^2$
20. $4\pi R^2 - 2\pi RH$
21. $y^2 - y - 12$

Do the indicated operations, and express in simplest form.

30. $\frac{a}{a + 1} + \frac{5}{6a + 6}$
31. $\frac{h^2}{h^2 - 4} - \frac{h}{h + 2}$
15-4 Radicals

Simplify each expression.

1. \( \sqrt{28} \)

2. \( \frac{3}{2} \sqrt{180a^2} \)

3. \( \frac{1}{2} \sqrt{4b^{16}} \)

4. \( 6 \sqrt{\frac{1}{3}} \)

5. \( 2 \sqrt{\frac{7}{8}} \)

6. \( \sqrt{\frac{3}{3}} \)

7. \( 3 \sqrt{4a^2 + 4b^2} \)

8. \( \frac{3}{\sqrt{2}} \)

9. \( \frac{\sqrt{2}}{2\sqrt{3}} \)

10. \( \frac{20}{\sqrt{8}} \)

Combine the expressions.

11. \( \frac{3\sqrt{2} + 6}{\sqrt{18}} \)

12. \( \frac{a}{\sqrt{2}a} \)

13. \( \frac{1}{\sqrt{2} + 1} \)

14. \( \frac{6}{3 - \sqrt{5}} \)

Perform the indicated operations, and simplify the results.

21. \( (\sqrt{6} - 2)(\sqrt{3}) \)

22. \( (4\sqrt{6})(3\sqrt{2} - 2\sqrt{3}) \)

23. \( (3\sqrt{2} + \sqrt{3})(3\sqrt{2} - \sqrt{3}) \)

24. \( (2\sqrt{30})(6\sqrt{\frac{3}{5}}) \)

Find to the nearest tenth.

25. \( \sqrt{29} \)

26. \( \sqrt{425} \)

27. \( \sqrt{37.8} \)

28. \( \sqrt{8.24} \)

29. By substitution show that \( 2\sqrt{3} \) is a root of \( x^3 - 2x^2 - 12x + 24 = 0 \).

30. If \( \frac{\sqrt{6}}{\sqrt{6} + 1} - \frac{\sqrt{6}}{\sqrt{6} - 1} = k\sqrt{6} \), find \( k \).

Express as a rational fraction in lowest terms.

31. \( 1.393939... \)

32. \( 0.57 \)
15-5 Equations

Find the solution set of each equation.

1. \[3a + 7 = 5a - 11\]
2. \[2c = 3(2c - 1)\]
3. \[\frac{2f}{3} + \frac{3f}{4} + \frac{17}{6} = 0\]
4. \[\frac{g}{2} - \frac{g + 2}{3} = 1\]
5. \[\frac{1}{2p} - \frac{1}{6p} = \frac{1}{2}\]
6. \[4(2d + 1) - 2(2d - 1) = 0\]
7. \[\frac{2}{2t + 3} + 2 = \frac{10}{3}\]
8. \[2[3(2k + 1) - (k - 3)] = 5 - 3(2k - 5)\]

Solve for \(x\).

9. \[\frac{1 - v}{6v - 3} - \frac{2}{3} = \frac{1}{2v - 1}\]
10. \[\frac{10}{w^2 - 25} + \frac{w}{w - 5} = \frac{30 + w}{w + 5}\]
11. \[\frac{2y}{2y - 1} - \frac{5}{1 - 2y} = 3\]
12. \[\frac{6}{x + 3} + \frac{2}{3 - x} = 0\]
13. \[\frac{1}{1 - c^2} + \frac{1}{c - 1} = 0\]
14. \[4.8(x - 2) - 3(9x + 1) = 0\]

Find the solution set, expressing irrational results to the nearest tenth.

15. \(2ax = 6b\)
16. \(mx + n^2 = m^2 - nx\)
17. \[\frac{2}{n} - \frac{2}{m} = \frac{1}{x}\]

18. \(x^2 + 64 = 16x\)
19. \(3x^2 = 75\)
20. \(h^2 - 3h = 0\)
21. \(3t^2 - 9t + 1 = 0\)

Solve for both variables.

22. \(\sqrt{2y} + 1 = 9\)
23. \(\sqrt{9x^2 - 1} = 3x\)
24. \(\sqrt{s} + \frac{2}{\sqrt{s}} = \frac{s + 2}{\sqrt{s}}\)
25. \(\sqrt{11x + 20} + x = 2\)

26. \(x - y = 1\)
\(3x - 2y = 0\)
27. \(2x - y = -6\)
\(6x + 4y = 3\)
28. \(10t + u = 8(t + u) + 7\)
\(u = t - 6\)

29. \(\frac{m}{2} + 3n = -15\)
\(\frac{m}{4} - 5n = 38\)
30. \(.3r + .2s = 9.5\)
\(.2r + .5s = 15.5\)

Solve by graphing.

31. \(x + y = 1\)
\(3x + y = 9\)
32. \(2x - y = -5\)
\(2x + 3y = -12\)
33. \(y = |x| + x\)
\(x + 4y = 2\)
34. For what value of $k$ will $kx - 3y = 1$ pass through $(4, -3)$?
35. For what value of $k$ will $2x + 3ky = k$ pass through $(-2, 3)$?
36. Find the $x$-intercept of $2x - 3y = 6$.
37. Find the $y$-intercept of $4x - 3y = 12$.

In Exercises 38–43, find an equation of the line
38. having the slope $-4$ and passing through $(5, \frac{1}{2})$.
39. having the slope $\frac{3}{2}$ and passing through $(-1, -6)$.
40. parallel to the line $3x - y = 2$ and passing through $(2, 4)$.
41. parallel to the line $2x + 3y = 3$ and having the $y$-intercept $3$.
42. passing through points $(-1, 10)$ and $(1, -2)$.
43. passing through points $(1, 6)$ and $(-1, -9)$.
44. Find $k$ so that the slope of the graph of $6x + ky = k$ will be $2$.
45. Graph points $(x, y)$ satisfying both $x + 2y = k$ and $2x - y = k$.

15–6 Functions and Variation

Solve each equation for the indicated variable.
1. $v^2 = 2gs$ for $s$
2. $R = \frac{3}{2}(F - 32)$ for $F$
3. $v = \sqrt{2gh}$ for $h$
4. $A = \frac{1}{2}h(B + b)$ for $B$

In Items 5–10, write a formula for the function shown by each table.

5. | $x$ | 1 | 3 | 4 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

6. | $x$ | -1 | 1 | 2 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-1</td>
<td>5</td>
</tr>
</tbody>
</table>

7. | $x$ | 1 | 2 | 4 | 5 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

8. | $n$ | 1 | 2 | 4 | 5 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>2</td>
<td>7</td>
<td>17</td>
</tr>
</tbody>
</table>

9. | $t$ | 2 | 4 | 6 | 8 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

10. | $x$ | 1 | 2 | 3 | 5 |
    |---|---|---|---|
    | $y$ | 1 | -2 | -5 | -11 |

11. Graph $d = 25t$ if $0 < d \leq 110$. What is the domain of this function?
12. A room's dimensions are $l$, $w$, and $h$. It has two $a \times b$ windows and a $c \times d$ door. Write an equation for the wall surface $S$ of the room.
13. In the formula $V = \frac{4}{3}\pi R^3$, find $V$ when $R = 3\frac{1}{2}$, and $\pi = \frac{22}{7}$. 
Write as proportions.

14. \( x \) varies directly as \( y \)
15. \( m \) varies inversely as \( n^2 \)
16. \( n \) varies directly as \( c \) and inversely as \( d \).

Write as equations using the constant \( k \).

20. \( h \) varies directly as \( t \)
21. \( V \) varies inversely as \( P \)
22. \( \frac{x_1}{x_2} = \frac{y_2}{y_1} \)
23. \( A_{1} s_{2}^2 = A_{2} s_{1}^2 \)

24. If \( f \) is proportional to \( w \), and \( f = 81 \) when \( w = 18 \), find \( w \) when \( f = 90 \).
25. If \( s \) varies as \( t^2 \), and \( s = 64 \) when \( t = 2 \), find \( t \) when \( s = 100 \).
26. If \( x \) is proportional to \( y \), and \( y \) varies directly as \( z \), find \( z \) when \( x = 8 \), if \( x = 6 \) when \( z = 9 \).
27. If \( r \) varies inversely as \( s \), and \( r = 6 \) when \( s = \frac{3}{2} \), find \( s \) when \( r = 9 \).
28. \( F \) varies inversely as \( d^2 \); \( d = 6 \) when \( F = 28 \); find \( F \) if \( d = 9 \).
29. A six-inch pulley is belted to a fourteen-inch pulley which is running at 420 r.p.m. How fast is the smaller pulley running?
30. A gear containing 46 teeth is meshed with a gear having 48 teeth. If the latter runs at 600 r.p.m., how fast does the first run?
31. If a machine digs a trench 6 feet deep and 32 yards long in 3 hours, how fast would it dig one 7 feet deep and 20 yards long?
32. a. If 4.6 land miles \( m \) equal 4 nautical miles \( n \), find \( k \) in \( m = k n \).
   b. How many nautical miles are in 207 land miles?

15-7 Inequalities

Graph the solution sets.

1. \( 2y + 5 \geq 6y - 7 \)
2. \( 5 < 2n - 3 \leq n + 2 \)
3. \( |2t + 7| > 3 \)
4. \( |2k - 5| > 5 \)
5. \( x^2 + 2x > 3 \)
6. \( x + 2y = 4 \)
7. \( x + 2y > 4 \)
8. \( 3x > 2y - 5 \)

15-8 Problems

1. How large is a portion of 100 which is 4 more than 3 times the rest?
2. If the smaller of two numbers in the ratio $\frac{3}{4}$ is increased by 4, and the larger, decreased by 4, their ratio is $\frac{5}{5}$. Find the original numbers.

3. Find the sides of a right triangle which are consecutive even numbers.

4. Find two numbers in the ratio $\frac{3}{4}$ if the sum of their reciprocals is 2.

5. The sum of two numbers is 38. When the larger is divided by the smaller, the quotient is 3 and the remainder, 2. Find the numbers.

6. If 7 is added to both numerator and denominator of a fraction, the result equals $\frac{4}{5}$. If 7 is subtracted from both numerator and denominator, the result equals $\frac{3}{5}$. Find the original fraction.

7. Dan is 3 times as old as Ethelynn. In 4 years, he will be twice as old as she is then. How old are they?

8. Mrs. Parker has a yard 30 feet by 36 feet in which a walk borders a garden measuring 720 square feet. How wide is the walk?

9. One angle of a triangle is $7^\circ$ less than $\frac{1}{3}$ of the second angle, and the third is $1\frac{1}{2}$ times as large as the second. How large is the smallest?

10. What angle has a supplement $10^\circ$ less than 3 times its complement?

11. When a vertical ten-foot pole casts a six-foot shadow, how tall is a tree which casts a fifteen-foot shadow?

12. The sides of a triangle are 8, 10, and 12. Find the perimeter of a similar triangle whose longest side is 54.

13. Bert, sitting 6 feet from the middle of a seesaw, balances Ned, sitting 7 feet from the middle. How heavy is Ned if Bert weighs 140 pounds?

14. If 245 grams of potassium chlorate produce 96 grams of oxygen, how many grams, to the nearest tenth, of oxygen do 100 grams produce?

15. A house plan shows a living room as $1\frac{1}{5}$ inches by $1\frac{1}{5}$ inches. If the scale reads 1 inch = 10 feet, what are the dimensions of the room?

16. Rod A, with 100 equal parts, is the same length as Rod B, with 80 equal divisions. What height on Rod B corresponds to a height of 60 on A?

17. If a line $2\frac{1}{2}$ inches long represents 100 feet, what distance is represented by a line $1\frac{1}{8}$ inches long?

18. Two trains leave a town at the same time and travel in opposite directions. One goes 18 m.p.h. faster than the other. If they are 205 miles apart after $2\frac{1}{2}$ hours, how fast is each train traveling?

19. Sam cycles to Dick’s house at 12 m.p.h. and returns by bus at 36 m.p.h. If the round trip is 1 hour, how far from Dick does Sam live?

20. Ben’s boat goes 5 m.p.h. in still water. How far does he travel in a three-hour round trip on a river flowing at 1 mile an hour?

21. Starting at the same time, Mark cycles north, and Carl rides south. Mark rides 3 m.p.h. faster than Carl. If in one hour the boys are 15 miles apart, how fast is each?
22. The total income of two sums invested at 5% and 3\(\frac{1}{2}\)% is $65. Were the rates interchanged, the income would be $71. Find each sum.

23. Make a bar graph of one day's fruit sales, by pounds: apples, 43; peaches, 31; pears, 18; cherries, 16; and grapes, 25.

24. Make a circle graph of this budget: rent, 20%; food, 40%; clothes, 10%; insurance, 10%; housewares, 5%; other, 10%; savings, 5%.

25. Mrs. Adams buys 2 yards of silk and 8 yards of cotton for $8.30. Mrs. Sherman pays $7.85 for 3 yards of the silk and 5\(\frac{1}{2}\) yards of the cotton. What is the cost per yard of each?

26. The total income of two parts of $1000 invested at 7% and 2% is $32. What amount is invested at each rate?

27. How many pounds of nuts worth 44\(\frac{1}{2}\) a pound added to 10 pounds worth 80\(\frac{1}{2}\) a pound produce a mixture worth 59\(\frac{1}{2}\) a pound?

28. A child's bank opens when it contains $10. If 52 quarters and dimes open it, how many coins of each kind are there?

29. How many milliliters of water added to 12 milliliters of an 80% acid solution will make the result a 60% solution?

30. How many pints of pure disinfectant added to 85 pints of a 5% disinfectant solution will result in a 15% solution?

31. Find the number which is 1 less than 9 times the sum of its two digits, and whose units digit is 6 less than its tens digit.

32. How long will it take two machines together to do a job which one does in 12 hours and the other, in 18 hours?

33. Some boys buy a $96 tent. When two boys cannot pay, each of the others pays $4 more. What was the original number of boys?

34. If the first of three consecutive even numbers is divided by 4, the second by 6, and the third by 8, the sum of the quotients equals 29. Find the numbers.

35. Find the diagonal of a cube whose edge measures 4 inches.

36. Fence posts are placed at equal intervals around a field whose perimeter is 720 feet. There would be 10 fewer posts if they were 1 foot farther apart. How many posts are there?

37. Make a broken-line graph of this day's temperature variation.

<table>
<thead>
<tr>
<th>Hour</th>
<th>9</th>
<th>11</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-4</td>
<td>0</td>
<td>5</td>
<td>6</td>
<td>0</td>
<td>-5</td>
</tr>
</tbody>
</table>

38. The tens digit of a number is 2 less than the units digit. Three times the square of the tens digit increased by the sum of its two digits equals the number, itself. Find the number.
39. Find a number which is 28 times the sum of its three digits, whose units and hundreds digits are equal, and whose tens digit is one more than the sum of the other two.

40. In triangle \(ABC\), \(m \angle A\) is 20° more than twice that of \(\angle B\). If \(m \angle C\) is at least 1°, find the largest possible measure of angle \(A\).

41. Mr. House buys flagstones to cover a patio of 160 square feet. If the patio may be between 8 and 10 feet wide, what limitations are there on its length, assuming that Mr. House uses all the flagstones?

42. A master mechanic works at least twice as fast, but not as much as three times as fast, as his apprentice. If together they do a job in 6 hours, how fast can the master do the job alone?

15–9 Indirect Measurement; Vectors

Find each trigonometric function.

1. \(\sin 49°\)
2. \(\tan 75°\)
3. \(\cos 31°\)

Find each angle to the nearest degree.

4. \(\sin x = .8250\)
5. \(\tan y = 1.2\)
6. \(\cos A = .2275\)
7. \(\tan B = \frac{5}{12}\)

Make a sketch, and solve.

8. In triangle \(ABC\), \(m \angle C = 90°\), \(m \angle A = 52°\), and \(AB = 100\). Find \(CB\) to the nearest tenth.

9. In triangle \(ABC\), \(m \angle C = 90°\), \(m \angle A = 27°\), and \(AB = 100\). Find \(AC\) to the nearest tenth.

10. In triangle \(\hat{ABC}\), \(m \angle C = 90°\), \(CB = 4\), and \(AC = 5\). Find \(m \angle A\) to the nearest degree.

11. In triangle \(ABC\), angle \(A\) measures 39° and angle \(B\) measures 73°. If the altitude to side \(AB\) measures 10 inches, find

   a. \(AB\) to the nearest inch;
   b. the area of the triangle to the nearest square inch; and
   c. the perimeter of the triangle to the nearest inch.

12. A pilot begins his descent 4300 feet from the runway, at an altitude of 750 feet. Find the angle of descent, to the nearest degree.

13. As shown, the Great Pyramid has an edge of 609 feet making a 52° angle with the base. How high, to the nearest foot, is the pyramid?
14. A twenty-four-foot ladder leaning against a wall is 8 feet from it at the ground. Find the angle that the ladder makes with the ground, to the nearest degree.

15. Find, to the nearest inch, the hypotenuse of a right triangle if an acute angle measures 52° and the side opposite this angle is 20 inches.

16. From the top of a lighthouse 120 feet high the angle of depression of a ship is 8°. How far, to the nearest foot, is the ship from the lighthouse?

17. Tex and Slim pull a steer so that their ropes are at right angles. Tex pulls with 180 pounds of force, and Slim, with 150 pounds. Find the magnitude of the resultant, and the angle it makes with the larger force.

18. When a weight of 50 pounds is hung on a rope, supported at both ends, the tension in one part of the rope is 3 times that in the other, and the parts of the rope make a right angle at the point of suspension. Find the tension in each part of the rope, to the nearest pound.

19. In a west wind, an airplane headed north actually flies at 370 knots in a direction 4° from north. Find the wind's speed, to the nearest knot.

20. A ferry heads south at 12 miles an hour, but the river flowing east at 5 miles an hour changes its course. Find how fast the ferry moves and the direction of motion.

ALGEBRAIC PRINCIPLES

This section contains two tests which should help you discover how well you understand the principles of algebra.

15–10 A True-False Test

Determine whether or not each statement is true. Unless otherwise stated, the set of all real numbers is the domain of each variable.

1. The factor common to \((x - 1)^2\) and \((x^2 - 1)\) is \((x - 1)\).

2. \(\frac{x - y}{m} = \frac{x}{m} - \frac{y}{m}, m \neq 0\)

3. \(\frac{a}{m - n} = -\frac{a}{m}, n \neq m\)

4. \(a^{3m} \div a^{3m} = a^m, a \neq 0, a \neq 1\)

5. \(\frac{2x - y}{2m} = \frac{x - y}{m}, m \neq 0, y \neq 0\)

6. \((a + b)^2 = a^2 + b^2\)

7. \(2^m \cdot 2^n = (4)^{m+n}\)

8. \(\frac{a^3}{a^3} = 0\)

9. \(a^2 - 1\) is a polynomial

10. A quadratic equation always has 2 real roots.

11. The graph of \(3x - y = 6\) passes through the point \((4, -6)\).

12. \(x + y = 2\) and \(2x + 2y = 6\) have a common solution.
13. The graphs of $3x = 2y$ and $2x = 3y$ meet at the origin.
14. When each side of a square is doubled, the area is quadrupled.
15. When each side of an equilateral triangle is doubled, its perimeter is 6 times as long as at first.
16. If the radius of a circle is multiplied by 4, the circumference becomes 8 times as long.
17. As the complement of an angle increases, so does its supplement.
18. If $n$ men do a job in $6h$ hours, $2n$ men can do the job in $3h$ hours.
19. If $0.3m = 0.06$, then $m = 0.002$.
20. If $\frac{2\sqrt{3}}{m} = \frac{m}{3\sqrt{3}}$, the positive value of $m$ is $3\sqrt{2}$.
21. \{rational fractions\} is finite.
22. \{bacteria in one liter of a culture\} is finite at a given instant.
23. If the perimeter of a triangle is 18, and one side is twice another, the only possible integral lengths of the sides are 4, 6, and 8.
24. The difference between the supplement and the complement of any angle is 90 degrees.
25. The sum of the complements of two angles of a triangle equals the third angle.
26. The graph of $x + ay = b$, with $a \neq 0$, has a slope equal to $a$.
27. Several lines with equal slope may be parallel or coincident.
28. Straight lines whose slope equals 2 are represented by $y = 2x + b$.
29. If a train does 45 m.p.h. from $A$ to $B$, but only 30 m.p.h. from $B$ to $A$, its average rate for the round trip is 37.5 m.p.h.
30. In triangle $ABC$, if $\sin A = \cos B$, then angle $C$ measures 90°.
31. If $a \neq 0$, the reciprocal of the reciprocal of $a$ is $a$.
32. A straight-line graph represents direct variation between the variables.
33. The graph of $x^2 - 4x + 3 = y$ has two $x$-intercepts.
34. If an open sentence has no roots, it is not an equation.
35. If $n \geq 5$ and $n \leq 5$, then $n = 5$.
36. \{-1, 0, 1\} is closed under addition.
37. If a peddler has $n$ articles of the 40 he had yesterday, the replacement set for $n$ is \{1, 2, 3, ..., 40\}.
38. An irrational number can be expressed as a decimal numeral if you consider enough places.
39. To show that $a \cdot 3b = 3ab$, you must use the associative, commutative, and distributive properties as reasons.
40. If a point moves so that the sum of its distances from two perpendicular axes is always 5, its path is described by $x + y = 5$. 
15-11 A Completion Test

Supply the expression that completes each statement correctly.

1. If the perimeter of a rectangle is 36 and one side is 6, the other side is \(?\).

2. If the quotient of two numbers is \(-1\), their sum is \(?\).

3. The weight \(w\) of a wire varies directly as its length \(s\) and as the square of its diameter \(d\). If \(k\) is the constant, \(w = \)?.

4. The heat \(H\) you get from an electric heater varies inversely as the square of your distance \(d\) from the heater. The formula expressing this relationship is \(\frac{H_1}{H_2} = \)?.

5. If the middle of three consecutive integers is \(m\), the first number is \(?\) and the third is \(?\).

6. If the abscissa of a point on the graph of \(2x - 3y = 7\) is 2, its ordinate is \(?\).

7. Cement, sand, and gravel in the ratio of 2 to 3 to 5 are used in a concrete. For 25 pounds of mixture, you need \(?\) pounds of cement.

8. If \(k = 5 - 3 \cdot 2\), the value of \(k\) is \(?\).

9. If \(-m = 8\), the absolute value of \(m\) is \(?\).

10. If the area of a square is 28 square inches, each side, to the nearest tenth of an inch, is \(?\) inches.

11. If \(\frac{3 - m}{8} = \frac{(m + 7)}{12}\), then \(m = \)?.

12. If the sides of a rectangle are \(2\sqrt{5} - 3\) and \(2\sqrt{5} + 3\), the area is \(?\).

13. If Tom averages 82\% on three tests, he needs a mark of \(?\) on the fourth test to raise his average to 85\%.

14. If \((2\sqrt{6})(3\sqrt{8}) = x\sqrt{3}\), then \(x = \)?.

15. If a man travels 4 hours at 5 m.p.h. and then travels \(h\) hours at \(r\) m.p.h., his average rate is \(?\) m.p.h.

16. If \(m = \frac{1}{a - 1}\), then as \(a\) increases from 2 on, \(m \)?.

17. If \(m = \frac{4}{4 - a}\), then as \(a\) increases from 1 to 3, \(m \)?.

18. If \(y\) varies inversely as \(x\), and \(y = 8\) when \(x = 6\), then \(y = \)? when \(x = 12\).

19. Supplies lasting 16 persons 60 days last 20 persons \(?\) days.

20. The largest possible diameter, expressed as an integer, of a circular table top which can be carried through a doorway 7 by 4 feet is \(?\).
21. If the line \( y = ax + b \) passes through the origin, \( b \) equals zero.

22. If the line \( ay = 3x - 4 \) passes through the point \((0, 2)\), \( a \) equals \( \frac{3}{4} \).

23. If \( \frac{4x^2 + y^2}{4x^2} = 2 \), the positive value of \( \frac{y}{x} \) is \( \sqrt{2} \).

24. Four squares in line form a rectangle whose diagonal is 200 centimeters. The diagonal of one of the squares is \( \frac{200}{\sqrt{2}} \) centimeters.

25. If \( r \in \{ -3, -2, -1, 0, 1, 2, 3 \} \) and \( 2r + 3 \leq 3r + 1 \), the solution set of the inequality is \( \{ -3, -2, -1, 0 \} \).

26. If \( c - d = 7 \) and \( cd = 5 \), then \( c^2 + d^2 = 84 \).

27. If \( r, s, \) and \( t \) are directly proportional to \( 4, 7, \) and \( 15 \), in that order, and \( 2r + s = 45 \), then \( t = 15 \).

28. If \( (h + k)^2 = 20 \) and \( h \times k = 1 \), the positive value of \( h - k \) is \( \sqrt{19} \).

29. The distance between the points \((-2, 0)\) and \((4, 8)\) is \( \sqrt{80} \) units.

30. The lines \( x + 2y = 5 \) and \( 4x + ky = 5 \) are parallel when \( k = -2 \).

31. In solving \( n^2 - 5n = 2 \) by completing the square, you add \( \frac{25}{4} \) to each member of the equation.

32. To transform \( \frac{x}{3 + x} \) into a fraction whose denominator is \( x^2 - 9 \), you multiply numerator and denominator by \( \frac{1}{2} \).

33. The graph of \( ax + by = c \) is horizontal when \( a = 0 \) and \( b \neq 0 \).

34. Expressed as a rational fraction in lowest terms, \( .636363... \) is \( \frac{2}{3} \).

35. The set of ordered pairs \((x, 3x - 4)\) for all real numbers \( x \) is a special kind of relation, called a \( \text{linear} \). relation.

36. In the formula \( A = \pi r^2 \), if \( r \) is doubled, \( A \) is multiplied by \( 4 \).

37. The formula expressing \( y \) in terms of \( x \) for the corresponding values shown in the chart is \( y = \frac{3x - 4}{2} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

38. If in \( \triangle ABC \), \( m \angle C = 90^\circ \) and \( \sin A = .5 \), then \( AC : BC = \sqrt{2} \).

39. If the set of integers is the replacement set of \( x \), then the solution set of \( 4 < \frac{x - 1}{2} < 6 \) is \( \{ 5, 6 \} \).

40. If the square of a positive number is 3, its cube is \( \sqrt[3]{9} \).
### TABLE 1

#### FORMULAS

<table>
<thead>
<tr>
<th>Shape</th>
<th>Formula(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>$A = \pi r^2$, $C = 2\pi r$</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>$A = bh$</td>
</tr>
<tr>
<td>Right Triangle</td>
<td>$A = \frac{1}{2}bh$, $c^2 = a^2 + b^2$</td>
</tr>
<tr>
<td>Square</td>
<td>$A = s^2$</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>$A = \frac{1}{2}h(b + b')$</td>
</tr>
<tr>
<td>Triangle</td>
<td>$A = \frac{1}{2}bh$</td>
</tr>
<tr>
<td>Sphere</td>
<td>$A = 4\pi r^2$</td>
</tr>
<tr>
<td>Cube</td>
<td>$V = s^3$</td>
</tr>
<tr>
<td>Rectangular Box</td>
<td>$V = lwh$</td>
</tr>
<tr>
<td>Cylinder</td>
<td>$V = \pi r^2h$</td>
</tr>
<tr>
<td>Pyramid</td>
<td>$V = \frac{1}{3}Bh$</td>
</tr>
<tr>
<td>Cone</td>
<td>$V = \frac{1}{3}\pi r^2h$</td>
</tr>
<tr>
<td>Sphere</td>
<td>$V = \frac{4}{3}\pi r^3$</td>
</tr>
</tbody>
</table>

### TABLE 2

#### AMERICAN SYSTEM OF WEIGHTS AND MEASURES

<table>
<thead>
<tr>
<th>Length</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 inches</td>
<td>1 foot</td>
<td>12 inches</td>
</tr>
<tr>
<td>3 feet</td>
<td>1 yard</td>
<td>3 feet</td>
</tr>
<tr>
<td>5 1/2 yards</td>
<td>1 rod</td>
<td>5 1/2 yards</td>
</tr>
<tr>
<td>5280 feet</td>
<td>1 land mile</td>
<td>5280 feet</td>
</tr>
<tr>
<td>6076 feet</td>
<td>1 nautical mile</td>
<td>6076 feet</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Area</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>144 square inches</td>
<td>1 square foot</td>
<td>144 square inches</td>
</tr>
<tr>
<td>9 square feet</td>
<td>1 square yard</td>
<td>9 square feet</td>
</tr>
<tr>
<td>160 square rods</td>
<td>1 acre</td>
<td>160 square rods</td>
</tr>
<tr>
<td>640 acres</td>
<td>1 square mile</td>
<td>640 acres</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1728 cubic inches</td>
<td>1 cubic foot</td>
<td>1728 cubic inches</td>
</tr>
<tr>
<td>27 cubic feet</td>
<td>1 cubic yard</td>
<td>27 cubic feet</td>
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#### WEIGHT

<table>
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<tr>
<th>Ounce</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 ounces</td>
<td>1 pound</td>
<td>16 ounces</td>
</tr>
<tr>
<td>2000 pounds</td>
<td>1 ton</td>
<td>2000 pounds</td>
</tr>
<tr>
<td>2240 pounds</td>
<td>1 long ton</td>
<td>2240 pounds</td>
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<table>
<thead>
<tr>
<th>Dry Measure</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 pints</td>
<td>1 quart</td>
<td>2 pints</td>
</tr>
<tr>
<td>8 quarts</td>
<td>1 peck</td>
<td>8 quarts</td>
</tr>
<tr>
<td>4 pecks</td>
<td>1 bushel</td>
<td>4 pecks</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Liquid Measure</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
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<td>16 fluid ounces</td>
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<td>16 fluid ounces</td>
</tr>
<tr>
<td>2 pints</td>
<td>1 quart</td>
<td>2 pints</td>
</tr>
<tr>
<td>4 quarts</td>
<td>1 gallon</td>
<td>4 quarts</td>
</tr>
<tr>
<td>231 cubic inches</td>
<td>1 gallon</td>
<td>231 cubic inches</td>
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#### CAPACITY

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<tr>
<th>Milliliters (ml)</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
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<td>1000 milliliters</td>
<td>1 liter</td>
<td>1000 milliliters</td>
</tr>
<tr>
<td>1000 liters (l)</td>
<td>1 kiloliter</td>
<td>1000 liters</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Milligrams (mg)</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1 gram</td>
<td>1000 milligrams</td>
</tr>
<tr>
<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
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#### METRIC SYSTEM OF WEIGHTS AND MEASURES

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<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
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<td>10 millimeters (mm)</td>
<td>1 centimeter</td>
<td>10 millimeters</td>
</tr>
<tr>
<td>100 centimeters</td>
<td>1 meter</td>
<td>100 centimeters</td>
</tr>
<tr>
<td>1000 meters</td>
<td>1 kilometer</td>
<td>1000 meters</td>
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<table>
<thead>
<tr>
<th>Capacity</th>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000 milliliters</td>
<td>1 liter</td>
<td>1000 milliliters</td>
</tr>
<tr>
<td>1000 liters (l)</td>
<td>1 kiloliter</td>
<td>1000 liters</td>
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<table>
<thead>
<tr>
<th>Weight</th>
<th>Unit</th>
<th>Equivalent</th>
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<tr>
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<td>1 gram</td>
<td>1000 milligrams</td>
</tr>
<tr>
<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
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<table>
<thead>
<tr>
<th>Grams</th>
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<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
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<table>
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<tr>
<th>Ounce</th>
<th>Unit</th>
<th>Equivalent</th>
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<tbody>
<tr>
<td>1000 milligrams</td>
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<td>1000 milligrams</td>
</tr>
<tr>
<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
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<table>
<thead>
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<th>Pound</th>
<th>Unit</th>
<th>Equivalent</th>
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</thead>
<tbody>
<tr>
<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
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<table>
<thead>
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<th>Kilogram</th>
<th>Unit</th>
<th>Equivalent</th>
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<tbody>
<tr>
<td>1000 grams</td>
<td>1 kilogram</td>
<td>1000 grams</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ton</th>
<th>Unit</th>
<th>Equivalent</th>
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INDEX

Numerals in boldface refer to the pages on which terms are defined or explained.

Abscissa, 338-340, 343
Absolute value, 123
distance on a number line, 494
of zero, 123
Actuaries and mathematics, 491
Acute angle, 497
Addition
  associative property, 73-75, 117, 241, 522
  in clock arithmetic, 152-153
  closure property, 71-73
  commutative property, 73-75
  of complex numbers, 521
  of directed numbers, 124-128
  of fractions, 297-302
  identity element, 77, 117
  on the number line, 116-120
  of polynomials, 197-200
  of radical expressions, 417-419
  of vectors, 510
  Additive identity element, 77, 117
  Additive inverse, 120-121
Age problems, 385-386
Ahmes, 224
Algebra of logic and sets, 105-107
Algebraic expression, 35, 36-39, 49-55
  expansion of, 213-214
Amicable numbers, 278-279
Angle(s), 496-498
  acute, 497
  central, 357
  complementary, 173-177
  corresponding, 498
  of depression, 502
  directed, 173
  of elevation, 502
  initial side, 173
  measure of, 173, 496
  naming, 496
  obtuse, 497
  right, 411
  straight, 496
  supplementary, 173-177
  of triangles, 175, 497
  vertex of, 173, 496
Approximations
  decimals, 402-403
  square roots, 408-411, 471
tangents of angles, 503
Area formulas, 211-213
Associative property
  of addition, 73-75, 117, 241
  in factoring, 241
  of multiplication, 74-75
Assumptions, 69
  geometric, 493-495
  See also Axioms and Properties
Average, arithmetic, 142-144
Average deviation, 142-144
Axioms, 69-73
  See also Properties
Axis, 337-338
Bar, 20-21, 401
Bar graph, 354-357
Base
  with percentage, 290-291
  of a power, 40-42, 203
Bearing, 509
Between, 2
Binary number system, 195
Binomial(s), 197
  factoring differences of squares, 246-248
  multiplication of, 253-255, 419-420
  squaring, 248-251
Boundary line, 351
Braces, 11, 20-22
Brackets, 20-22
Broken-line graph, 354-357
Center of a circle, 357
Checking
  division, 221
  roots of equations, 81-85, 312-313, 467-469
Circle graph, 357-358
Civic planners and mathematics, 424
Clock arithmetic, 152-153
Closure
  addition, 71-73
  complex numbers, 522
  multiplication, 71-73
Codes, 33
Coefficient, 40
  fractional, 306-308
  numerical, 241
Collinear, 493-495
Combining similar terms, 79
Common denominator, 299-302
Common factor(s)
  greatest, 283
  greatest monomial, 241
  identifying, 241-244
  in reducing fractions, 283-285
Commutative property
  addition, 73-75, 117, 241
  complex numbers, 522
INDEX

in factoring, 241
multiplication, 73–75
Comparing numbers
equality, 5–7
inequality, 7–10, 114–116
number scale, 2
order relationship, 114–116
Complementary angles, 173–177
Completeness, property of, 407
Completing squares, 469–473
Complex fraction, 304–305
Complex number, 520–522, 521
Component, vector, 512–514
Conjugate, 419, 521
Conjunction, 105
Consecutive integers, 170–172
even, 170
odd, 170
Consistent equations, 368
Constant, 36
of proportionality, 442
Coordinate(s), 334
Cartesian, 493
first, 334, 435
in planes, 337–340
of points, 2–5
second, 334, 436
See also Graphs and Graphing
Coordinate geometry, 365
Coordinate plane, 338
Correspondence, 2
number pairs and points in a plane, 338, 340
numbers and points on lines, 2–5, 16–17, 114, 116, 120–121, 338, 407
one-to-one, 13
of triangles, 498
Cosine function, 505–506
County agents and mathematics, 271
Cryptography, 33
Cube, volume of, 213
Cubic equation, 264
Decimal forms
fractions, 286
nonterminating, 401
numbers, 381–382
rational numbers, 400–403
repeating, 401
Degree
angles, 173, 496
linear equations, 341
monomials, 197
polynomial equations, 264
polynomials, 197
remainders, 222
Denominator
lowest common, 299–302
rationalizing, 415–417, 420
Density, property of, 398–400
Dependent equations, 368
Derived equation, multiplication by zero in, 84
Descartes, René, 365
Deviation, average, 142–144
Difference, 129
of squares, 246–248
Digit problems, 381–383
Directed line segments, 509
Directed numbers
addition, 116–120, 125–126
associative property, 117
closure, 140
commutative property, 117
division, 138–142
multiplication, 133–138
operating with, 112–144
opposites, 120–123
subtraction, 128–132
Direct variation, 442–447
Discriminant, 474, 478
Disjoint sets, 31
Disjunction, 31, 105
Displacement, 116
Distance
between points, 494
from zero, 2
Distributive property, 76–79
division of polynomials, 219
factoring, 237–248
multiplication of polynomials, 209
Divisibility of integers, 328–330
Division
directed numbers, 138–142
exponents in, 217
fractions, 295–297
polynomials by monomials, 219–221
polynomials by polynomials, 221–223
property of equality, 83–85
property of pairs of factors, 408
radicals, 414–417
square roots, 404–407, 409
transformation by, 83
by zero, 77, 84, 140, 344, 431–432
Domain
of definition, 435–438
relations, 439–440
variables, 36
Electrical engineers and mathematics, 151
Element of a set, 10–11
Empty set, 13–14, 92, 165
End points, 494
Enrichment materials, See Extra for Experts, History of Algebra, Just for Fun, Optional Sections, and Vocational Applications
Equality properties
addition, 80-83
axioms, 69-70
division, 83-85
multiplicative, 83-85, 374
reflexive, 69-70
subtraction, 80-83
symmetric, 69-70
transitive, 70
use in transformation, 80-95

Equation(s), 45
absolute value in, 124, 342-343
checking roots, 81
consistent, 368
cubic, 264
dependent, 368
Diophantine, 393-395
equivalent, 81, 157, 201, 313, 421-422
fractional, 312-314
fractional coefficients in, 306-308
graphing, to solve inequalities, 482-484
inconsistent, 368
independent, 368
indeterminate, 393
linear, 264-267, 340-343, 367-379
of lines, 341, 349-350
member of, 45
multiplication by zero, 84
polynomial, 264-267, 489
radical, 421-422
review, 530-531
simultaneous, 368-379
slope-intercept form, 346-348
solving, 86, 264-267, 367-372, 374-377
squaring both members, 421-422, 431-432
transformation, 81, 82, 80-95, 157-159
with variable in both members, 91-94
See also Open Sentences, Properties, Solution Sets, and System of Equations
Equivalent equations, 81, 84, 157, 201, 313, 369, 421-422
Eratosthenes, sieve of, 364
Estimating square roots, 408-411, 471
Euclid, 279
Euler, 279
Evaluating algebraic expressions, 35-39
Evaluation exercises, 37-39, 42-44, 123, 124, 138, 141-142, 202, 214
Evolution, 403
Exponent(s), 40
in division, 217; negative, 232-233;
powers, 204-206; zero as, 218-219
See also Powers
Expression(s)
algebraic, 36
evaluation of, 37-39
expansion of, 213-214
mixed, 302-304
numerical, 5
open, 36
radical, 414-423
simplifying, 21-24
variable, 36

Extractions of a root, 403

Factor(s), 40
common, 241-244, 283-285
integral, 238
prime, 238-240
See also Common factor
Factor theorem, 489-490
Factoring
combining types, 261-263
difference of squares, 246-248
distributive property, 237-248
grouping, 241-243
over a given set, 237-241
polynomials, 239-253
product of a binomial sum and difference, 257-259
product of a binomial sum or difference, 255-257
quadratic trinomials, 259-260
review, 528-529
solving equations by, 263-267
trinomial squares, 251-253

Fallacies, algebraic, 431-432
False statement, 7-10, 92

Formula(s), 436
areas, 43, 213, 288, 416, 452
distance, 178
interest, 308
law of lever, 448
miscellaneous, 38-39, 43-44, 269, 416, 423, 441
percentage, 290
Pythagorean, 412
quadratic, 473-475
volumes, 43, 44, 213
work, 314
writing, 436, 444, 450, 454

Fractions
addition, 297-302
complex, 304-305
defining, 281-282
division, 295-297
in lowest terms, 283
multiplication, 292-294, 296-297
multiplication property, 283
open sentences, 306-308
INDEX

547

ratios, 281–291
reducing, 283–285
zero in denominator, 281

Fulcrum, 448

Function(s), 438–442
cosine, 505–506
review, 531–532
sine, 504–506
tangent, 501–504
value of, 439

Geometry
angles, 172–177
assumptions, 493–495
coordinate, 365
formulas, 43, 44, 213, 288, 441, 452
introduction, 493–501
lines, 493
planes, 493
points, 493
square roots interpreted by, 411–414

Graph(s)
bar, 354–357
broken-line, 354–357
circle, 357–358
direct variation, 442
inequalities, 350–353
inequalities, 350–353, (Trans-Vision) 356, 379–381, 482–484
inverse variation, 448
linear equations, 341, 340–343, 367–370
numbers, 2
open sentences, 45
ordered pairs, 337–339
pictographs, 356–357
quadratic equations, 353–354, 477
relations, 435, 439
sets, 16–17
solution sets, 45
statistical, 354–357

Graphing
numbers on lines, 1–5
ordered pairs in planes, 337–340
sets, 16–17
solution sets of inequalities, 307, 480
systems of linear equations, 367–370
Grouping, 10–21, 74

Half-plane(s), 351, 496
intersection, 380–381

Human Equation, The, 25, 62, 109, 155, 185, 224, 331, 365, 388, 433, 463, 519
Home economists and mathematics, 320

Hyperbola, 448

Hypotenuse, 411–414

i, 520
Identity, 92
Identity element
additive, 77, 117, 521
multiplicative, 77, 138, 521
Inclusion, symbols of, 20–21, 41, 201
Inconsistent equations, 368
Indeterminate equations, 393
Indirect measurement, 498–514
Inequalities, 45–49
equivalent, 161
graphs, 350–353, 379–381, 480
limits, 461–462
members, 45
properties, 159–163
quadratic, 479–484
solving, 46–49, 158–166, 379–381, 479–484

Leonardo of Pisa, 433

Line
equation, 349–350
length, 494
number, 1
slope, 343–346

Linear equation(s), 264–267, 367–379
graph, 340–343
and straight lines, 340–350
slope intercept form, 346–348

See also Equalities, Equations, Properties and Systems of Equations
Linear programming, 461-462
Lowest common denominator, 299-302

Machinists and mathematics, 96
Means of proportions, 443
Measure of an angle, 173, 496
Member
equations, 45
inequalities, 45
sets, 10-11

Merchandisers and mathematics, 235
Methods, see Rules
Mixture problems, 182-184, 310-311
Monomial, 197
as common factor, 241-244
division by, 215-218, 219-221
multiplication of, 203-204, 207-208
powers, 204-206

Motion problems, 316-319, 383-385
Multiple, 35-36, 328-330
Multiple roots, 265
Multiplication
in addition and subtraction method, 374-377
- associative property, 74-75, 522
of binomials, 253-255, 419-420
closure property, 71-73, 522
commutative property, 73-75
of directed numbers, 133-138
fractions, 292-294
identity element, 77, 138
inverse of, 138
-1 by -1, 134
polynomials, 209-210
properties of directed numbers, 134
property of equality, 83-85, 374
property of fractions, 283
property of inequality, 161, 431-432
property of -1, 134
property of 1, 77
property of zero, 77
of radicals, 414-417
rule of exponents in, 203
of square roots, 404-407, 409
of sum and difference of two numbers, 245-246
transformation by, 83
zero in, 84, 313
Multiplicative identity element, 77, 138
Multiplicative inverse, 138

Navigators and mathematics, 523
nth root, 403
Negative number(s), 111-144
exponents, 232-233
multiplication, 133-138
opposites, 120-123
square roots of, 404, 466
Noncollinear, 494
Nonterminating decimals, 401
Null set, 13-15, 92, 165
Number(s)
absolute value, 123-124
axioms, 69-79
binary system, 195
comparing, 114-116
complex, 520-522
decimal, 381-383
of different real roots of quadratic equations, 477-479
directed or signed, 112-144
factoring, 237-240
grouping, 10-19
irrational, 403-407
magnitude, 112
names, 2, 5
negative, 112-144, 232-233, 404, 466
operations, 19-24
order property, 159-160
ordered pairs, 333-340, 435-442
positive, 112
prime, 238, 364
property of divisors of pairs, 408
property of nonzero products of two real, 480
property of square roots of equal, 465-467
rational, 397-403
relationships among, 1-10
review of properties, 525-526
roots, 403-407
rounding off, 402-403
Number line, 1-2
extending, 111-116
Number pairs, 333-340, 435-442
Number scale, 1-2
Number systems, (Trans-Vision) 180, 194, 394, 407
Numerical, 2, 5
Numeration systems, 194-195
Numerical coefficient, 40, 241, 306-307
Numerical trigonometry, 501-509
One
as exponent, 40
multiplicative property of, 77
One-to-one correspondence, 13
on a line, 2-5, 16-17, 114, 116, 120-122, 338, 407
in a plane, 338-339
Open expression, 36
See also Open Sentence, Equations and Inequalities
Open sentence(s), 44-49, 367-387
directed numbers, 157-166
fractional coefficients, 306-308
graphs, 45
roots, 334-337
solution set, 334-337
in two variables, 333-337
INDEX

See also Equation, Inequalities, Linear Equations, Quadratic Equations, Solution Sets, and Systems of Equations

Operation(s)
- closure, 71
- directed numbers, 116–144
- inverse, 81
- order, 23–24
- rational, 397
- review of fundamental, 528–529
See also Addition, Division, Multiplication, and Properties

Opposite
- of directed numbers, 120–123
- of the sum of two numbers, 121

Opposite rays, 495


Order
- directed numbers, 114
- inequalities, 161
- of magnitude, 2
- property of numbers, 159, 163
- rational numbers, 398–400
See also Commutative Property

Order relations, 1–5

Ordered pairs of numbers, 333–340, 435–442

Ordinate; 338–340, 343

Origin, 337

Parabola, 353–354, 476

Parallel lines, 367

Parentheses, 20

Per cent, 289

Percentage, 289, 290–292

Perfect number, 278

Perfect squares, 407

Periodic decimal, 401

Petroleum chemists, 460

Pi (π), 169, 397

Plane
- half-, 496
- points in, 333–340

Plane coordinate system, 338

Polynomial equation(s)
- degree of, 264
- solved by factoring, 264–267, 489
- standard form of, 264

Polynomials, 197
- addition, 198–200
- degree, 197
- division, 219–223
- factoring, 239–243
- multiplication, 206–210
- powers, 213–214
- prime, 256
- quadratic trinomials, 248–261
- squaring binomials, 248–251
- subtraction, 200–203

Postulate, 69

Power(s), 40
- of polynomials, 213–215
- of products, 203–206
- raising to a, 403
See also Exponents

Principle, substitution, 71, 367, 375
See also Properties

Problem(s)
- age, 385–386
- analysis of, 166–184
- angles, 172–177
- area, 211–213
- arithmetic averages, 143–144
- consecutive integers, 170–172
- digits, 381–383
- investment, 308–310
- mixture, 182–184, 310–311
- per cent and percentage, 291
- Pythagorean theorem, 413–414
- review, 532–535
- similar triangles, 500–501
- solving, 57, 166–169, 269
- trigonometric functions, 506, 507–508
- vector, 511–512, 514
- work, 314–316

Product(s)
- of means and extremes, 443
- nonzero, property of, 480
- of powers, 203–206
- of primes, 238
- special, 245–251
- of the sum and difference of two numbers, 245–246
See also Multiplication

Proof, 80, 85, 127, 136

- addition property of equality, 80
- factors whose product is zero, 263
- multiplicative property of —1, 134

Properties
- addition, of equality, 80–83
- addition, for the set of directed numbers, 125
- additive, of inequality, 160
- additive, of zero, 77
- associative, 73–75, 117
- closure, 71–73
- commutative, 73–75, 117
- completeness, 407
- complex numbers, 522
- density, 398–399
- directed numbers, 125, 134
- distributive, 75–79, 237–248
- division, of equality, 83–85
- irrational numbers, 407–411
INDEX

of multiplication for set of directed numbers, 134
multiplcative, of equality, 83–85, 374–376
multiplcative, of fractions, 283
multiplcative, of inequality, 161
multiplcative, of −1, 134
multiplcative, of 1, 77
multiplcative, of zero, 77
nonzero product of two real numbers, 480
number, review, 525–526
number scale, 1–2
opposites, 121
order, of directed numbers, 114
order, of numbers, 159–160
pairs of divisors, 408
product of square roots, 404–407, 409
quotients, 216
quotients of square roots, 404–407, 409
reflexive, 69–70
square roots of equal numbers, 465–467
substitution, 71, 374
subtraction, of equality, 80–83
sum and product of the roots of a quadratic equation, 467–469
symmetric, of equality, 69–70
transitive, of equality, 70
transitive, of inequality, 160
Proportion, 443–446
Proportionality, constant of, 442
Psychometrists and mathematics, 192
Pythagorean theorem, 411–414
Quadrant, 338–339
Quadratic equation(s), 264
general, 473
graph, 353–354, 477
solving, 465–479
sum and product of roots, 467–469
Quotient(s), 138
powers, 215–218
property of, 216
property of square roots of, 404–407, 409
See also Division
Radical(s), 403
addition and subtraction of, 417–419
in equations, 421–422
expressions involving, 414–422
multiplication and division, 414–420
simplification, 414–417
Radicand, 403
Range
of relations, 439–440
of values, 435–438
Ratio(s), 286, 281–291
Rationalizing a denominator, 415–416, 420
Ray, 172–173, 495–498
Real number system, 407
(Trans-Vision) 420
Reciprocal, 138, 295
Region, 107
Relation, 435, 438–440
Replacement set, 36, 44–49, 436
Representation of numbers on a line, 1–5
See also Graphing
Restrictions on replacement set of variable, 51–54, 281–282, 284
Resultant, 510
Reviews
general, 525–536
Right angle, 411, 496
Right triangle, 411–414
Rise, 343
Root(s), 45–49
checking, 81–85, 312–313, 467–469
double or multiple, 265, 478
extracting, 403
negative, 404
of numbers, 403–407
of open sentences in two variables, 334–337
principal, 404–407
of quadratic equations, 476–479
square, 408–413, 465–467, 471
See also Solution sets
Root index, 403
Roster, 11, 435
Rounding off decimals, 402–403
Rule(s)
absolute value of indicated product, 135
addition of fractions, 298
addition of polynomials, 198
addition of signed numbers, 117, 125–126
arithmetic average (mean), 142
comparing measures of quantities, 286
division of fractions, 295
division of polynomials by monomials, 220
division of signed numbers, 139–140
exponents, 203, 205, 217
factoring, 251, 261
multiplication by −1, 134
multiplication of fractions, 292
multiplication of polynomials, 209
INDEX

multiplication of polynomials by monomials, 207
multiplication of binomials, 253
multiplication of signed numbers, 135
multiplication of sum and difference of two numbers, 245
involving per cent, 289, 290
for products, 135
involving reciprocals, 139
of relations, 435
rounding a decimal, 402
solving equations, 86, 313, 366, 465, 467
subtraction of polynomials, 201
subtraction of signed numbers, 129–130
for a set, 11
See also Properties

Scientific notation, 276–278
Segment, line, 494
Sense of an inequality, 161
Sentence
algebraic, 45
compound, 105–107
See also Open Sentence
Set(s), 10–19
algebra of, 105–107
arithmetic of, 30–31, 60–61
disjoint, 31
factoring over a given, 237–240
finite, 13
graphing, 16–17
infinite, 13
intersection, 30–31
null, 13–15, 92, 165
of polynomials, 256
of rational numbers, 397–403
replacement, 36, 44–49
solution, 45–49, 467–469
specifying, 10–12, 16
subset, 18–19
union, 60–61
universal, 31
Venn diagrams, 31, 61, 106–107
Signed numbers, 112–144
See also Directed Numbers
Similar terms, 77–79
Similar triangles, 498–501
Similarity, 498
Sine function, 504–506
Slope, 343–346
Slope-intercept form, 346–348
Solution sets, 45–49
checking, 81–85, 312–313, 467–469
graphing, 45
of open sentences in two variables, 334–337
Solving equations, 44–49
by factoring, 264–267
steps in, 86
systems, 367–379
Solving open sentences, 45–49, 334–337
Square(s)
completing trinomial, 469–473
factoring, 246–248, 251–253
magic, 153–154; perfect, 407
Square root(s)
approximation, 408–411, 471
of both members of an equation, 421
geometric interpretation, 411–413
negative numbers, 466
principal, 404–407
product property, 404–407, 409
property of, of equal numbers, 465–467
quotient property, 404–407, 409
Squaring
binomials, 248–251
both members of equations, 421–422, 431–432
Structure, review, 525–526
See also Properties
Subscripts, 443
Subset(s)
improper, 18
proper, 18
relation to sets, 18–19
of sets, 30–31
Substitution
method, 378–379
principle, 71, 378
Subtraction
of directed numbers, 128–132
of polynomials, 200–203
property of equality, 80–83
of radicals, 417–419
solving systems of equations, 370–377
Supplementary angles, 173–177
Surveyors and mathematics, 32
Symbols, See table following Index
Systems of equations
addition and subtraction method, 370–377
graphing, 367–370
linear, 367–381
simultaneous, 368
substitution, 378–379
System
of numeration, 194–195
of positive integers, 397
of rational numbers, 397–403
Tangent function, 501–504
Term, 37
constant, 254
linear, 254
lowest, 283
quadratic, 254
similar or like, 77-79
unlike, 77
Tests
algebraic principles, 536-539
Theorem(s), 411, 493
factor, 489-490
Pythagorean, 411—413
about triangles, 497-499
Transforming equations, 80-95, 157-159
Transitive property of equality, 70
Trans-Vision inserts, 180, 356, 420
Triangle(s), 497-514
measurement, 501-503
naming, 497
right, 411-414
similar, 498-501
sum of angles, 175
Trigonometry, numerical, 501-509
Trinomial(s), 197
completing a, square, 469-473
factoring a, square, 251-253
quadratic, 248-260
Truth set, 105-107
Truth table, 106
Uniform-motion problems, 178-182, 316-319, 383-385
Union of sets, 60-61
relation to disjunction, 105
Universal set, 31
Universal, 31, 60-61
Value
of discriminant, 474, 477-478
of expression, 37
of a function, 439
of a variable, 36
See also Absolute Value
Variable(s), 35-60, 36
Variable expression, 36
Variation
combined, 453-456
direct, 442-447
inverse, 447-452
joint, 452-456
review, 531-532
Vector(s), 509-514
component, 512
equivalent, 509
magnitude, 509
resolving, 512-514
review, 535-536
Venn diagrams, 31, 61, 106-107
Vocational applications, 32, 96, 110, 151, 156, 192, 235, 236, 271, 320, 332, 366, 396, 424, 435, 460, 465, 491, 523
x-axis, 337-338, 477
y-axis, 337-338
y-intercept, 347
Zero
absolute value, 123
additive property, 77
denominator of a fraction, 281
in division, 77, 84, 140, 344, 431-432
exponent, 218-219
multiplication by, 313, 84
multiplicative property of, 77
product of factors, 263-264

LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Page</th>
</tr>
</thead>
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<tr>
<td>$</td>
<td>a</td>
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<tr>
<td>$\rightarrow$</td>
<td>because or meaning</td>
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<td>$\land$</td>
<td>conjunction</td>
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<td>$\ldots$</td>
<td>continue unendingly</td>
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<td>$\lor$</td>
<td>or and so on through</td>
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<td>$\subseteq$</td>
<td>is an element of</td>
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<tr>
<td>$\not\subseteq$</td>
<td>is not an element of</td>
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<td>$\neq$</td>
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<td>$\geq$</td>
<td>is greater than or equal to</td>
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<td>$i$</td>
<td>$\sqrt{-1}$</td>
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<td>$\leq$</td>
<td>is less than or equal to</td>
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<td>$\overline{AB}$</td>
<td>line segment $AB$</td>
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<td>$m\angle$</td>
<td>measure of an angle</td>
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<td>$-2$</td>
<td>negative 2</td>
<td>112</td>
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<td>$n$th root</td>
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<td>$+2$</td>
<td>positive 2</td>
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<td>$a'$</td>
<td>$a$-prime</td>
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<td>set</td>
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<td>$\sim$</td>
<td>similar</td>
<td>498</td>
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<td>$\cup$</td>
<td>union</td>
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<tr>
<td>$\overline{AB}$</td>
<td>vector $AB$</td>
<td>509</td>
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</tbody>
</table>
Chapter 1. Symbols and Sets

Pages 3–5  Written Exercises  A 1. R: $\frac{1}{2}$; S: $\frac{1}{3}$; T: $\frac{1}{4}$  2. R: $\frac{1}{2}$; S: $\frac{1}{2}$; T: $\frac{3}{4}$  3. R: $\frac{5}{6}$; S: $\frac{1}{3}$; T: $\frac{1}{2}$  4. R: $\frac{1}{3}$; S: $\frac{3}{5}$; T: $\frac{1}{4}$  5. R: $\frac{5}{6}$; S: $\frac{1}{3}$; T: $\frac{1}{2}$  6. R: $\frac{1}{2}$; S: $\frac{1}{2}$; T: $\frac{3}{4}$  7. $\frac{1}{2}$  8. None


Pages 9–10  Written Exercises  A 1. =  2. <  3. =  4. <  5. =  6.  7.  8.  9. >  10. =  11. >  12. =  13. <  14. >  15. <  16. >  17. >  18. >  19. <  20. =  21. <  22. >  23. <  24. =  25. 1  26. Any num. other than 7  27. 5  28. 0  29. 0  30. 2  31. 7  32. 0  33. Any num. < 5  34. Any num. 35. 5  36. Any num. > 4  37. Any num. other than 12  38. Any num. $\geq 8$ and $< 16$  39. Any num. > 0  40. 8  41. 41  42. 12  43. Any num. other than 0  44. $\frac{1}{4}$  45. Any num. > 1 and $< 3$  46. Any num. > $\frac{1}{4}$ and $< \frac{3}{4}$  47. Any num. > 0 and $< 1$  48. Any num. > $\frac{1}{4}$ and $< \frac{3}{4}$  49. Any num. $\geq \frac{1}{8}$ and $< \frac{3}{8}$  50. Any num. $> \frac{3}{8}$ and $< \frac{5}{8}$  51. Any num. > 1.53327 and < 1.54327  52. Any num. $> \frac{16}{8} + \frac{6}{2}$  53. Any num. > $\sqrt{6}$ and $< \sqrt{7}$  54. Any num.  55. Pages 14–15  Written Exercises  A 1. {8, 9}, finite  2. {0, 1, 2, 3, 4, 5}, finite  3. {3, 9}, finite  4. {0, 3, 5, 6}, finite  5. {1, 2, 3, 5, 9, 7}, finite  6. {1, 3, 5, 7, 9}, finite  7. {10, 12, 14, 16, 18}, finite  8. {Feb.}, finite  9. {1, 2, 3, . . . , 1775}, finite  10. {202, 204, 206, . . . , 998}, finite  11. {3, 6, 9, 12, 15}, finite  12. {25, 50, 75, . . . }, inf.  13. {1964, 1968, 1972, . . . }, inf.  14. {40}, finite  15. {\frac{1}{3}}, finite  16. {1, 3, 5, . . . }, inf.  17. {2, 4, 6, . . . }, inf.  18. 0, finite  19. {1, 3, 5}, finite  20. {0, 1, 4, 9, 16}, finite  21. 0, finite  22. 0, finite  23. f  24. d  25. h  26. j  27. a  28. g  29. b  30. i  31. c  32. k  33. e  34. l

Page 19  Written Exercises  A 1. {3}, {15}, {10}  2. {3, 15}, {3, 10}, {10, 15}  3. {3, 15, 10}  4. 0  5. 0; {15}; {10, 15}  6. 0  7. {10}  8. {3}, {15}, {3, 15}  9. {the nos. of arith.}  10. {the nos. of arith.}  11. {all fractions}  12. {all men}  13. {all cities of Alaska}  14. {the islands of Hawaii}  15. Pages 22  Written Exercises  A 1. 0  2. 25  3. 510  4. 510  5. 2  6. 8  7. 18  8. 2  9. 360  10. 360  11. 4  12. 1  13. B  14. 3 \frac{1}{8}  15. 16  2  17. 51  18. \frac{4}{3}  19. 56  20. 11 \frac{5}{8}  21. 1000  22. 725,500  23. 3 \frac{5}{8}  24. 14 \frac{3}{8}  25. Pages 24  Written Exercises  A 1. 5  2. 3  3. 23  4. 51  5. 9  6. 9  7. 6  8. 15  9. 1  10. 5  11. 9  12. 5  13. 23  14. 23  15. 7  16. 7  17. 1  18. 1  19. 1  20. 21  23. 1  24. 1  25. 13  26. 1

Pages 27–28  Chapter Test  1. \frac{1}{4}  2. \frac{5}{6}  3. \frac{3}{4}  4. 5  5. \frac{5}{6}  6. \frac{7}{8}  7. T  8. F  9. T  10. F  11. Any num. other than 2

A 12. Any num. > 20  13. Any num. > 2 and < 4  14. Any num. $> \frac{27}{20}$ and $< \frac{37}{20}$  15. {6, 12, 18, 24}  16. {100, 101, 102, . . . , 999}  17. Inf.  18. Finite  19. 0  20–22. no. lines  23. {4, 8}  24. 1  25. 13  26. 1

Pages 28–30  Chapter Review  1. number  2. zero  3. magnitude  4. distance  5. \frac{3}{4}  6. \frac{3}{8}  7. E  8. numerical, numeral  9. equal  10. \neq  11. \neq  12. greater  13. 5 < 7 or 7 > 5  14. 1  15. 1  16. set  17. member or element  18. \in  19. not  20. roster  21. rule  22. 12, 15, 18  23. infinite  24. empty  25. finite  26. empty or null  27. 3  28. no. line  29. subset  30. {20}  31. brackets, braces, bar  32. one  33. 14  34. \frac{3}{4}  35. inclusion, divisions, multiplications, additions, subtractions  36. 0  37. 27  38. 19  39. 44  40. 1

566
Chapter 2. Variables and Open Sentences

Pages 37–38 Written Exercises A 1. 2; 12 2. 2; 11 3. 1; 12 4. 1; 1296 5. 1; 1296 6. 1; 0 7. 2; 60 8. 5; 2 9. 2; 9 B 10. 2; 15 11. 1; 0 12. 1; 225 13. 1; 1 14. 1; 23 15. 1; 9 16. 1; 1 17. 1; 11 18. 1; 2 19. 1; 1


Pages 42–43 Written Exercises A 1. b 2 2. c 3 3. a 4. a 2 5. 7n 6. 14n 3 7. n 3 8. 8n 3 9. E 10. F 3 11. R 3 12. s 3 13. 5y 3 14. 8z 15. 5/2 16. 1/2 y 3 17. (2P) 2 18. (8w) 3 19. (xy) 3 20. (ab) 3 21. (a - 1) 3 22. (1 - a) 3 23. (r + 2) 3 24. (t + 7) 2 B 25. 1/2 26. 1/4 27. 36 28. 200 29. 9 30. 4 31. 35 22. 33. 34. 35. 36. 9 C 37. 38. 39. 36. 40. 12 41. 11 42. 18 43. 30 44. 1 45. 0 46. 1/2 47. 1/2 48. 3439 49. 139 50. 0 51. 37

Pages 43–44 Problems A 1. 225 sq cm 2. 3375 cu cm 3. 22,680,000,000,000,000,000 g(m/sec) 2 4. 2.25 amp 5. 4281.25 ft B 6. 57,876.48 cu in 7. 1,065,312,000,000K 8. 125,000 g(cm/sec) 2 9. 58,875 cu in 10. 3/4 candle power/sq ft 11. 600 ft lb/sec C 12. 3.32 ft/sq mil 13. .006003 (gm/cm) 2 14. 285,600 amp 2 (ohm/sec) 15. 2.0125

Pages 47–49 Written Exercises A 1. {4} 2. {4, 6, 8} 3. {0} 4. {2, 6, 8} 5. {2, 4, 6} 6. {2} 7. {8, 9} 8. {9, 10} 9. a. x 6 10. y 7 11. b. x 6 12. y 0 10. a. ø 10. [1] 11. [1] 12. [1] 13. [1] 14. [3] 15. 0 16. [0] 17–22. no. lines 23. ø 24. ø 25–28. no. lines 29. ø 30. 5 31–34. no. lines 35. 7 36. 37–40. no. lines B 41. True for all values given 42. False for all values given 43. False for all values given 44. True only for d = 4, e = 0; d = 4, e = 1; d = 7, e = 0; d = 7, e = 1 45. True for all values given 46. True only for x = 0, y = 3, z = 5; x = 0, y = 3, z = 3 47. True only for t = 1, v = 1 48. False for all values given 49. True only for n = 3, n = 1; m = 9, n = 1 50. True only for y = 1 C 51. True only for t = 4, s = 4; t = 8, s = 4 52. True only for h = 4, d = 6; h = 4, d = 8; h = 8, d = 8 53. a. F b. F 54. a. T b. T 55. a. T b. F 56. a. F b. T

Page 54 Written Exercises A 1. x + 5 2. d + c 3. 2x + 4 4. 5y + 2x 5. 15 - n 6. 3x - 2 7. z - 1 8. x - d 9. 2x - 5 10. 5k - 5 11. (3x)y 12. 6a 13. r + 6 3 14. 3(r - 6) or r - 6 3 15. 1/2(x + y) or x + y 2 16. a - b 3 17. 5(2 + y) 18. a(a - b) 19. x(x + 2) + 5 20. 2u - v 2 21. 5 4x + 3k 22. (a + b) - ab 23. 1/(2(a + b)) 24. mn 3(m - n) 25. 3(r - s) 2(r + s)

Pages 54–55 Problems A 1. x, 2x 2. t, 3f 3. 90w 4. 5t 3. 5w 6. 3n 7. 5m, 3m 8. 10s 9. 400 - q 10. h + 10,000 11. (a) w; (b) 2w + 6 12. 3n - 17 13. (a) s; (b) 2x - 48; (c) (2x - 48) + s + s 14. 57 - w 15. s + 3/8s = 108 16. (3c + 7c) + 5 17. 5c + 2(c + 28) 18. (a) 2x; (b) 2x + 3


Page 64 Chapter Test 1. B 2. 5 3. 7 4. 7 5. 36 6. 5 7. (1, a) 8. (1, 2, 3, 4, 6, 12, m, n) 9. (b, p, q) 10. 8q 11. 3 12. Num. coeff: 3; base: (x + 1); exp: 1 13. h - h - h or h 3 14. y 15. 9, 12 16. F 17. F 18. {1} 19. 6, 7, . . . ., 10 20–21. no. lines 24. 523 < 12e 25. 20 ft

Pages 65–67 Chapter Review 1. symbol 2. replacement, domain 3. 99 4. product, quotient 5. coefficient 6. n 7. s, s 8. exponent 9. c 10. 3, a 2 11. 1 12. 3375; 45 13. n 14. domain or replacement set 15. left 16. variable 17. inequality 18. True only for y = 4 19. True for all values given 20. True only for y = 0; y = 1; y = 2; y = 4 21. False for all values given 22. solution 23. root 24. root 25. (b) 26. x > 2 27. Three times a no. x; x ≥ 0 28. Three greater than the quotient of a no. x and 3; x ≥ 0 29. The product of a no. x diminished by 1 and 3; x ≥ 1 30. a - 5 31. 2b + 3 32. 3c - 3 33. d + e 3 34. 3f 35. 100 - 5n, n ∈
Chapter 3. Axioms, Equations, and Problem Solving

Pages 72-73 Written Exercises A 1. Closed for +, −, ×; not closed for ÷: 0 ÷ 0 undefined. 2. Closed for ÷, ×; not closed for +, −: 1 ÷ 1 and 1 − 1 not in set. 3. Not closed for +, −, ×: 2 + 2, 2 − 2, 2 ÷ 2 not in set. 4. Not closed for +, −, ×: 3 + 3, 3 − 3, 3 ÷ 3, 3 × 3 not in set. 5. Not closed for +, −, ×: 2 + 3, 1 − 3, 3 ÷ 2, 2 ÷ 3 not in set. 6. Closed for ×; not closed for +, −: 1 + 1, 0 − 1 not in set, 1 ÷ 0 undefined. 7. Not closed for +, −, ×: 1 + 2, 1 − 2, 1 ÷ 2, 2 × 2 not in set. 8. Closed for +, ×; not closed for −: 15 ÷ 20, 25 ÷ 10 not in set. 9. Not closed for +, −, ×: 1 ÷ 1, ½ ÷ ½, ½ ÷ ½ not in set. 10. Closed for ×; not closed for +, −: 1 ÷ 3, 3 ÷ 3 not in set. 11. Closed for +, ×; not closed for −: 2 ÷ 2 not in set. 12. Not closed for +, −, ×: 1 ÷ ½, ½ ÷ ½, ½ ÷ ½ not in set.

Page 79 Written Exercises A 1. 56x 2. 85y 3. 65a 4. 23b 5. 30a 6. 7x 7. 12x + 6 8. 15n + 3 9. 3x + 4r 10. 4x + 2y 11. 2a + 3b + 4c − d 12. 6a − 5y − 3w − 4 13. 10m + 21 14. 26x + 10 15. 7b 16. 4x + 30 17. 16a + 4b + 1 18. s + 5t + 2 19. 6a + 11b 20. 12x + 6xy 21. 10a + 10b 22. 23m + 16n 23. 36 + 12a 24. 5a + 4 25. 10d + 17 26. 62e + 6 27. 48w + 4v 28. 26r + 40r 29. 5a + 6b + 1 30. 4r + 19s + 1 31. 9a + 28 32. 38b + 7 33. 16r + 30s + 6 34. 20m + 18n + 55 35. 1850 36. 1900 37. 35. 38. 7790 39. 0 40. 41. 75 42. 570 43. 326,326 44. 141 45. 290 46. 0


Pages 89-91 Problems A 1. 35 2. 27 3. 12 4. 63 5. 36 ft 6. 20 ft 7. 89. 8. Pat, 92; Mary, 65 9. 82 10. 26 11. 5.3 12. 11.2 13. 38 14. 36 15. 23 16. 83 17. Emma, 11 yr; Bob, 22 yr; Kent, 27 yr 18. Width, 37 ft; length, 74 ft. 19. Width, 24 ft; length, 72 ft. 20. Width, 9 ft; length, 27 ft 21. 112 g 22. 70c, 35c 23. Skirt, $6; jacket, $9 24. (a) 2 lb; (b) 1.6 lb 25. (a) 165; (b) 330 26. 480 lb; 240 lb; 120 lb 27. A, 2000; B, 4000; C, 16,000 28. 37.5 kg 29. 260 in; 30 in; 59 in 30. A, 1845; B, 1230; C, 1945 31. 10 nickels; 40 dimes; 20 pennies 32. 20


Pages 94-95 Problems A 1. 1 2. 5 3. 6 4. 3 ½ lb 5. $120; $40; $120 6. 14; 32 7. 18 yr; 9 yr 8. 30; 43 9. 16 10. Standard, $.15; deluxe, $.15 11. 13 ½ ft; 11 ½ ft 12. Width, 17 in; length, 25 in 13. 45 4¢ stamps; 5 7¢ stamps 14. 14 in 15. 44 16. 135 17. 6 18. Juneau, 30°; Omaha, 60° 19. S2 20. Onions, 4¢; potatoes, 8¢; apples, 12¢ 21. 9 yr; 8 yr; 24 yr
Pages 98–100 Chapter Test 1. Reflex. 2. Symmetric. 3. No, $2 \times 2$ not in set. 4. Yes. 5. (a) Comm. for add.; (b) Assoc. for add.; (c) Comm. for mult.; (d) Assoc. for mult.; (e) Distrib.; (f) Mult. prop. of 1; (g) Comm. for add.; (h) Distrib.; (i) Distrib. 8. $2 \times 9.390$ 10. $\{164\}$ 11. $\{24\}$ 12. $\{4\}$ 13. $\{32\}$ 14. Yes 15. $\$34.71$ 16. $\{1\}$ 17. Pencil, $\$0.99$; pen, $\$1.79$ 18. $\{3\}$ 19. 12 in


Pages 102–104 Cum. Review 1. Inf. 2. Finite 3. Finite 4. Inf. 5. (Kansas, Kentucky} 6. (nos. between 0 and 48 which are each 1 less than a multiple of 6} 7. (the no. 1 and the whole nos. between 1 and 82 which are powers of 3} 8-10. no. lines 11. 2 12. 13.8 13. (0, 1, 2, 3, 9, 10, 12, 20} 14. (3, 9, 12} 15. (2} 16. Transitive or subst. princ. 17. Distrib. 18. Symmetric 19. Comm. for mult. 20. (a) Distrib. (b) Comm. for add.; (c) Assoc. for add.; (d) Distrib.; (e) Subst. princ.; (f) Mult. prop. for 0; (g) Additive identity 21. An $+ 21$ 22. $d — 4$ 23-24. no. lines 25. Closed 26. Not closed; 3 $\div 4$ not in set 27. 30 28. $31$ 29. 1 30. 2 31. $2n — 3$ 32. $x$ 33. $s + 5$ 34. $30 — x$ 35. $(R + r)^2$ 36. $2(R^2 + r^2)$ 37. (3} 38. $\emptyset$ 39. (1, 2, 3} 40. (2, 3} 41. (0, 1} 42. (0, 1, 2, 3} 43. a. $t + 5$; b. $t$ (0, 1, 2, 3,...) 44. 27 $-$ 3 45. (16} 46. (1} 47. (0} 48. (1} 49. (4} 50. $\emptyset$ 51. 51 52. 11, 9 53. $\$1.49$

Chapter 4. The Negative Numbers

Page 128 Problems 1. $34.30 2. $8.75 3. $-19.32 4. $264 5. 275 ft 6. —1 yd 7. \{-5, -4, -3, -1, 0, 1, 2, 3, 4\}; not closed under add. 8. —5 lb 9. —27.5°F


Page 132 Problems A 1.32 ft 2. —8° 3. —13° 4. —274 ft 5. 27 yr 6. 75 yr 7. \{-5, -4, -3, -1, 0, 1, 2, 3, 4\}; not closed under add. 8. —5 lb 9. —27.5°F


Pages 143-144 Problems 1. $2023\frac{1}{3} 2. 8007\frac{1}{3} M 3. —5860 ft 4. 17\frac{1}{2} g 5. —1\frac{1}{6} 6. 6\frac{7}{6} 7. —.187 8. 14.9 9. —9b 10. —32 11. —.36 12. —49c 13. —1.3 14. —.14 15. .015 16. —2.3 17. —.19 18. .022 B 19. —10 20. —10 21. —4 22. 3 23. 75 24. —4 25. \( \frac{2}{6} \) 26. —e + b 27. —.8 28. —41 29. 30. 0 31. C 32. 30. 33. —3.2 34. \( \frac{3}{8} \) 35. —89 36. 2

Pages 145-146 Chapter Test 1. a. —3; b. 0 2. 3 3. no. lines 4. 2.97 ≤ d ≤ 3.03; d ∈ \{directed nos. between 2.97 and 3.03 inclusive\} 5. —38 6. —45 ft 7. a. \{-3\}; b. \{—3, —1\} 8. 2, —2 9. —62 10. —35 11. 10 12. —13.7° 13. 826 14. —4a 15. 45 16. 5ab^4 — 40b^5 — 3a^2b + 9a^2b^2 17. a. \{-12\}; b. \{\frac{3}{8}\} 18. a. —8; b. \frac{9}{19} 19. —\frac{9}{16} x^3y^5 20. 1\frac{1}{37}^a

Pages 146-150 Chapter Review 1. right 2. minus 3. positive, negative 4. —4.5 5. directed, signed 6. C 7. D 8. Q 9. M 10. greater 11. right 12. negative 13. less 14. —2 15. 2 16. 0 17. —3 18. \{1, 4\} 19. —(2, —4, 5) 20. \{-2, 0\} 21. right 22. left 23. —7 24. —16 25. 0 26. [9] 27. \{0\} 28. negative, same 29. 0 30. 0 31. 0, additive inverse 32. opposites 33. \{-1\} 34. \{0, 1, 4\} 35. \{-4, —2, —1, 0, 1\} 36. distance 37. magnitude 38. \{-3\}; 3 39. —3.2 40. 12 41. —\frac{3}{2} 42. .085 43. \{4, —4\} 44. 0 45. \{-5, —4, —2, 0, 1, 4\} 46. \{-5, —4, —2, 4\} 47. positive 48. negative 49. a. difference; b. negative; c. negative 50. —30 51. 0 52. 212 53. —114 54. n + 33 55. 38 — t 56. opposite or inverse 57. 5 58. —21 59. —38 60. —16 61. —39 62. r + 109 63. —s + 15 64. —7° 65. 48 yr 66. positive 67. negative 68. opposite 69. —266 70. —266 71. —13 72. \frac{b}{4} + 8 73. —3 + \frac{k}{2} 74. 10a — 31b 75. 31m — 47mn 76. 0 77. 22,360 78.,90 79. 80. 80. positive 81. negative 82. divisor 83. 0 84. \frac{8}{3} 85. 1 86. —(9) 87. \{\frac{15}{8}\} 88. \{-\frac{3}{8}\} 89. 1 90. 0, multiplicative inverse 91. reciprocal 92. 1 93. —\frac{1}{3} 94. \frac{1}{11} 95. \frac{2}{3} 96. \frac{a — b}{2} 97. No, 2 + \frac{1}{2} not in set 98. Yes, except for 0 99. deviation; directed 100. 3
Chapter 6. Working with Polynomials

Pages 199–200 Written Exercises A 1. $-3n - 4h$ 2. $-7r - 2.1s$ 3. $\frac{3}{2}x^3 - \frac{3}{2}y^3$ 4. $\frac{1}{2}m^2 + \frac{3}{2}n^2$ 5. $\frac{3}{2}m - \frac{2}{3}n = \frac{1}{2}$ 6. $\frac{1}{2}x - \frac{3}{2}y + \frac{1}{2}z$ 7. $x - 2y$ 8. 0 9. $-3yz + 10$ 10. $-13ab + 4$ 11. $10x - 10$ 12. $13z + 1$ 13. $7xy + 8.8$ 14. $9xz + 13.3$ 15. 1.3 $16. 9n - 3r + 2$ 17. $12h^2 - 4.3dh + .4d^2$ 18. $-2xy - .13y^2$ 19. $8x^3 - 60x^2 + 150x - 125$ 20. $2y^3 - 7y^2 + 5y - 1$ 21. Correct 22. Incorrect; $5y - 73$ 23. Incorrect; $-4$ 24. Incorrect; $11x - 9$ 25–29. Answers are correct as given in text. B 30. $12x^3 + 4x^2 - 3$ 31. $5y^2 - 10y + 2$ 32. $t^2 + 2t - t^2 + t + 3t + 2$ 33. $-s^4 - s^3 + 10s + 15$

Pages 202–203 Written Exercises A 1. $-15^{6}y^3$ 2. $-48x^3y^6$ 3. $27a^3$ 4. $60a^2b^2$ 5. $-30xyz$ 6. $216n^3$ 7. $72x^2y^2$ 8. $-14 abc$ 9. $\frac{2}{9}g^2r^2t$ 10. $-48s^3t^3$ 11. $\frac{7}{2}g^2x^2$ 12. $-r^3s$ 13. $-a^2b^3$ 14. $r^4t^3$ 15. $12x^4y^6$ 16. $10a^6b^6$ 17. $0.064r^5t^15$ 18. $7.29z^4y^9$ 19. $e^{0}d^4f^6$ 20. $w^5x^2y^6$ 21. $-a^4b^4$ 22. $-h^4k^7$ 23. $-72d^3m^4$ 24. $-1.28b^4c^4$ 25. $162a^2b^2$ 26. $-34p^3q^2$ 27. 0 28. 0 29. $91c^{10}d^6$ 30. $-8m\sqrt{n^7}p^5$ 31. $5x^4 + 4x^3$ 32. $r^2 - 2r^3$ 33. $10m^5 + n^9 + 10m^5 + r^4$ 34. $a^m + n + 2am^b + b + m^8$

Pages 206 Written Exercises A 1. $-8^{6}m$ 2. $-27^{6}s$ 3. $-48x^3y^2$ 4. $-150m^2n^2$ 5. $27^{6}b$ 6. $216r^4$ 7. $64m^{10}$ 8. $375x^3y^4$ 9. $20a^2b^2$ 10. $-63b^2c^2$ 11. $2x^2y^3$ 12. $-0.9a^6b^3$ 13. $0.0146a^{10}b^6$ 14. $k^{8}b^6$ 15. $81s^{8}t^9$ 16. $-11x^3y^2$ 17. 0 18. $a^3 - 32a^8$ 19. $-4b^3$ 20. $1.1t^2r^5$ 21. $74m^6k^7$ 22. $x^5y^2 + 7x^3y^4$ 23. $e^2d + 3e^6d^4$ 24. $2.8k^3m^3 - 3.2k^2m$ 25. $7a^2b^3 - 7a^2b^5$

Pages 207–208 Written Exercises A 1. $-5x^2 + 15x - 35$ 2. $-6a^2 - 12a - 6$ 3. $a^3 - 2a^2b + ab^2$ 4. $x^3 - 2x^2y + 2x^2$ 5. $-ab^2 + b^3 + b^2c$ 6. $-r^3 - r^2s + r^2t$ 7. $-5x^4 + x^3 - 2x^2$ 8. $-7y^2 + 3y^3 + y^5$ 9. $-4x^4 + 6x^2 + 8y^6$ 10. $-3x^5 + 18x^4 - 27x^3$ 11. $15x^2y - 6x^3y^5 + 9x^3y^4 + 3x^3y^6$ 12. $15x^2 + 20r^6s^3 - 5r^4s^2 - 5s^2r^3$ 13. $-2a^2b^3 + 2a^2b^4 - 4a^2b^6 + 2a^2b^7$ 14. $-3c^6d^3 - 23c^6d^7 + 3c^6d^8 + 6c^6d^9$ 15. $3$ 16. $15x^2 - 3y^2$ 17. 0 18. $a^3 - 32a^8$ 19. $-4b^3$ 20. $1.1t^2r^5$ 21. $74m^6k^7$ 22. $x^5y^2 + 7x^3y^4$ 23. $e^2d + 3e^6d^4$ 24. $2.8k^3m^3 - 3.2k^2m$ 25. $7a^2b^3 - 7a^2b^5$

Pages 208–209 Problems A 1. $d = (490 + 4x) mi$ 2. 200 mph 3. 3.5 hr 4. 7 hr 5. 30 ft, 50 ft 6. 120 ft 7. $8.450, 8.350 8. 6.1 mi/min; 10.7 mi/min 9. 550, 660 10. 5 hr, 7 hr

Page 210 Written Exercises A 1. $x^2 + 9x + 14$ 2. $a^2 + 15a + 56$ 3. $y^2 - 4y - 45$ 4. $n^2 - 5n - 36$ 5. $x^2 - 13x + 42$ 6. $y^2 - 11y + 24$ 7. $10x^2 + 17x + 3$ 8. $56n^2 + 95n + 36$ 9. $35x^2 - x - 6$ 10. $60w^2 + w - 10$ 11. $12y^2 - 34y + 24$ 12. $42x^2 - 83x + 40$ 13. $a^2 + 2ab + b^2$ 14. $x^2 - 2xy + y^2$ 15. $x^2 - y^2$ 16. $c^2 - d^2$ 17. $m^2 - 10m + 25$ 18. $x^2 + 18x + 81$ 19. $m^2 + 6mn + 9n^2$ 20. $m^2 - 8mn + 16n^2$ 21. $r^2 - 9rs + 22. 9a^2 - b^2$ 23. $a^2 + 9.7ab - 1.5b^2$ 24. $2a^2 + 5.5ab - 1.5b^2$ B 25. $m^3 - 3.3m^2 + 1.3m - 12$ 26. $2m^3 - 1.5n^2 - 1.82n + 3$ 27. $a^3 - b^3$ 28. $a^3 + b^3$ 29. $a^4 - 7a^2 - 8$ 30. $y^4 + 35y^2 - 200$
Page 260
Written Exercises
A 1. (2y + 1)(y + 3)
B 2. (3x + 1)(x + 2)
C 3. (3n - 1)(n - 1)

Pages 262-263
Written Exercises
A 1. 3(x + 3)(x - 3)
B 2. 14(y + 2)(y - 2)
C 3. 2(w + 9)²
D 4. 8(z + 7)²
E 5. (6x - 5)(x - 1)
F 6. (5x - 3)(r - 2)
G 7. (7n - 1)²
H 8. (5x - 2)²
I 9. -5a(x² + y²)
J 10. -r(t² + r²)

Pages 262-266
Written Exercises
A 1. (2} 2. (8} 3. {-5} 4. {-1} 5. {-2} 6. {-3} 7. {-12}
B 8. {directed nos.} 9. {directed nos.} 10. {±} 11. {any no. other than 0} 12. {any no. other than 0} 13. {3, 5}

Pages 266-267
Written Exercises
A 1. {-10, 9} 2. {-7, 2} 3. {5, -3} 4. {-5} 5. {-1} 6. {-3} 7. {-12}
8. {directed nos.} 9. {directed nos.} 10. {±} 11. {any no. other than 0} 12. {any no. other than 0} 13. {3, 5}

Page 273
Chapter Test
1. 72
2. 7yz²
3. 16a(5a - b)
4. (x - 2)(x + 1)
5. (y² + 3)(y - 5)
6. 81x⁶
7. r²s² - t²
8. 2(2m + 5)(2m - 5)
9. (x + 12)(x - 12)
10. 9a² + 6ab + b²
11. 25k² - 20km + 4m²
12. (5x + 9y)²
13. x = 4 or 4
14. (n + 14)(n + 3)
15. (r - 18)(r + 5s)
16. (m + 9)(m - 4)
17. (k - 9)(k + 2)
18. (9n + 8)(4n + 7)
19. (3a + 4b)(a - 9b)
20. 10(3t - 5)(t + 2)
21. {2, -5}

Page 576
ANSWERS
Chapter 8. Working with Fractions

Page 282 Written Exercises A 1. \( \frac{5}{y} ; y \neq 0 \) 2. \( \frac{-7}{z} ; z \neq 0 \) 3. \( \sqrt{a^2} \); no exclusions 4. \( \frac{y}{y} \); no exclusions 5. \( \frac{x}{z} \); no exclusions 6. \( \frac{3y}{1} \); no exclusions 7. \( \frac{b - 1}{b} ; b \neq 0 \) 8. \( \frac{a}{a - 1} ; a \neq 1 \) 9. \( \frac{k}{6k - 12} ; k \neq 2 \) 10. \( \frac{f}{3f + 21} ; f \neq -7 \) 11. \( \frac{12x^2 + 4}{3} \); no exclusions 12. \( \frac{3y^2 - 2}{1} \); no exclusions 13. \( \frac{g - 2}{14g + 7} \); \( g \neq -\frac{1}{4} \) 14. \( \frac{h - 5}{33h - 11} ; h \neq \frac{1}{3} \) 15. \( \frac{1}{x(x - 1)} ; x \neq 0 \) and \( x \neq 1 \) 16. \( \frac{1}{y(y + 4)} ; y \neq 0 \) and \( y \neq -4 \) 17. \( \{6, 2\} \) 18. \( \{-7, -2\} \) 19. \( \{\frac{1}{4}, -3\} \) 20. \( \{-\frac{4}{9}, 1\} \) 21. \( \{-2, 2\} \) 22. \( \{-3, -3\} \) 23. \( \emptyset \) 24. \( \emptyset \) 25. \( \{-4, 4\} \) 26. \( a \neq 0 \) and \( a \neq b \) 27. \( c \neq -d \) 28. \( r \neq s \) B C

Pages 284–285 Written Exercises A 1. \( \frac{1}{h} ; h \neq 0, k \neq 0 \) 2. \( \frac{t}{6} ; t \neq 0 \) 3. \( \frac{3}{4mn} ; m \neq 0, n \neq 0 \) 4. \( \frac{5x^2}{9x^3y^2} ; x \neq 0, y \neq 0 \) 5. \( \frac{a}{a} \neq -b \) 6. \( \frac{c - d}{c + d} ; c \neq -d \) 7. \( \frac{4}{x - y} ; |x| \neq |y| \) 8. \( \frac{r - 5}{3} ; r \neq -5 \) 9. \( \frac{2a + 3b}{2ab} ; a \neq 0, b \neq 0 \) 10. \( \frac{5r + 3b}{5rs} ; r \neq 0, s \neq 0 \) 11. \( x - 4; x \neq 4 \) 12. \( -1 - a; a \neq 1 \) 13. \( \frac{m}{m - 2} ; m \neq 2 \) 14. \( \frac{x + 4}{x - 4} ; x \neq 4 \) 15. \( \frac{3}{a + 1} ; a \neq -1 \) 16. \( \frac{5}{x + 2} ; x \neq -2 \) 17. \( s(r + 1); r \neq 1 \) 18. \( 2(d - 1); d \neq -1 \) 19. \( \frac{x^2 + 1}{x + 1} ; x \neq -1 \) 20. \( \frac{x - 4}{x^2 - 4} ; |x| \neq 2 \) 21. \( \frac{3a - 3}{a + 1} ; a \neq -3 \) and \( a \neq 1 \) 22. \( \frac{4a + 1}{4 - a} ; |a| \neq 2 \) 23. \( \frac{c - 3}{-c - 6} ; |c| \neq 6 \) 24. \( \frac{3d + 3}{d - 8} ; d \neq 8 \) and \( d \neq 3 \) 25. \( c + 2d \) \( \frac{1}{c - 2d} ; |c| \neq 2d \) 26. \( \frac{3a + b}{3a - b} ; |b| \neq 3a \) 27. \( \frac{r}{r - 1} ; r \neq 1 \) and \( r \neq 3 \) 28. \( \frac{t}{t - 1} ; t \neq 1 \) and \( t \neq 7 \) 29. \( \frac{a - 4}{a + 1} ; a \neq -1 \) 30. \( \frac{b + 6}{b - 1} ; b \neq 1 \) 31. \( \frac{z - 5}{z + 4} ; z \neq 3 \) and \( z \neq -4 \) 32. \( \frac{m - 4}{m - 3} ; m \neq 3 \) and \( m \neq -2 \) 33. \( \frac{x - 2}{x - 2} ; |x| \neq 2 \) 34. \( \frac{3n + 1}{3n - 1} ; n \neq \frac{1}{3} \) and \( n \neq -\frac{1}{3} \) 40. \( \frac{3n + 1}{3n - 1} ; n \neq \frac{1}{3} \) and \( n \neq -\frac{1}{3} \) 41. \( \frac{2n + 3}{2n - 3} ; n \neq \frac{1}{3} \) and \( n \neq -\frac{1}{3} \) 42. \( \frac{m^2 + 2}{2(m + 2)} \); \( m \neq 0 \) and \( m \neq -2 \) and \( m \neq -1 \) 43. \( \frac{5(t - 2)}{4(1 - 2t)} ; t \neq 0 \) and \( |t| \neq \frac{1}{3} \) 44. \( \frac{(x + 5)^2}{2(x^2 + 25)} ; x \neq 0 \) 45. \( \frac{(p + 2)^2}{5(p^2 + 4)} ; p \neq 0 \) 46. \( \frac{2x^2(z + 5x)}{z - 3x} ; z \neq -4x \) and \( z \neq 3x \) 47. \( \frac{2n(m - 4n)}{3m(m - 3n)} ; m \neq 0 \) and \( m \neq 3n \) and \( m \neq -2n \)
Pages 287–288 Written Exercises A 1. $\frac{3}{8}$ 2. $\frac{1}{2}$ 3. $\frac{3}{8}$ 4. $\frac{3}{4}$ 5. $\frac{2}{3}$ 6. $\frac{3}{8}$ 7. $\frac{3}{2}$ 8. $\frac{3}{10}$ 9. $\frac{3}{11}$ 10. $\frac{9}{10}$ 11. $\frac{1}{3}$ 12. $\frac{1}{12}$ 13. $\frac{1}{2}$ 14. $\frac{1}{3}$ 15. $\frac{1}{4}$ 16. $\frac{1}{2}$ 17. $\frac{1}{18}$ 18. $\frac{1}{9}$ 19. $\frac{3}{8}$ 20. $\frac{1}{8}$ B 21. $\frac{1}{22}$ 22. $\frac{1}{5}$ 23. $\frac{1}{4}$ 24. $\frac{1}{12}$ 25. $\frac{1}{5}$ 26. $-\frac{2}{1}$

Pages 288–289 Problems A 1. $\frac{93}{2}$ 2. $-\frac{1}{5}$ 3. 10, 25 4. 6 lb 5. 64 in. 6. 2 7. copper, 4$\frac{1}{2}$ lb; silver, 40$\frac{1}{2}$ lb 8. Ross, $\$11,000$; Morgan, $\$13,200$ 9. $\$1.05$ for 21 lb 10. second worker 11. 12 cm; 18 cm 12. 9 cm; 12 cm B 13. 959,850 14. 33,880 mi 15. (For prob. concerning sulfuric acid) 10 lb hydrogen, 160 lb sulfur, 320 lb oxygen 16. Sand, 50 lb; silt, 225 lb; clay, 225 lb. 17. $\$560$ 18. $\$3.00$

Pages 290–291 Written Exercises A 1. 17.5 2. 0.05 3. 4 4. 900 5. 4.5 6. 72 7. 24 8. 57 9. 3 10. 36 11. 1.27 12. 3 13. 80 14. 68 15. 42 16. 6.5 17. 195 18. 42 19. 216 20. 1700 21. 75% 22. 60% 23. 400% 24. 250% 25. $\frac{5}{2}$ or $\frac{1}{2}\frac{1}{2}$ 26. 1.25% or 1$\frac{1}{4}$% 27. 1200% 28. 2000%

Pages 291 Problems A 1. 1470 2. 1.25 lb 3. 0.4 qt 4. 151.2 lb 5. $\$580$ 6. $\$950$ 7. $\$53$ 8. $\$12.50$ 9. $\$700 10. $\$60 11. $\frac{1}{5}$% 12. 24.5% or 24$\frac{1}{2}$% B 13. 20% 14. 15% 15. $\$16.50$ 16. 5 lb 17. $\$600$ 18. $\$3.05$

Pages 293–294 Written Exercises A 1. $\frac{1}{3}\frac{1}{6}$ 2. $\frac{9}{4}$ or $\frac{3}{1}$ 3. $\frac{1}{4}$ 4. $\frac{4}{5}$ 5. $\frac{2}{3}$ 6. $\frac{5}{12}$ 7. $\frac{7}{16}$ 9. $\frac{1}{a}$ 10. $\frac{1}{r}$ 11. $\frac{3}{5}$

Pages 295–296 Written Exercises A 1. $\frac{3}{12}$ 2. $\frac{1}{14}$ 3. $\frac{1}{14}$ 4. $\frac{3}{14}$ 5. $\frac{4}{14}$ 6. $\frac{5}{14}$ 7. $\frac{6}{14}$ 8. $\frac{7}{14}$ 9. $\frac{8}{14}$ 10. $\frac{3}{14}$

Pages 296–297 Written Exercises A 1. $\frac{y}{10}$ 2. $\frac{b}{8}$ 3. $\frac{c}{3}$ 4. $\frac{2r}{s}$ 5. $\frac{5}{x}$ 6. $\frac{3}{a + 2}$ 7. $\frac{4(m + n)}{n(3m - n) - n(5m + n)}$

Pages 298–299 Written Exercises A 1. $\frac{1}{3}\frac{1}{2}$ or $\frac{1}{15}$ 2. $\frac{3}{10}$ 3. $\frac{3}{4}$ 4. $\frac{2}{3}$ 5. $\frac{5}{a + 12}$ 6. $\frac{7}{3}$

Pages 301–302 Written Exercises A 1. $\frac{8b}{2a}$ 2. $\frac{x + 6a}{3x}$ 3. $\frac{3}{2c}$ 4. $\frac{5a + 12}{5a}$ 5. $\frac{-a + 17}{10}$ 6. $\frac{9n - 20}{12}$ 7. $\frac{8h - 3}{12}$ 8. $\frac{-1a}{18}$ 9. $\frac{3x^2 - 2x + 1}{x^3}$ 10. $\frac{3a^2 + 2a}{a^2}$ 11. $\frac{c + a^2 - 2a}{abc}$ 12. $\frac{4r - 2s + st}{rst}$ 13. $\frac{ax + bx - ex}{abc}$ 14. $\frac{ax - ay + az}{xyz}$ 15. $\frac{3x + 2y - xy}{x^2y^2}$ 16. $\frac{4s - 2t + rt}{r^2s^2}$ 17. $\frac{2a - b}{a^2 - b^2}$ 18. $\frac{c + 2d}{c^2 - d^2}$ 19. $\frac{2}{1}$
ANSWERS

16. \( \frac{2h^2 + 10h + 13}{(h + 2)(h + 3)} \)
17. \( \frac{4x^2 + 2x}{(x + 1)(x - 1)} \)
18. \( \frac{2w^2 + 2wv}{2w + v} \)
19. \( x + 2 + \frac{5}{4x} \)
20. a. \( \frac{1}{ty} \) b. \( \frac{2}{x + a} \) c. \( \frac{k}{5k + n} \)

21. \( x \geq -8 \)
22. \$700
23. 5 lb
24. \( 5 \)
25. 40 min
26. 10 mph

Pages 323–326 Chapter Review
1. zero
2. \( y = \pm 3 \)
3. \( x = 2 \)
4. \( x = 0 \)
5. \( a = b \)
6. \( \frac{2x}{3y} \)
7. \( \frac{1}{a + b} \)
8. \( \frac{4x + 3y}{4xy} \)
9. \( a - 2 \)
10. \(-1 \)
11. \( 1 - x \)
12. \(-2 - y; \) \(-\frac{2}{3} \)
13. same unit
14. 3:1
15. 1:6
16. 8:9

17.
18.
19.
20. a. \(- \)
b. \(- \)
c. \(- \)
21. \( x > -8 \)
22. \$700
23. 5 lb
24. \( \{5\} \)
25. 40 min
26. 10 mph

Pages 326–328 Cum. Review
1–8
1. multiplication
2. \( -3 < y < 3 \)
3. \( \{ -4 \} \)
4. \( a < b \) 5. sometimes
6. \(-2a + k \)
7. \( 10x^2 \)
8. \( \{3, -4\} \)
9. \( a < 0 \)
10. \( t = 0 \)
11. \( t \neq 3 \)
12. \( t < 0 \)
13. \( t \neq -1 \)
14. F
15. F
16. T

17. F
18. F
19. \( (y + 2) - (y - 2) \)
20. \( \frac{27(x - 3)}{z + 3} \)
21. 3:7
22. \( x^2 + 2x + 4 \)
23. \( \frac{49x^2y}{121a^2b} \)
24. 1, -1
25. \( (x - 3) \)

26. \{ -4 \}
27. \( \{0, 4\} \)
28. \( \{ all directed nos. except 0 and -1 \} \)
29. \( 7k(k - 3)(k + 3) \)
30. \( (12 - h)(5 + h) \)
31. \( (4c - 5d)(3r - 2s) \)
32. \( zr(2z - 5x)^2 \)
33. \( (7n - 7k - 5)(7n - 7k + 5) \)
34. \( \frac{1}{6}((3x + 1)(2x - 7)) \)
35. \( \frac{1}{6}(2y^2 + y - 1) \)
36. \( \frac{5x^2 - 66x + 189}{(x - 3)^2(x - 1)} \)
37. \( \frac{2a - 3b}{a - b} \)
38. \( \frac{40r^4}{9q^2} \)
39. \( v \)
40. \$240
41. 7\( \frac{1}{2} \) cm
42. 8 lb; 12 lb
43. \( \frac{9}{4} \)
44. 300 mph
45. 2 hr

Chapter 9. Graphs

Pages 336–337 Written Exercises
A
1. \( a = -1, b = 4 \)
2. \( a = -3, b = -1 \)
3. \( a = \frac{3}{4}, b = 3 \) or \( b = -3 \)
4. \( a = -3, b = 0 \) or \( b = -1 \)
5. \( a = -2 \) or \( a = 1, b = -6 \) or \( b = 6 \)
6. \( a = 1 \) or \( a = -3, b = 0 \)
7. \( \{(-6, -3), (0, 3), (6, 9)\} \)
8. \( \{(-6, 10), (0, 4), (6, -2)\} \)
9. \( \{(-6, 38), (0, 8), (6, -22)\} \)
10. \( \{(-6, -11), (0, 7), (6, 25)\} \)
11. \( \{(-6, -24), (0, -10), (6, 4)\} \)
12. \( \{(-6, 15), (0, 6), (6, -3)\} \)
13. \( \{(2, 2)\} \)
14. \( \{(-1, -1)\} \)
15. \( \{(5, 0)\} \)
16. \( \{(3, 1)\} \)
B
17. \( \{(4, -3)\} \)
18. \( \{(2, -3)\} \)
19. \( \{(0, 1), (0, 1), (1, 1)\} \)
20. \( \{(-1, -3), (0, -3), (0, 0)\} \)
21. \( \{(-1, 0), (-1, 1), (-1, 2), (3, 1), (3, 2), (4, 2)\} \)
22. \( \{(-2, 0), (3, 1), (3, 0)\} \)
C
23. \( \{(-3, 1), (0, 4)\} \)
24. \( \{(0, -3), (4, -2)\} \)
25. \( \{(1, 2)\} \)
26. \( \{(-1, 7), (3, 3)\} \)

Page 342 Written Exercises
B
13. \( a = (0, 0); \) \( b = (0, -10) \)
14. \( a = (7, 0) \)
15. \( a = (5, 0); \) \( b = (0, -12) \)
16. \( a = (-18, 0); \) \( b = (0, 4) \)
17. \( a = (0, 0); \) \( b = (0, 0) \)
18. \( a = (0, 0); \) \( b = (0, 0) \)

Pages 345–346 Written Exercises
B
13. \( a = -1 \)
14. \( a = -1 \)
15. \( a = 7 \)
16. \( a = -5 \)
17. \( a = 0 \)
18. \( a = 3 \)

Page 348 Written Exercises
1. \( y = 3x + 1 \)
2. \( y = -2x + 5 \)
3. \( 4y = -x - 8 \)
4. \( 3y = x - 9 \)
5. \( 7y = -4x \)
6. \( y = 7 \)
7. \( y = -10 \)
8. \( y = -x + 2 \)
9. \( y = x - 1 \)
10. \( y = 7x \)
ANSWERS

Pages 390–393  Chapter Review  1. same axes  2. ordered pair; also, solution set  3. root  4. (6, 2)  5. (—6, —4)  6. (3, 5)  7. ((5, 1))  8. {{(—30, —100)}}  9. {{(4, 3)}}  10. equals  11. y + 5 = 2x; x + y = 100 12. l = 5w; 2l + 2w = 72 13. 5x ft; 1½ ft  14. (5, 7)  15. (4, y)  16. (2, 3)  17. (8, —3)  18. (—2, —22)  19. (—(4, 7))  20. half-plane  21. (0, 10)  22. (0, —5)  23. position (place)  24. 3, 1  25. 97  26. 84  27. greater
28. 205  29. 16 mph  30. 20 mph  31. 16 yr, 20 yr  32. 13 yr, 18 yr  33. 42 yr, 16 yr  34. 4  35. §

Chapter 11.  The Real Numbers

Pages 399–400  Written Exercises A
24. (a) 27. a. T; b. F; c. F; d. T

Page 403  Written Exercises A
1. .08 2. .06 3. .21875 4. 1.125 5. —.90 6. —1.6 7. —.285714 8. —.2125

Pages 405–407  Written Exercises A
1. 22 2. 21 3. 36 4. 48 5. —34 6. —28 7. ±Jg. 8. ±aV

Pages 410–411  Written Exercises A
1. 1.9 2. 2.1 3. 3.2 4. 5.5 5. 42 6. 73 7. —17.8 8. —20.2 9. 156

Pages 413  Written Exercises A
1. Yes 2. Yes 3. Yes 4. Yes 5. 25.00 in 6. 40.00 M 7. 6.25 mi 8. 4.14 yd

Pages 416–417  Written Exercises A
1. ±.9, ±.3 2. ±5.2, ±3.9 3. 13.8 in, 9.2 in 4. 11.8 in 5. 894.4 sq cm

Pages 416–417  Problems B
1. ±.9, ±.3 2. ±5.2, ±3.9 3. 13.8 in, 9.2 in 4. 11.8 in 5. 894.4 sq cm
Chapter 14. Geometry and Trigonometry*

See page 587 for answers to Chapter 14 for the 1962 edition of this book.


Page 500* Written Exercises A 1. △ABC is a right triangle. 2. △JKL is a right triangle 3. △DEF is not a right triangle 4. △GHK is not a right triangle 5. △PQR is a right triangle 6. △XYZ is not a right triangle 7. AC = BC = √2, or 1.414, approx. 8. 90° 9. 80° 10. 90° 11. Their sum does not equal 180° 12. Yes

Page 506* Written Exercises A 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. MK 12. (1, -1)

Pages 510-511* Written Exercises A 1. √2 2. √2 3. √3 2. 4. -1/2 5. 1 6. 0 7. -1 8. 0 9. (b) 10. 11. 12. 13. 14. 1 15. 0 16. 17. 8. 18. 19. 1/2 20. 1/2 21. 22. 22. 23. 23. 1/2

Page 512* Written Exercises A (In Ex. 1-8, the answers are given in the order: sine, cosine, tangent.)

1. 0.0872, 0.9962, 0.0349, 0.9994, 0.0349 3. 0.4540, 0.8910, 0.5095 4. 0.5878, 0.8090, 0.7265 5. 0.7431, 0.6691, 1.1106 6. 0.8572, 0.5150, 0.16643 7. 0.0175, 0.9998, 0.0175 8. 0.8387, 0.5446, 1.5399 9. 33° 10. 57° 11. 29° 12. 78° 13. 30° 14. 41°

Page 513* Problems A 1. 30 ft 2. 42 ft 3. 10 ft 4. 29 ft 5. 37 ft 6. 19 ft 7. (a) 2403 ft (b) 689 ft 8. 23 ft

Page 516* Problems A 1. 386.2 ft 2. 363.2 ft 3. 100.7 ft 4. 86.6 ft 5. 321.7 ft 6. 4813.5 ft 7. 255.8 ft 8. 3456.2 ft

Page 517* Problems A 1. 35° 2. 31° 3. 7° 4. (a) 165.6 mi (b) 150.1 mi 5. 28°, 62°, 90° 6. 868.1 ft 7. 4.8 ft 8. 129.8 ft

Pages 518-519* Written Exercises A 1. 95° 2. 9 ft 3. 15 ft 4. 56.5 meter 5. 8.5 meters 6. 9 in 12 in 7. 8 8. 9. 10. DE/AD and BG/AG

Page 520* Problems A 1. 33.75 ft 2. 30 ft 3. 17.5 ft 4. 30.9 ft


Page 524-525* Problems A 1. 141 lb at 45° 2. Approx. 13 mph at 67° 3. Approx. 151 mph at a bearing of 262° or at an angle of 188° 4. Approx. 211 mph at a bearing of 276° 5. Approx. 534 lb at an angle of 68° (or 112°) with the ground. 6. Approx. 4326 lb at 324°

Page 526* Problems A 1. 36 lb 2. 17 lb 3. 12 lb 4. 130 lb 5. 1128 lb 6. 635 lb 7. Force of the wind, 4285 lb; pull on the rope, 5230 lb B 8. 177 lb

Page 528* Chapter Test 1. S 2. Ray 3. None of these 4. 71° 5. d 6. 2 7. 2.4 8. 12 ft 9. 85 ft 10. 162 lb

11. Horizontal, 38 lb; vertical, 140 lb


*The asterisks identify page references to the 1965 edition of the text.
Chapter 14. Geometry and Trigonometry


Pages 497–498 Written Exercises 17. 80° 18. 60° 19. 90° 20. 90° 21. 55° 22. 66° 23. 20° 24. 40°

Page 500 Written Exercises 1. 95° 2. 85° 3. 9 ft 4. 15 ft 5. .26 M 6. 8.5 M 7. 9 in, 12 in 8. 15 M, 22.5 M

Page 502 Problems 1. 30 ft 2. 24.5 ft 3. 30 ft 4. 17.5 ft 5. 30.6 ft 6. 34 ft

Pages 503–504 Problems 1. 30 ft 2. 42 ft 3. 101 ft 4. 87 ft 5. 322 ft 6. 4813 ft 7. 35° 8. 31° 9. 75 ft 10. 320 ft

Pages 505–506 Written Exercises 1–6. .8 7–14. 8 15. 8 16. 8 17. 8 18. 8

Page 506 Problems 1. 386.3 ft 2. 363.2 ft 3. 10.4 ft 4. 29.0 ft 5. 37.4 in 6. 25.9 ft 7. 255.8 ft


Pages 507–508 Problems A 1. (a) 169.1 mi; (b) 61.6 mi 2. (a) 10.6 ft; (b) 33.9 ft 3. (a) 2403.3 ft; (b) 689.0 ft 4. 128.6 ft 5. 119.2 ft 6. 1177.6 ft 7. 7° 8. (a) 165.6 mi; (b) 150.1 mi 9. 28°; 52°; 90° 10. 868.1 ft B 11. 4.8 yd 12. 129.8 ft

Page 511 Written Exercises 1. No 2. No 3. Yes 4. Yes 5. Yes 6. Yes 7. Yes 8. No 9. \(-\overline{MN}\) or \(\overline{NM}\) 10. \(-\overline{KL}\) or \(\overline{LK}\) 11. \(-\overline{KL}\) or \(\overline{KN}\) 12. \(-\overline{LM}\) or \(\overline{ML}\) 13. \(\overline{VS}\) 14. \(\overline{SV}\) 15. \(\overline{WR}\) 16. \(\overline{TW}\) 17. \(\overline{ST}\) 18. \(\overline{WS}\)

(b) 122 lb at an angle of 35° (b) 518 lb at an angle of 170° (b) 100 kg at an angle of 307° (b) 166 g at an angle of 245°

Page 511–512 Problems 1. 142 lb at an angle of 45° 2. 13 mph at a bearing of 67° 3. 151 mph at a bearing of 262° or an angle of 188° 4. 211 mph at a bearing of 276° 5. 534 lb at an angle of 68° with the ground 6. 4326 lb at a bearing of 324°

Page 513 Written Exercises 1. 106 lb horizontally; 106 lb vertically downward 2. 105 lb horizontally; 182 lb vertically downward 3. 376 mph east; 137 mph north 4. 222 mph east; 611 mph north 5. 18 mph east; 18 mph north 6. 11 mph west; 11 mph south 7. (14.7, 8.5) 8. (4.2, 4.2) 9. (10.0, 7) 10. (6.8, 0) 11. (9, 9) 12. (16, 12) 13. (13, -2) 14. (1, 9)

Page 514 Problems 1. 36 lb 2. 17 lb 3. 12 lb 4. 130 lb 5. 1128 lb 6. 308 lb 7. 4284 lb; 5230 lb 8. 88 lb 9. 17 lb 10. 93 lb 11. 104 lb horizontally; 57 lb vertically 12. 137 lb horizontally; 6 lb vertically

Pages 515–516 Chapter Test 1. F 2. T 3. 115° 4. A ray 5. a line segment 6. a point 7. none of these 8. 68 9. 240 ft 10. .3 mi vertically; 2.0 mi horizontally 11. 12 ft 12. 161 or 162 lb 13. 56° 14. 38 lb horizontally, 140 lb vertically


Chapter 15. Comprehensive Review and Tests

Properties of Numbers: Structure


Order prop. of numbers 11. Closure prop. for addition 12. Def. of subset 13. Def. of opposite of zero; no other number has this prop. 15. Def. of multiple 16. Trans. prop. of inequality 17. Def. of equivalent equations

Algebraic Representation

\[
\frac{a + n}{3} \quad 2. p = 12n \quad 3. c = \sqrt{n^2 + 25} \quad 4. (12f - 12t - 3) \quad 5. \frac{25n}{c} \quad 6. \frac{c - nt}{p} \quad 7. (a) 1.2m dollars; (b) .9m dollars \quad 8. 12.5 \leq n < 16 \quad 9. 49\pi < A < 64\pi \quad 10. 9 \quad 11. -8
\]